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The Bound State of Weakly Coupled Schrödinger Operators in One and Two Dimensions

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We study the unique bound state which $(-d^2/dx^2) + \lambda V$ and $-\Delta + \lambda V$ (in two dimensions) have when λ is small and V is suitable. Our main results give necessary and sufficient conditions for there to be a bound state when λ is small and we prove analyticity (resp. nonanalyticity) of the energy eigenvalue at $\lambda = 0$ in one (resp. two) dimensions.

1. INTRODUCTION

It is well known that a one-dimensional quantum mechanical attractive square well binds a single state no matter how small the coupling, and that a three-dimensional square well that is too shallow has no bound states. It is not so well known that two dimensions is like one dimension in that shallow attractive square wells have a bound state in that case. (We first learned this from Mark Kac to whom we are grateful). Given these facts, it is easy to see by a variational argument that if V is everywhere nonpositive, strictly negative on an open set and $V(x) \to 0$ at infinity, then $-\Delta + \lambda V$ will have a bound state no matter how small λ is. Several questions are suggested by this situation. The lowest eigenvalue $e(\lambda)$ if $-\Delta + \lambda V$ is easily seen under wide circumstances to be real analytic for $\lambda > 0$ by an application of Kato–Rellich perturbation theory (see [6, 8]). What about analyticity at $\lambda = 0$? Suppose V is not everywhere nonpositive. When does $-\Delta + \lambda V$ have a bound state for all small positive λ? Surprizingly, these questions do not seem to have been answered before and our goal is to answer them. Our results include:

1. *Conditions for a Bound State*

In one dimension, if $\int (1 + |x|^2)|V(x)|\,dx < \infty$, we prove that $-d^2/dx^2 + \lambda V$ has a bound state for all small positive λ if and only if $\int V(x)\,dx \leqslant 0$. In two dimensions, if $\int |V(x)|^{1+\epsilon}\,d^2x < \infty$ (some $\epsilon > 0$) and $\int (1 + x^2)^\epsilon\,|V(x)|\,d^2x < \infty$,

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then $-\Delta + \lambda V$ has a bound state for all small positive λ if and only if $\int V(x)\, dx \leqslant 0$. In either dimension, if $\int V(x)\, dx = 0$, $-\Delta + \lambda V$ has a bound state for *all* $\lambda \neq 0$. In either case, if λ is small there is only one bound state.

2. *Analyticity and Nonanalyticity at* $\lambda = 0$

Suppose that $-\Delta + \lambda V$ has a unique bound state for all λ small and positive. Let $e(\lambda)$ be its energy. In one dimension, if $\int e^{a|x|} \,|\, V(x)|\, dx < \infty$ for some $a > 0$, then $e(\lambda)$ is analytic at $\lambda = 0$. In two dimensions, $e(\lambda)$ is *never* analytic obeying a bound $|\, e(\lambda)| \leqslant \exp(-(\alpha\lambda)^{-1})$ (resp. $|\, e(\lambda)| \leqslant \exp(-\alpha\lambda^{-2})$) for $\alpha > 0$ if $\int V d^2x < 0$ (resp. $\int V d^2x = 0$).

3. *Threshold Behavior*

In two dimensions $\lambda^n e(\lambda) \to 0$ as $\lambda \to 0$ for any n. In one dimension, $e(\lambda) \sim c\lambda^2$ with $c \neq 0$ if $\int V d\,x < 0$ and $e(\lambda) \sim c\lambda^4$ with $c \neq 0$ if $\int V\, dx = 0$. This should be compared with three dimensions where, if V is spherically symmetric, $e(\lambda) \sim c$ $(\lambda - \lambda_0)$ with $c \neq 0$ for p wave and higher and $e(\lambda) \sim c\ (\lambda - \lambda_0)^2$ for s-waves (generically $c \neq 0$). We will study this type of "threshold" behavior as a general perturbation phenomena in a future article [12].

Our interest in this problem and our general approach is motivated by work of Abarbanel, Callan, and Goldberger [1] who derived Eq. (2) below and formally found the expansion

$$(-E(\lambda))^{1/2} = -\tfrac{1}{2}\lambda \int V(x)\, dx - \tfrac{1}{4}\lambda^2 \int V(x)\,|\, x - y\,|\, V(y)\, dx\, dy + O(\lambda^3)$$

by expanding (2) "by hand." In particular, it is a pleasure to thank M. L. Goldberger for raising this problem and for his encouragement.

2. One-Dimensional Case

Let E be a negative eigenvalue of $(-d^2/dx^2) + V$ so that $((-d^2/dx^2) + V)$ $\phi = E\phi$. Then $\psi = |\, V\,|^{1/2}\, \phi$ formally solves the equation $|\, V\,|^{1/2}\, ((-d^2/dx^2) - E)^{-1}$ $V^{1/2}\psi = \psi$, where $V^{1/2}$ is shorthand for $|\, V\,|^{1/2}$ sign $V = V/|\, V\,|^{1/2}$. One can make this formal argument rigorous if $\int |\, V(x)|\, dx < \infty$. (This is the condition for V to be a form bounded perturbation of $-d^2/dx^2$; one can follow the argument in [9].) Let K_α be the operator, $|\, V\,|^{1/2}\, (-d^2/dx^2 + \alpha^2)^{-1}\, V^{1/2}$ with integral kernel:

$$K_\alpha(x, y) = |\, V(x)|^{1/2}\, (2\alpha)^{-1} \exp(-\alpha\,|\, x - y\,|)\, V^{1/2}(y) \tag{1}$$

if $\int |\, V(x)|\, dx < \infty$ and $\alpha > 0$, then (1) is easily seen to be trace class $(\mathrm{tr}(|\, K_\alpha\,|) < \infty)$ so the theory of trace class determinants [2, 4, 10] is applicable. Then -1 is an eigenvalue of K_α if and only if $\det(1 + K_\alpha) = 0$, so we have:

Proposition 2.1. *Let* $\int |\, V(x)|\, dx < \infty$. *Then* $-\alpha^2$ (with $\alpha > 0$) *is an eigenvalue of* $(-d^2/dx^2) + \lambda V$ *if and only if*:

$$\det(1 + \lambda K_\alpha) = 0. \tag{2}$$

Next we note the elementary:

Proposition 2.2. *If* $\int |\, V(x)|\, dx < \infty$, *then there are constants* Λ_0, c *so that for* $|\,\lambda\,|$ *small* $< \Lambda_0$, $-d^2/dx^2 + \lambda V \geqslant -c\lambda$. *In particular, any negative eigenvalues of* $-d^2/dx^2 + \lambda V$ *approach zero as* λ *goes to zero.*

Proof. There is a basic Sobolev type estimate

$$|\, \phi(x)|^2 \leqslant d(\|-\phi'\|_{L^2}^2 + \|\,\phi\,\|^2). \tag{3}$$

(See, e.g., Kato [6] or Reed–Simon [7] for proofs.)

By (3):

$$\begin{aligned} |(\phi, (-\phi'' + \lambda V\phi))| &\geqslant \|\,\phi'\,\|_2^2 - \lambda \int |\, V(x)|\,|\, \phi(x)|^2 \\ &\geqslant (1 - \lambda d_1)\,\|\,\phi'\,\|_2^2 - \lambda d_1\,\|\,\phi\,\|^2, \end{aligned}$$

with $d_1 = d \int |\, V(x)|\, dx$. Take $c = d_1$ and $\Lambda_0 = d_1^{-1}$. ∎

Now define:

$$L_\alpha(x, y) = |\, V(x)|^{1/2}\, V^{1/2}(y)/2\alpha \tag{4}$$

$$M_\alpha(x, y) = (2\alpha)^{-1}\,|\, V(x)|^{1/2}\, [e^{-\alpha|x-y|} - 1]\, V^{1/2}(y) \tag{5}$$

so that

$$K_\alpha = L_\alpha + M_\alpha\,.$$

Now:

Proposition 2.3. *If* $\int (1 + |\, x\,|^2)\,|\, V(x)|\, dx < \infty$, *then as* $\alpha \downarrow 0$, M_α *converges in Hilbert-Schmidt norm* ($\|\, A\,\|_{HS} = (\mathrm{Tr}(A * A))^{1/2}$) *to the operator*:

$$M_0(x, y) = -1/2\,|\, V(x)|^{1/2}\,|\, x - y\,|\, V^{1/2}(y). \tag{6}$$

Proof. M_0 is Hilbert–Schmidt since

$$\int |\, M_0(x, y)|^2\, dx\, dy \leqslant \int |\, V(x)|\,|\, V(y)|\, (|\, x\,|^2 + |\, y\,|^2) < \infty.$$

Now, $M_\alpha(x, y) \to M_0(x, y)$ pointwise and $| M_\alpha(x, y)| \leqslant | M_0(x, y)|$ since $| e^{-y} - 1 | \leqslant y$ for $y \geqslant 0$; so by the dominated convergence theorem,

$$\int |(M_\alpha - M)(x, y)|^2 \, dx \, dy \to 0. \quad \blacksquare$$

The main technical result of this section is:

THEOREM 2.4. *Suppose that* $\int (1 + | x |^2) \, | V(x)| \, dx < \infty$. *Then, for* λ *small,* $-d^2/dx^2 + \lambda V$ *has at most one negative eigenvalue. There is such an eigenvalue if and only if*

$$\alpha = -(\lambda/2)(V^{1/2}, (1 + \lambda M_\alpha)^{-1} \, | V |^{1/2}) \tag{7}$$

has a solution $\alpha > 0$ *and the eigenvalue is then* $E = -\alpha^2$.

Proof. If there is a negative eigenvalue, $\alpha = -E^{1/2}$ must go to zero as $\lambda \downarrow 0$. Thus for λ small $\| \lambda M_{\alpha(\lambda)} \| < 1$, so $1 + \lambda M_{\alpha(\lambda)}$ is invertible. Thus

$$\det(1 + \lambda K_\alpha) = \det(1 + \lambda M_\alpha) \det(1 + (1 + \lambda M_\alpha)^{-1} \lambda L_\alpha).$$

Since $1 + \lambda M_\alpha$ is invertible, $\det (1 + \lambda M_\alpha) \neq 0$, so $-\alpha^2$ is an eigenvalue if and only if

$$\det(1 + (1 + \lambda M_\alpha)^{-1} \lambda L_\alpha) = 0. \tag{8}$$

Now $(1 + \lambda M_\alpha)^{-1} \lambda L_\alpha$ is rank one and, if A is rank 1, $\det (1 + A) = 1 + \operatorname{tr} A$ so (8) is equivalent to $\operatorname{tr} ((1 + \lambda M_\alpha)^{-1} \lambda L_\alpha) = -1$, which is equivalent to (7).

This proves the entire theorem except for the assertion that (7) has at most one positive solution for λ small and fixed and thus, the Schrödinger operator has at most one eigenvalue. Since there is a one–one correspondence between eigenvalues and solutions of (7) and the number of eigenvalue can only go up if V is replaced by $- | V |$, we need only show that (7) has at most one solution when $V \leqslant 0$. In that case, (7) is equivalent to:

$$\alpha = G(\alpha, \lambda), \tag{9a}$$

$$G(\alpha, \lambda) = (\lambda/2)(| V |^{1/2}, (1 + \lambda M_\alpha)^{-1} \, | V |^{1/2}). \tag{9b}$$

From (9), we see that any solution α for λ small must obey

$$\alpha = (\lambda/2) \int | V(x)| \, dx + O(\lambda^2),$$

so if V is not identically zero, $| \alpha^{-1} | \leqslant C_1 \lambda^{-1}$ for λ small and α *any* solution. Now,

M_α is easily seen to be real analytic as an operator valued function in the region $\operatorname{Re} \alpha > 0$, so by the Cauchy integral formula,

$$\left\| \frac{\partial M_\alpha}{\partial \alpha} \right\| \leqslant C_2 \, | \alpha |^{-1}$$

for all real α and λ small. An explicit calculation shows that

$$\left(| V |^{1/2}, \frac{\partial M_\alpha}{\partial \alpha} \, | V |^{1/2}\right)$$

stays bounded as $\alpha \to 0$ because

$$\int | V(x)| \, | x - y |^2 \, | V(y)| \, dx \, dy < \infty, \qquad \text{(by hypothesis).}$$

It follows that for *all* α with $\alpha^{-1} \leqslant C_1 \lambda^{-1}$ and all small λ:

$$\begin{aligned}
\left| \frac{\partial G}{\partial \alpha} \right| &= \left| \frac{\lambda}{2} \left(| V |^{1/2}, (1 + \lambda M_\alpha)^{-1} \frac{\partial M_\alpha}{\partial \lambda} (1 + \lambda M_\alpha)^{-1} \, | V |^{1/2}\right) \right| \\
&\leqslant \frac{\lambda}{2} \left(| V |^{1/2}, \frac{\partial M_\alpha}{\partial \lambda} \, | V |^{1/2}\right) + C_3 \lambda^2 \left\| \frac{\partial M_\alpha}{\partial \alpha} \right\| \\
&\leqslant C_4 \lambda.
\end{aligned}$$

Thus, for all sufficiently small λ and $\alpha^{-1} \leqslant C_1 \lambda^{-1}$, we have $| \partial G / \partial \alpha | \leqslant 1/2$ if $\alpha_1(\lambda)$ and $\alpha_2(\lambda)$ are two solutions of (9), all α in between obey $\alpha^{-1} \leqslant C_1 \lambda^{-1}$ and so $| \partial G / \partial \alpha | < 1/2$. But then

$$| \alpha_2 - \alpha_1 | = \left| \int_{\alpha_1}^{\alpha_1} (\partial G / \partial \alpha) \, d\alpha \right| \leqslant (1/2) \, | \alpha_2 - \alpha |,$$

so $\alpha_2 = \alpha_1$. $\blacksquare$

THEOREM 2.5. *Let* V *obey* $\int (1 + x^2) | V(x)| \, dx < \infty$, V *not a.e. zero. Then* $-(d^2/dx^2) + \lambda V$ *has a negative eigenvalue for all positive* λ *if and only if*

$$\int V(x) \, dx \leqslant 0. \tag{10}$$

If it does have an eigenvalue, then it is unique and simple and obeys

$$\begin{aligned}
\alpha(\lambda) &\equiv (-E(\lambda))^{1/2} \\
&= -(\lambda/2) \int V(x) \, dx - (\lambda^2/4) \int V(x) \, | x - y | \, V(y) \, dx \, dy + o(\lambda^2).
\end{aligned} \tag{11}$$

Proof. Writing

$$\begin{aligned}(1+\lambda M_\alpha)^{-1} &= 1-\lambda M_\alpha+\lambda^2 M_\alpha^{\,2}(1+\lambda M_\alpha)^{-1}\\ &= 1-\lambda M_0-\lambda(M_\alpha-M_0)+\lambda^2 M_\alpha^{\,2}(1+\lambda M_\alpha)^{-1}\end{aligned}$$

we see that (7) has a unique solution for λ small with $|\alpha|$ small and it is given by (11). The question of whether this solution represents an eigenvalue is equivalent to whether this solution is strictly positive for λ small. If $\int V(x)\,dx < 0$, it clearly is positive. If

$$\int V(x)\,dx = 0,$$

then

$$\int V(x)\,|x-y|\,V(y)\,dx\,dy < 0,$$

so again the solution is strictly positive. To see this later assertion, we need only note that if

$$\int V\,dx = 0,$$

then:

$$\begin{aligned}-\int V(x)\,|x-y|\,V(y)\,dx\,dy &= \lim_{\alpha\downarrow 0} 2\int V(x)\left(\frac{e^{-\alpha|x-y|}-1}{2\alpha}\right)V(y)\\ &= \lim_{\alpha\downarrow 0} 2\int |\hat{V}(k)|^2\,\frac{dk}{k^2+\alpha^2}\\ &= 2\int |\hat{V}(k)|^2\,\frac{dk}{k^2} > 0,\end{aligned}$$

so long as V is not identically zero. (This last integral if finite since $\hat{V}(0) = 0$ and $\hat{V}$ is a C^2 function). ∎

Remarks. 1. There is a close connection between the negativity of $\int V(x)|\,x-y\,|\,V(y)\,dx\,dy$ when $\int V(x) = 0$ and the positive definiteness of $e^{-\alpha|x-y|}$; see, e.g., Gel'fand–Vilenkin [3].

2. Notice that if there is an eigenvalue it is either of order λ^2 or λ^4 for λ small.

3. Notice that if $\int V(x)\,dx = 0$, there is an eigenvalue for all λ, positive or negative; $\lambda \neq 0$.

4. In the theorem we asserted that there was an eigenvalue for all $\lambda > 0$ but only proved this for all small λ. This suffices since the number of eigenvalues can only increase as λ increases. This is because, if $(\psi, ((-d^2/dx^2) + \lambda_0 V)\,\psi) \leqslant 0$, then $(\psi, V\psi) < 0$ so $(\psi, ((-d^2/dx^2) + \lambda V)\,\psi) < 0$ if $\lambda > \lambda_0$. Thus, by the min–mix principle, the number of negative eigenvalue increases.

5. One can systematically develope an asymptotic series to all orders for $\alpha(\lambda)$(and thus; $E(\lambda)$) so long as $\int |\,x\,|^n\,|\,V(x)|\,dx < \infty$ for all n. If $\int |\,x\,|^n\,|\,V(x)|\,dx = \infty$ for some n and $V \leqslant 0$, then some derivatives of $E(\lambda)$ will diverge as $\lambda \downarrow 0$.

6. Theorems of the form of 2.4 and 2.5 should hold if we know that $\int (1 + |\,x\,|)|\,V(x)|\,dx < \infty$ but are definitely false if we only require that $\int |\,V(x)|\,dx < \infty$. For if $V(x) \leqslant -cx^{-2+\epsilon}$ near infinity ($c, \epsilon > 0$), then V will have infinitely many negative eigenvalues even if $\int V(x) > 0$!

THEOREM 2.6. *Suppose that $\int e^{a|x|}\,|\,V(x)|\,dx < \infty$ for some $a > 0$. Then, if $\int V(x) \leqslant 0$, the unique negative eigenvalue $e(\lambda)$ for $-d^2/dx^2 + \lambda V$ occuring for small λ is analytic in λ at $\lambda = 0$.*

Proof. Since $e(\lambda) = (\alpha(\lambda))^2$ we need only show α is analytic. But under the hypothesis, M_α is analytic near $\alpha = 0$ (say in Hilbert–Schmidt norm), so that $F(\alpha, \lambda) = \alpha + (\lambda/2)(V^{1/2}, (1 + \lambda M_\alpha)^{-1}\,|\,V\,|^{1/2})$ is analytic near $\alpha = \lambda = 0$. Since $(\partial F/\partial\alpha)(0, 0) = 1$, and $F(0, 0)$, the unique solution of $F(\alpha, \lambda) = 0$, $\alpha(\lambda)$ for λ small is analytic by the implicit function theorem for functions of several variables [5]. ∎

Remarks. 1. If $\int |\,x\,|^n\,|\,V(x)|\,dx < \infty$, but $\int e^{a|x|}\,|\,V(x)|\,dx = \infty$ for all $a > 0$, it seems quite likely that $e(\lambda)$ will have an asymptotic series to all orders but not be analytic in λ.

2. It is something of a pure coincidence that $e(\lambda)$ is analytic at $\lambda = 0$. For example, if p^2 in $p^2 + \lambda V$ is replaced by $p^\beta + \lambda V$ and $\beta > 1$, then $e(\lambda)$ is not analytic near $\lambda = 0$ (see [12]) for most β.

3. Two-Dimensional Case

This case is not too different in principle from the one-dimensional case although two technical complications occur:

(1) K_α is no longer trace class.

(2) The kernel of $(-\Delta + \alpha^2)^{-1}$ is not as simple as $(2\alpha)^{-1}\exp(-\alpha\,|\,x-y\,|)$.

The first problem will be solved by using the theory of modified determinates [2, 4, 10] and the second by the following:

LEMMA 3.1. *Let $G(\mathbf{x}, \alpha)$ be the integral kernel for $(-\Delta + \alpha^2)^{-1}$ in two dimensions. Then, there will exist entire functions f and g such that*

$$G(\mathbf{x}, \alpha) = f(\alpha\,|\,x\,|)\ln\alpha\,|\,x\,| + g(\alpha\,|\,x\,|). \tag{12}$$

Moreover:

(i) $f(0) = -1/2\pi$,

(ii) *$g(z)$ and $f(z)$ are bounded by*

$C_1 \exp(-C_2(z))$ *on the half axis*, $z \in [0, \infty)$ *for some* $C_1, C_2 > 0$.

Proof. Since $G(x, \alpha) = G(\alpha x, 1)$ the estimate need only be proven for $G(x, 1)$. $G(x, 1)$ is up to a constant a Bessel function of imaginary argument (up to a constant $G(i \mid x \mid, 1)$ is the zeroth order Neumann function) so that (12) and (ii) follow from the theory of such functions [13]. The evaluation of $f(0)$ can be found, e.g., in [11, p. 174]. ∎

Now let

$$M_\alpha(x, y) = \mid V(x)\mid^{1/2} k_\alpha(x - y)\, V(y)^{1/2}, \tag{13}$$

where

$$k_\alpha(x) = f(\alpha \mid x \mid) \ln \mid x \mid + g(\alpha \mid x \mid) + \ln \alpha[f(\alpha \mid x \mid) - f(0)].$$

Thus

$$\mid V \mid^{1/2} (-\Delta - \alpha^2)^{-1}\, V^{1/2} = M_\alpha + L_\alpha,$$

with

$$L_\alpha(x, y) = \mid V(x)\mid^{1/2}\, V(y)^{1/2} (-1/2\pi) \ln \alpha. \tag{14}$$

PROPOSITION 3.2. *Suppose that for some* $\delta > 0$:

$$\int \mid V(x)\mid^{1+\delta}\, dx < \infty; \qquad \int \mid V(x)\mid (1 + x^\delta)\, dx < \infty$$

Then, for any $\alpha \geqslant 0$, M_α *is Hilbert–Schmidt and for* α *small* $\| M_\alpha - M_0 \|_{HS} \leqslant C \alpha^\epsilon$ *for some* $\epsilon > 0$.

Proof. On account of the definition of k_α, we need only prove that:

$$\int \mid V(x)\mid \mid V(y)\mid \mid \ln \mid x - y \mid \mid^2\, dx\, dy < \infty$$

to see that M_α is Hilbert–Schmidt. In the region $\mid x - y \mid \leqslant 1$, the integral is finite by Young's inequality and the fact that $V \in L^{1+\delta}$ by hypothesis. In the region $\mid x - y \mid \geqslant 1$, the integral is finite on account of

$$\mid \ln(x - y)\mid^2 \leqslant C(1 + x^\delta)(1 + y^\delta); \qquad (x - y) \geqslant 1.$$

The only term that is not Lipschitz continuous at $\alpha = 0$ is the term from $\ln \alpha$ $[f(\alpha(x)) - f(0)\mid$. Since $\ln\ \alpha\, [f(\alpha \mid x \mid) - f(0)] \leqslant V(\alpha x)^{\delta/2} \ln \alpha$, this term is Hölder continuous. ∎

Remarks. 1. For later purposes we note:

$$M_0(x, y) = \mid V(x)\mid^{1/2} (-\tfrac{1}{2} \ln (x - y) + g(0))\, V(y)^{1/2}. \tag{15}$$

2. $g(0)$ is an explicit multiple of Euler's constant.

THEOREM 2.3. *Let* V *obey* $\int \mid V(x)\mid^{1+\delta}\, dx < \infty$; $\int \mid V(x)\mid (1 + x^\delta)\, dx < \infty$ *for some* $\delta > 0$. *Then* $-\Delta + \lambda V$ *has at most one negative eigenvalue for* λ *small and positive. It occurs if and only if there is for* λ *small, a positive solution*, $\alpha(\lambda)$, *of*

$$(\ln \alpha)^{-1} = +(2\pi)^{-1} \lambda(V^{1/2}, (1 + \lambda M_\alpha)^{-1} \mid V \mid^{1/2}), \tag{16}$$

where $0 < \alpha(\lambda) < 1$. *In that case, the eigenvalue is* $-\alpha(\lambda)^2$.

Proof. The modified determinent $\det_2$ for A Hilbert–Schmidt is defined by [2, 4, 10]:

$$\det_2 (1 + A) = \det ((1 + A)\, e^{-A}),$$

which is well defined since $(1 + A)\, e^{-A} - 1$ is trace class. $\det_2 (1 + A) = 0$ if and only -1 is an eigenvalue of A. Moreover, if A is Hilbert–Schmidt and B is trace class:

$$\det_2 ((1 + A)(1 + B)) = \det_2 (1 + A) \det (1 + B)\, e^{-\mathrm{Tr}(B)} e^{-\mathrm{Tr}(AB)}.$$

Thus, if $1 + A$ is invertible, $A + B + AB$ has -1 as an eigenvalue if and only if $\det (1 + B) = 0$. The proof now follows by mimicking the proof of Theorem 2.4. ∎

THEOREM 3.4. *Let* V *obey* $\int \mid V(x)\mid^{1+\delta}\, d^2x < \infty$, $\int \mid V(x)\mid(1 + x^\delta)\, d^2x < \infty$ *for some* $\delta > 0$. *Then* $-\Delta + \lambda V$ *has a bound state for all small positive* λ, *if and only if* $\int V(x)\, d^2x \leqslant 0$. *If* $\int V(x) < 0$, *then the eigenvalue* $E(\lambda)$ *obeys*:

$$E(\lambda) \sim -\exp\left(\left[(\lambda/4\pi) \int V(x)\, dx\right]^{-1}\right),$$

where

$$E(\lambda) \sim -\exp(-(a\lambda)^{-\rho}), \qquad (a > 0),$$

means that for λ *small*

$$-\exp(-(a - \epsilon)^{-1} \lambda^{-1}) \leqslant E(\lambda) \leqslant -\exp(-(a + \epsilon)^{-1} \lambda^{-1})$$

If $\int V(x) = 0$, *then* $E(\lambda) \sim -\exp(-(c\lambda)^{-2})$ *for suitable* $c > 0$.

Proof. For $\alpha(\lambda)$ to go to zero (rather than infinity), the right side of (16) must be negative for λ small and positive. The result now follows as in the one-dimensional case. ∎

Note added in proof. The lowest order perturbation terms for weak coupling have been obtained in Landau and Lifshitz, "Quantum Mechanics," pp. 156–157. I should like to thank J. Klauder for bringing this to my attention.

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