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On the Absorption of Eigenvalues by Continuous Spectrum in Regular Perturbation Problems

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We consider the family of operators $A + \lambda B$ with A and B self-adjoint and B relatively form bounded. We consider situations where as $\lambda \downarrow \lambda_1$, some eigenvalue $\mu(\lambda)$ approaches the continuous spectrum of $A + \lambda B$. Typical of our results is the following. If B is relatively form compact, and $\mu(\lambda) \to \mu(\lambda_1)$, then either $(\mu(\lambda) - \mu(\lambda_1))/\lambda - \lambda_1 \to 0$ or $\mu(\lambda_1)$ is an eigenvalue of $A + \lambda_1 B$.

1. INTRODUCTION

Let A and B be self-adjoint operators so that $A \geqslant 0$ and B is A-form bounded with relative bound zero, i.e., $Q(B) \supset Q(A)$ (where $Q(C) = \{\psi \mid \int |\lambda| \, d(\psi, E_\lambda \psi) < \infty$ with E_λ as the spectral measure for C) and for any $a > 0$ there is a b with

$$|(\psi, B\psi)| \leqslant a(\psi, A\psi) + b(\psi, \psi) \tag{1}$$

for all $\psi \in Q(A)$. Under this hypothesis $A + \lambda B$ is an entire analytic family of type (B) in the sense of Kato [3]. In particular, the theory of Rellich [6] and Kato [2, 3] is applicable: If μ_0 is an eigenvalue of $A + \lambda_0 B$ which is discrete (i.e., an isolated point of spec$(A + \lambda_0 B)$) and of multiplicity k and either $k = 1$ or λ_0 is real, then for λ near λ_0, the only spectrum of $A + \lambda B$ near μ_0 is discrete, of total multiplicity k, and given by one or more functions analytic in λ near λ_0.

Let us restrict λ to be real henceforth and suppose μ_0 is a discrete eigenvalue of $A + \lambda_0 B$ (for simplicity, suppose $k = 1$). As λ varies, the eigenvalue μ_0 varies being given by a real analytic function $\mu(\lambda)$. The Kato–Rellich theory described above continues to be applicable so long as $\mu(\lambda)$ stays away from the nondiscrete spectrum of $A + \lambda B$. The questions which will concern us in this note involve the situation which occurs when $\mu(\lambda)$ approaches the nondiscrete spectrum as λ approaches some critical value of λ_1. A typical phenomenon that

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occurs is that the eigenvalue is "absorbed," i.e., as λ is continued past λ_1 the eigenvalue disappears; put differently, as λ is continued in the opposite direction the continuous spectrum "gives birth" to a new eigenvalue. Two specific questions concern us here. Can one tell by looking at $A(\lambda_1)$ and its relation to B that a new eigenvalue is about to appear? What is the "threshold" behavior, i.e., as $\lambda \downarrow \lambda_1$, $\mu(\lambda)$ will approach some point of continuous spectrum $\mu(\lambda_1)$, what is the behavior of $\mu(\lambda) - \mu(\lambda_1)$?

Our interest in this set of problems was aroused by some work we have done on the behavior of the Schrödinger Operator $-\Delta + \lambda V$ in one or two dimensions [9]. In that case, for suitable V, there is a single negative eigenvalue for λ small. In one dimension, this eigenvalue is analytic at $\lambda = 0$ so long as V falls off at infinity at a sufficiently fast rate. In two dimensions, the eigenvalue is never analytic at $\lambda = 0$. We feel that the results of the present note shed some light on this previous work.

We have fairly general and complete results in the case that B is relative A-form compact, i.e., $|B|^{1/2}(A+1)^{-1}|B|^{1/2}$ is compact. Some of these results are abstractions of ideas of Birman [1] and Schwinger [7]. These results appear in Section 2. In Section 3, we describe a meager result in case B is only form bounded.

In interpreting the results of this paper, the reader should bear in mind the following result (which follows as in [8, II.8 App. 2]): If $A \geqslant 0$, if $\mu_n(\lambda)$ is given by the min–max principle for $A + \lambda B$ and if $\mu_n(0) = 0$ (all n), then $\mu_n(\lambda)$ is monotone decreasing in λ. This means that discrete eigenvalues are monotone and that $\Sigma(\lambda) = \inf \sigma_{\text{cont}}(A + \lambda B) = \sup_n \mu_n(\lambda)$ is monotone. We also warn the reader that we systematically abuse notation and use $(\phi, A\phi)$ to stand for the value of a quadratic form a at $\langle \phi, \phi \rangle$.

2. Relatively Compact Perturbations

Suppose that $A \geqslant 0$ and that $0 \in \text{ess spec}(A)$. If B is relatively form compact, then, by a general theorem, $\text{ess spec}(A + \lambda B) = \text{ess spec}(A)$ for all real λ (see, e.g. [4]).

Theorem 2.1. *Let A and B obey the hypotheses of the last paragraph. Suppose that, for $\lambda \in (\lambda_1, \lambda_1 + \epsilon) \subset (0, \infty)$, $A + \lambda B$ has a largest negative eigenvalue $\mu(\lambda)$ which is nondegenerate. Suppose that $\mu(\lambda) \nearrow 0$ as $\lambda \downarrow \lambda_1$ and that no other eigenvalue converges to zero as $\lambda \downarrow \lambda_1$. Then either*

$$\text{(a)} \quad \lim_{\lambda \downarrow \lambda_1} (\lambda - \lambda_1)^{-1} \mu(\lambda) = 0$$

or

(b) 0 *is an eigenvalue of* $A + \lambda_1 B$.

In the latter case, suppose that 0 *is not an eigenvalue of* A. *Then* 0 *is a simple eigenvalue of* $A + \lambda_1 B$, *and*

$$\lim_{\lambda \downarrow \lambda_1} (\lambda - \lambda_1)^{-1} \mu(\lambda) = (\eta, B\eta)$$

where η *obeys* $(A + \lambda_1 B)\eta = 0$, $\| \eta \| = 1$. (*In particular, if* 0 *is not an eigenvalue of* A, *then* $\lim_{\lambda \downarrow \lambda_1} (\lambda - \lambda_1)^{-1} \mu(\lambda) \neq 0$ *if and only if* 0 *is an eigenvalue of* $A + \lambda_1 B$.)

Proof. Suppose first that 0 is not an eigenvalue of $A + \lambda_1 B$. Let $\eta(\lambda)$ be normalized eigenvectors for $A + \lambda B$ with eigenvalue $\mu(\lambda)$. Since B is relatively form A compact, it obeys a bound of the form (1) with a arbitrarily small. Thus, for $\lambda \in (\lambda_1, \lambda_1 + \epsilon)$, we have

$$(\psi, (A + 1)\psi) \leqslant c(\psi, [(A + \lambda B) + c]\psi)$$

for some fixed c. It follows that $\|(A + 1)^{+(1/2)} \eta(\lambda)\|$ remains bounded as $\lambda \downarrow \lambda_1$. Let ψ_0 be a weak limit point for $(A + 1)^{+(1/2)} \eta(\lambda)$ and let $\eta_0 = (A + 1)^{-(1/2)} \psi_0$. Then $\eta(\lambda) \rightarrow \eta_0$ weakly, so for any $\phi \in D(A)$

$$\begin{aligned}&(\phi, (A + \lambda_1 B)\eta_0) \\ &\quad = (\phi, (A + \lambda B)\eta(\lambda)) - (\phi, B\eta_0)(\lambda - \lambda_1) - (\phi, (A + \lambda B)(\eta(\lambda) - \eta_0)) = 0.\end{aligned}$$

The last equality follows by taking $\lambda \downarrow \lambda_1$ and noting that $(\phi, (A + \lambda B)\eta(\lambda)) = \mu(\lambda)(\phi, \eta(\lambda)) \rightarrow 0$, $|(\phi, B\eta_0)(\lambda - \lambda_1)| \rightarrow 0$ and $(\phi, (A + \lambda B)(\eta(\lambda) - \eta_0)) = (A\phi, \eta(\lambda) - \eta_0) + \lambda(B\phi, \eta(\lambda) - \eta_0) \rightarrow 0$. Thus η_0 is an eigenvector for $A + \lambda_1 B$ for eigenvalue zero so $\eta_0 = 0$. Thus since $\{(A + 1)^{-(1/2)} \eta(\lambda)\}$ lie in a compact and the only weak limit point is zero, $(A + 1)^{1/2} \eta(\lambda) \rightarrow 0$ weakly. Since $|B|^{1/2} (A + 1)^{-(1/2)}$ is compact by hypothesis, $|B|^{1/2} \eta(\lambda) \rightarrow 0$ in norm so $(\eta(\lambda), B\eta(\lambda)) \rightarrow 0$. By the Kato–Rellich perturbation theory, for $\lambda \in (\lambda_1, \lambda_1 + \epsilon)$, $d\mu(\lambda)/d\lambda = (\eta(\lambda), B\eta(\lambda))$ so

$$\frac{1}{\lambda - \lambda_1} \mu(\lambda) = \frac{1}{\lambda - \lambda_1} \int_{\lambda_1}^{\lambda} (\eta(\lambda), B\eta(\lambda))\, d\lambda \rightarrow 0$$

proving (a).

Now suppose $A + \lambda_1 B$ has zero as an eigenvalue and that zero is not an eigenvalue of A. It follows that $(\eta, B\eta) < 0$ for any η with $(A + \lambda_1 B)\eta = 0$; $\lambda_1 > 0$. By a simple argument using the min–max principle, one sees that for $\lambda > \lambda_1$, the spectral projection for $(-\infty, 0)$ associated to $A + \lambda B$ has a dimension at least as large as that for the interval $(-\infty, 0]$ associated to $A + \lambda_1 B$. It follows that 0 is a simple eigenvalue of $A + \lambda_1 B$ since, by hypotheses, only one eigenvalue is being absorbed.

For $\lambda > \lambda_1$, let P_λ denote the projection onto the orthogonal complement of the eigenvectors associated to eigenvalues less than $\mu(\lambda)$. Let η obey $\| \eta \| = 1$, $(A + \lambda_1 B)\eta = 0$. Then, by the min–max principle

$$\mu(\lambda) \leqslant (\lambda - \lambda_1)(P_\lambda \eta, BP_\lambda \eta).$$

Since $P_\lambda \eta \to \eta$ as $\lambda \downarrow \lambda_1$, it follows that

$$\overline{\lim_{\lambda \downarrow \lambda_1}}\, (\lambda - \lambda_1)^{-1} \mu(\lambda) \leqslant (\eta, B\eta).$$

By the mean value theorem, the set of limit points of $(\lambda - \lambda_1)^{-1} \mu(\lambda)$ is a subset of the limit points of $d\mu/d\lambda = (\eta(\lambda), B\eta(\lambda))$. It follows that there is a sequence $\lambda_n \downarrow \lambda_1$ so that

$$(\eta(\lambda_n), B\eta(\lambda_n)) \to \underline{\lim}(\lambda - \lambda_1)^{-1} \mu(\lambda).$$

By passing to a further subsequence, we can suppose that $\eta(\lambda_n) \to \eta_\infty$ weakly. As above, η_∞ is an eigenvector for $A + \lambda_1 B$ so $\eta_\infty = \alpha\eta$ with $|\alpha| \leqslant 1$. Thus since $(\eta(\lambda_n), B\eta(\lambda_n)) \to (\eta_\infty, B\eta_\infty)$ as above, we have

$$\underline{\lim}(\lambda - \lambda_1)^{-1} \mu(\lambda) = |\alpha|^2 (\eta, B\eta) \geqslant (\eta, B\eta)$$

(since $(\eta, B\eta) < 0$). Thus $\underline{\lim}(\lambda - \lambda_1)^{-1} \mu(\lambda) \geqslant (\eta, B\eta) \geqslant \overline{\lim}(\lambda - \lambda_1)^{-1} \mu(\lambda)$ so the limit exists and equals $(\eta, B\eta)$. ∎

Remark. We have also proven that after a change of phase, $\eta(\lambda) \to \eta$ as $\lambda \downarrow \lambda_1$ weakly and thus, in norm, since $\| \eta(\lambda) \| = \| \eta \| = 1$.

EXAMPLE. Let V be a spherically symmetric function on $\mathbb{R}^3$ which is short range in some sense ($\int_0^\infty |x|\, |V(x)|\, dx < \infty$ will do if Jost function techniques are used [5]). Thus there are no s-wave (spherically symmetric) zero-energy eigenvalues, but for p-wave and higher (angular momentum $l \geqslant 1$) the limit of negative eigenfunctions will be square integrable. It follows that when a new s-wave bound state appears its energy is $o(\lambda - \lambda_1)$ (in fact a more detailed analysis shows $O(\lambda - \lambda_1)^2$) but for new p-wave and higher bound states we have $O(\lambda - \lambda_1)$ behavior.

As a particular consequence of this theorem, we see that $d\mu(\lambda)/d\lambda$ is bounded as $\lambda \downarrow \lambda_1$. It follows that

COROLLARY 2.2. *Let A and B obey the hypothesis of Theorem 2.1. Suppose that for all real λ, the number $N(\lambda)$, of negative eigenvalue $E_1(\lambda), \ldots, E_{N(\lambda)}(\lambda)$ of $A + \lambda B$ is finite. Then for any $\gamma > 1$*

$$f_\gamma(\lambda) = \sum_{n=1}^{N(\lambda)} (-E_n(\lambda))^\gamma$$

is continuously differentiable.

Our next two results which abstract ideas of Birman [1] and Schwinger [7] provide some illumination of Theorem 2.1. We state them in the case $B \leqslant 0$ where the strongest results exist. Given A and B so that $A \geqslant 0$, B is A-form compact, and $A + \lambda B$ has only finitely many negative eigenvalues, we call $\lambda_1 \geqslant 0$ a *threshold coupling constant* if and only if for $\lambda > \lambda_1$ there are more negative eigenvalues than for $\lambda < \lambda_1$.

THEOREM 2.3. *Let $B \leqslant 0$, $A \geqslant 0$ with B A-form compact. Then λ_1 is a threshold coupling constant if and only if*

$$\lim_{E \uparrow 0} \| \, | B |^{1/2} (A + \lambda_1 B - E)^{-1} | B |^{1/2} \| = \infty.$$

Moreover, the largest negative eigenvalue $\mu(\lambda)$ for $\lambda > \lambda_1$ but near λ_1 is given by the implicit equation

$$\| \, | B |^{1/2} (A + \lambda_1 B - \mu)^{-1} | B |^{1/2} \| = (\lambda - \lambda_1)^{-1}. \tag{2}$$

Proof. By a simple argument (see, e.g. [8]), a number $E < 0$ which is not an eigenvalue of $A + \lambda_1 B$ is an eigenvalue of $A + \lambda B$ if and only if $(\lambda - \lambda_1)^{-1}$ is an eigenvalue of $| B |^{1/2} (A + \lambda_1 B - E)^{-1} | B |^{1/2} = K(E)$. Let $e_0 < 0$ be the largest strictly negative eigenvalue of $A + \lambda_1 B$. For $E > e_0$, $K(E)$ is positive definite on a fixed space of finite dimension and is bounded on the negative definite space as $E \uparrow 0$. Since $K(E)$ is compact, $K(E)$ has an eigenvalue going to plus infinity as $E \uparrow 0$ if and only if $\| K(E) \| \nearrow \infty$ and the eigenvalue is given by the norm. ▌

Equation (2) sheds some light on Theorem 2.1 and could probably be used as the basis for an alternative proof. For, if 0 is an eigenvalue of $A + \lambda_1 B$ and η is the corresponding eigenvectors then $(\eta, | B |^{1/2} \eta) \neq 0$ so $(\eta, K(E) \eta)$ has a first-order pole as $E \downarrow 0$, i.e., (2) has a solution $\mu(\lambda)$ with $\mu(\lambda)^{-1} > c(\lambda - \lambda_1)^{-1}$. Conversely, if (2) has such a solution $| B |^{1/2} (A + \lambda_1 B - \mu)^{-1} | B |^{1/2}$ has a μ^{-1} singularity at $\mu = 0$. Since $| B |^{1/2}$ is compact, this should imply the existence of a fixed η with $(\eta, (A + \lambda_1 B - \mu)^{-1} \eta) \geqslant c(\mu)^{-1}$ which implies the existence of a zero eigenvalue.

By an argument very similar to that proving Theorem 2.3, one proves

THEOREM 2.4. *Under the hypothesis of Theorem* 2.3, *suppose that $\lambda = 0$ is not a threshold coupling constant. Then* $\lim_{E \uparrow 0} | B |^{1/2} (A - E)^{-1} | B |^{1/2}$ *exists (denote it by $| B |^{1/2} A^{-1} | B |^{1/2}$). Suppose that it is compact. Then the threshold coupling constants λ_i are related to the eigenvalue γ_i of $| B |^{1/2} A^{-1} | B |^{1/2}$ by $\lambda_i = \gamma_i^{-1}$.*

Let us close this section by considering a class of examples which shows that when a new eigenvalue appears, it can happen that $\mu(\lambda) \sim c(\lambda - \lambda_i)^{\alpha}$ for any

$\alpha \geqslant 1$. On $L^2(-\infty, \infty)$ consider the operators $p^\beta - \lambda e^{-|x|}$. Then for η suitable $(\eta, K_\beta(E)\,\eta) \sim \text{const} \int (|\psi(p)|^2/(p^\beta - E))\, dp$ where $\psi(p) \neq 0$ near zero so $(\eta, K_\beta(E)\,\eta) \sim cE^{-1+1/\beta}$ so long as $\beta > 1$. By a detailed analysis along the lines of [9] we can show that $\| K(E)\| \sim cE^{-1+1/\beta}$ so there is a negative eigenvalue $\mu(\lambda)$ for λ near zero with $\lambda^{-1} \sim c\mu(\lambda)^{-1+1/\beta}$ or $\mu(\lambda) \sim d\lambda^{\beta/\beta-1}$. For $\beta = 1$, one can show that $\mu(\lambda) \sim c \exp(-1/d\lambda)$, see [9].

3. Relatively Bounded Perturbations

We have less to say in case B is only assumed relatively bounded rather than relatively compact. What we can extend is the result that the approach of eigenvalues to the continuum is no faster than linear.

Theorem 3.1. *Let A be a self-adjoint operator with* $0 = \inf \operatorname{ess\,spec}(A)$. *Let B be a symmetric quadratic form with $Q(B) \supset Q(A)$ and*

$$|(\psi, B\psi)| \leqslant a(\psi, A\psi) + b(\psi, \psi). \tag{3}$$

For $|\lambda| < a^{-1}$, let $\Sigma(\lambda) = \inf \operatorname{ess\,spec}(A + \lambda B)$ and let $\mu_i(\lambda)$ $(i = 1,\dots,)$ be given by the min–max *principle* [4] *so that the μ_i are eigenvalues if $\mu_i(\lambda) < \Sigma(\lambda)$ and all eigenvalues below $\Sigma(\lambda)$ occur as $\mu_i(\lambda)$. Then for any $\epsilon > 0$, the $\mu_i(\lambda)$ and $\Sigma(\lambda)$ obey*

$$|\mu_i(\lambda) - \mu_i(\lambda')| \leqslant c\,|\lambda - \lambda'|, \tag{4}$$

$$|\Sigma(\lambda) - \Sigma(\lambda')| \leqslant c\,|\lambda - \lambda'|, \tag{5}$$

all $\lambda, \lambda' \in (-a^{-1} + \epsilon, a^{-1} - \epsilon)$. In particular, the approach of any eigenvalue to Σ is at fastest linear in the coupling constant.

Remark. The idea of the proof follows an argument from Simon [8]. Since $\Sigma(\lambda) = \lim_{i\to\infty} \mu_i(\lambda)$, (4) implies (5). Given ϵ, for any $\lambda \in (-a^{-1} + \epsilon, a^{-1} - \epsilon)$, we see that $(\psi, (A + \lambda B)\,\psi) \leqslant 2(\psi, A\psi) + b\,|a|^{-1}(\psi, \psi)$ so $\Sigma(\lambda) \leqslant b\,|a|^{-1}$. On the other hand, $(\psi, (A + \lambda B)\,\psi) \geqslant (a\,\epsilon)(\psi, A\psi) - b\,|a|^{-1}(\psi, \psi)$. Thus, in the min–max principle defining μ_i we need only consider ψ's with $\|\psi\| = 1$ and

$$(\psi, A\psi) \leqslant 2b\epsilon^{-1}a^{-2}.$$

By (6) and (3), for such ψ, the function

$$e_\psi(\lambda) = (\psi, (A + \lambda B)\,\psi)$$

obeys

$$|e_\psi(\lambda) - e_\psi(\lambda')| \leqslant b(2a^{-1}\epsilon^{-1} + 1)\,|\lambda - \lambda'| \tag{7}$$

(7) and the min–max principle implies (4) with $c = b(2a^{-1}\epsilon^{-1} + 1)$. ∎

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