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Notes on Infinite Determinants of Hilbert Space Operators

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We present a novel approach to obtaining the basic facts (including Lidskii's theorem on the equality of the matrix and spectral traces) about determinants and traces of trace class operators on a separable Hilbert space. We also discuss Fredholm theory, "regularized" determinants and Fredholm theory on the trace ideals, $\mathscr{C}_p(p < \infty)$.

1. INTRODUCTION

This note represents an approach to the abstract Fredholm theory of trace class (and more generally $\mathscr{C}_p = \{A \mid \mathrm{Tr}(|A|^p) < \infty\}$) operators on a separable Hilbert space, $\mathscr{H}$. There are few new results here but there are a set of new proofs which we feel sheds considerable light on the theory discussed. In particular, we would emphasize our proof of Lidskii's theorem (see Sect. 4): It was this new proof that motivated our more general discussion here.

To help emphasize the differences between our approach and others, we remark on the differences in the definition of the infinite determinant $\det(1 + A)$ for trace class A. First, some notations (formal definitions of algebraic multiplicity, etc., appear later): Given a compact operator, A, $\{\lambda_i(A)\}_{i=1}^{N(A)}$ ($N(A) = 1, 2, \ldots,$ or ∞) is a listing of all the nonzero eigenvalues of A, counted up to algebraic multiplicity and $\{\mu_i(A)\}_{i=1}^{\infty}$, the singular values of A, i.e., eigenvalues of $|A| \equiv (A^*A)^{1/2}$ listed so that $\mu_1(A) \geqslant \mu_2(A) \geqslant \cdots \geqslant 0$. Throughout, the trace of an operator in the trace class is defined by

$$\mathrm{Tr}(A) = \sum_{n=1}^{\infty} (\phi_n, A\phi_n), \tag{1.1}$$

for any orthonormal basis $\{\phi_n\}_{n=1}^{\infty}$.

The only two systematic analytic treatments of $\det(1 + A)$ for abstract $A \in \mathscr{C}_1$ of which we are aware are those of Gohberg and Krein [7] and Dunford and Schwartz [4] who rely on the basic definitions (respectively)

$$\det(1 + \mu A) = \prod_{i=1}^{N(A)} (1 + \mu\lambda_i(A)), \tag{1.2}$$

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$$\det(1 + \mu A) = \exp[\mathrm{Tr}(\log(1 + \mu A))] \tag{1.3}$$

We use instead the definition (also used by Grothendiek [8] in his algebraic discussion of infinite determinants)

$$\det(1 + \mu A) = \sum_{n=0}^{\infty} \mu^n \, \mathrm{Tr}(\Lambda^n(A)) \tag{1.4}$$

where $\Lambda^n(A)$ is defined in terms of alternating algebra. Of course, any full treatment must, in the end, establish the equality of all three definitions. This equality is a consequence of the theorem of Lidskii [15]:

$$\mathrm{Tr}(A) = \sum_{i=1}^{N(A)} \lambda_i(A) \tag{1.5}$$

(and, as we shall see, the equality of the Definitions (1.4) and (1.2) implies (1.5)!). At first sight (1.5) seems trivial, but to appreciate its depth, the reader should consider trying to prove it for a quasinilpotent trace class operator (i.e., one with $\{0\}$ as spectrum). The formula (1.2) is very natural but it is quite difficult to work with analytically. For example, even after one proves absolute convergence of the product (and so analyticity of $\det(1 + \mu A)$ in μ), the analyticity of $\det(1 + A + \mu B)$ in μ is not easy to prove. We find the formula (1.3) quite unnatural as a general definition since $\mathrm{tr}(\log(1 + \mu A))$ is singular for those μ with $(1 + \mu A)$ noninvertible and is only determined modulo $2\pi i$. The main advantage of (1.3) is the small μ expansion which leads to the formula of Plemelj [22]:

$$\det(1 + A) = \exp\left(\sum_{n=1}^{\infty} (-1)^{n-1} \, \mathrm{Tr}(A^n)/n\right) \tag{1.6}$$

which converges if $\mathrm{Tr}(|A|) < 1$ (or more generally, if $\mathrm{Tr}(|A|^p) < 1$ for some p). While (1.6) is often called Plemelj's formula, we note that it occurs in Fredholm's original paper [5]! Equation (1.6) is a useful motivation in the theory of regularized determinants (see Sect. 6).

In distinction, the formula (1.4) has the following advantages:

(i) Once one has the basic bound $|\mathrm{Tr}(\Lambda^n(A)| \leqslant (\mathrm{Tr}(|A|))^n/n!$, the analytic properties of $\det(1 + A)$ are easy to establish; e.g., $\det(1 + A + \mu B)$ is obviously analytic as a uniformly convergent sum of polynomials.

(ii) The algebraic properties of the determinant, in particular, $\det(1 + A)\det(1 + B) = \det(1 + A + B + AB)$ follow from the functional nature of Λ^n. In the finite dimensional case, this is well known (see e.g., Lang [13]). This formula occurs in Fredholm's original paper [5] proven via computation of various derivatives. Grothendieck [8] proves our Theorem 3.9 by the algebraic method we discuss.

(iii) If A is an integral operator (which is trace class) with a continuous kernel, (1.4) reduces to the definition of Fredholm [5]. This fact, which is useful for an abstract Fredholm theory (!) is far from evident from (1.2) or (1.3). While Eq. (1.4) is essentially Fredholm's definition, it is analytically simpler because of the possibility of using invariance of the trace; in particular one avoids Hadamard's inequality in proving the convergence of (1.4).

To distinguish our proof of Lidskii's theorem, (1.5), from those in [4, 7] we might compare them in the finite-dimensional case. In that case, there are two says of seeing (1.5): One can pass to a Jordan normal form, whence (1.5) follows by inspection (and the invariance of trace), or one can consider the characteristic polynomial, whence Eq. (1.5) follows by using the fact that the sum of the roots of a monic polynomial $P(X)$ is the coefficient of its next to leading term. In essence, the proofs in [4, 7] are analogs of the Jordan normal form proof while ours is via a "characteristic polynomial": In brief, we prove Eq. (1.5) by "applying Hadamard factorization to Fredholm's determinant." A primary complication in the "normal form" proof of (1.5) is the lack of a normal form for quasinilpotent operators. This must be gotten around by a limiting argument [7] or by an argument that is essentially our proof in the special case where A is quasi-nilpotent [4]!. (In this case Hadamard's factorization theorem can be replaced by Liouville's theorem.) The only place that we need to appeal to a limiting argument from a finite rank approximation is in our proof that $\det(1 + A) \det(1 + B) = \det(1 + A + B + AB)$.

We should mention that Carleman [3](and also Hille and Tamarkin [10]) establish a Hadamard factorization of $\det_2(1 + A)$(see Sect. 6). In particular, had they choosen to look at the second term of the Taylor series in their equalities they would have for A Hilbert–Schmidt that

$$\operatorname{Tr}(A^2) = \sum_{i=1}^{N(A)} \lambda_i(A)^2$$

(but they did not choose to do this). Hille and Tamarkin [10] have similar formulas in the trace class case and one can easily prove Lidskii's theorem from their results (essentially by the method shown in Sect. 4).

The material we present here is "foundational" and so it is important to have some care in how one proves the basic facts about trace class operators and trace ideals, lest one introduce a circularity. Thus, let us sketch the basic definitions and facts, primarily following the discussion in Reed and Simon [24, 25, 26]:

(1) [24, Sect. VI.5]. The closure in the norm topology of the finite rank operators on $\mathscr{H}$ is called the *compact operators*, $\mathscr{C}_\infty$. Any operator $A \in \mathscr{C}_\infty$ has a spectrum which is countable with only zero being a possible accumulation point.

Any $\lambda \in \sigma(A)$ which is nonzero is an eigenvalue. Any $A \in \mathscr{C}_\infty$ has a *canonical expansion*:

$$A = \sum_{n=1}^{\infty} \mu_n(A)(\phi_n, \cdot)\, \psi_n \tag{1.7}$$

where $\mu_n(A)$ are the *singular values* (eigenvalues of $|A| \equiv (A^*A)^{1/2}$) and $\{\phi_n\}_{n=1}^{\infty}$ (resp. $\{\psi_n\}_{n=1}^{\infty}$) are orthonormal sets (the ϕ_n are eigenvectors for A^*A and the ψ_n for AA^*). We order the $\mu_n(A)$ by $\mu_1(A) \geqslant \mu_2(A) \geqslant \cdots \geqslant 0$.

(2) ([26, Sect. XII.1, 2]; see also [19]). Given $\lambda \in \sigma(A)$ with $A \in \mathscr{C}_\infty$ and $\lambda \neq 0$, one defines the *spectral projection* P_λ by

$$P_\lambda = (2\pi i)^{-1} \oint_{|E-\lambda|=\epsilon} dE(E - A)^{-1} \tag{1.8}$$

for all small ϵ. Then P_λ is a finite-dimensional (nonorthogonal) projection so that A leaves $P_\lambda\mathscr{H}$ and $(1 - P_\lambda)\,\mathscr{H}$ invariant. Moreover, $\sigma(A \upharpoonright P_\lambda\mathscr{H} = \{\lambda\}$, $\sigma(A \upharpoonright (1 - P_\lambda)\mathscr{H}) = \sigma(A)\backslash\{\lambda\}$ and $\operatorname{Ran} P_\lambda = \{\psi \mid (A - \lambda)^n \psi = 0 \text{ for some } n\}$. We call dim (Ran P_λ) the algebraic multiplicity of λ. A list of all nonzero eigenvalues counting algebraic multiplicity of A is denoted by $\{\lambda_i(A)\}_{i=1}^{N(A)}$.

Remark. To define Eq. (1.8) all that is required is that λ be an isolated point of $\sigma(A)$ and the further properties of P_λ all hold whenever P_λ is finite-dimensional. Both conditions automatically hold if $A \in \mathscr{C}_\infty$ and $\lambda \neq 0$.

(3) [24, Sect. VI.6]. For any positive self-adjoint operator, A, the sum $\sum_{n=1}^{\infty} (\phi_n, A\phi_n)$ is independent of orthonormal basis and denoted $\operatorname{Tr}(A)$. The *trace class* $\mathscr{C}_1$ (called $\mathscr{I}_1$ in [24]) is those operators with $\operatorname{Tr}(|A|) < \infty$. One shows that $A \in \mathscr{C}_1$ if and only if A is compact with $\sum_{n=1}^{\infty} \mu_n(A) < \infty$. $\operatorname{Tr}(|A|) = \sum_{n=1}^{\infty} \mu_n(A)$ is called the *trace norm*, $\|\cdot\|_1$. $\mathscr{C}_1$ is a $*$-ideal in $\mathscr{L}(\mathscr{H})$ and one has

$$\| A + B \|_1 \leqslant \| A \|_1 + \| B \|_1, \; \| A^* \|_1 = \| A \|_1$$

and (1.9)

$$\| AB \|_1 \leqslant \| A \|_1 \| B \|_\infty$$

For $A \in \mathscr{C}_1$ and any orthonormal basis $\{\phi_n\}_{n=1}^{\infty}$, the sum $\sum_{n=1}^{\infty} (\phi_n, A\phi_n)$ is absolutely convergent and defines a number $\operatorname{Tr}(A)$, the *trace* of A, independent of basis ϕ_n. $\operatorname{Tr}(\cdot)$ is a $*$-linear functional on $\mathscr{I}_1$ with

$$|\operatorname{Tr}(A)| \leqslant \| A \|_1. \tag{1.10}$$

For any unitary U, $\| UAU^{-1} \|_1 = \| A \|_1$ and $\operatorname{Tr}(UAU^{-1}) = \operatorname{Tr}(A)$.

(4) [24, Sect. VI.6; 25, Appendix to Sect. IX.4]. The trace ideal, $\mathscr{C}_p$ $(1 \leqslant p < \infty)$ is defined as those A with $|A|^p \in \mathscr{C}_1$. Then $A \in \mathscr{C}_p$ if and only if A is compact and $\sum_{n=1}^{\infty} \mu_n(A)^p = \operatorname{Tr}(|A|^p) \equiv \| A \|_p^p < \infty$. From Eqs.

(1.9) and (1.10) and a simple complex interpolation argument, one easily finds that ($p^{-1} + q^{-1} = 1$).

$$|\operatorname{Tr}(AB)| \leqslant \| A \|_p \| B \|_q \tag{1.11}$$

(Hölder's inequality for operators), from which it follows that

$$\| A \|_p = \sup_{\substack{B \neq 0 \\ B \in \mathscr{C}_q}} (|\operatorname{Tr}(AB)|/\| B \|_q^q). \tag{1.12}$$

(Take $B = |A|^{p-1}U^*$ if $A = U|A|$ to get equality.) From (1.12), the triangle inequality for $\|\cdot\|_p$ follows. $\mathscr{C}_p$ is a *-ideal in $\mathscr{L}(\mathscr{H})$.

(5) [30]. In one place we need the existence of a Schur "basis," i.e., for any $A \in \mathscr{C}_\infty$, an orthonormal *set* (not necessarily complete), $\{\eta_n\}_{n=1}^{N(A)}$ so that

$$\lambda_n(A) = (\eta_n, A\eta_n). \tag{1.13}$$

One obtains (1.13) by writting a Jordan normal form for A on each F_λ (λ an eigenvalue which is nonzero) and then applying a Gram–Schmidt procedure.

Remark. It is with some reluctance that we use this device since it requires a "Jordan normal form" for A. We emphasize it enters in our proof of Lidskii's theorem only in the proof that $\sum_{i=1}^{N(A)} \lambda_i(A) < \infty$, something that can be proven by other means [7].

(6) ([24, Sects. II.4 and VIII.10]; see also [18]). Given $\mathscr{H}$, a separable Hilbert space, the n-fold antisymmetric product $\wedge^n\mathscr{H}$ is defined. If $\{\phi_i\}_{i=1}^\infty$ is an orthonormal basis for $\mathscr{H}$, then $\phi_{i_1} \wedge \cdots \wedge \phi_{i_n}$ ($i_1 < i_2 < \cdots i_n$) is a basis for $\wedge^n\mathscr{H}$. Given $A: \mathscr{H} \to \mathscr{H}$, one defines $\wedge^n(A): \wedge^n\mathscr{H} \to \wedge^n\mathscr{H}$ so that $\wedge^n(A)(\phi_1 \wedge \cdots \wedge \phi_n) = A\phi_1 \wedge \cdots \wedge A\phi_n$ for any $\phi_1, \ldots, \phi_n \in \mathscr{H}$. $\wedge^n(\cdot)$ is a functor, i.e., $\wedge^n(AB) = \wedge^n(A)\wedge^n(B)$ and $\wedge^n(A^*) = \wedge^n(A)^*$. Thus, e.g., $|\wedge^n(A)| = \wedge^n(|A|)$. If $\mathscr{H}$ is finite-dimensional, with $\dim(\mathscr{H}) = m$ then $\dim \wedge^n(\mathscr{H}) = \binom{m}{n}$, and $\wedge^m(A)$ on the one-dimensional space $\wedge^m\mathscr{H}$ is just multiplication by $\det(A)$. $\wedge^n\mathscr{H}$ is a natural subspace of $\otimes^n \mathscr{H}$, the n-fold tensor product.

We conclude this introduction with a sketch of the contents of these notes. In our proof of Lidskii's theorem, we need to know that for $A \in \mathscr{C}_1$, $\sum_{i=1}^{N(A)} |\lambda_i(A)| < \infty$ (so that the definition (1.2) converges). As noted in [7], this follows easily from Eq. (1.1) and the existence of a Schur basis, but we give an alternate proof of the more general Weyl [36] inequalities:

$$\sum_{i=1}^{N(A)} |\lambda_i(A)|^p \leqslant \| A \|_p^p \tag{1.14}$$

in Section 2. (For $p = 1$, these inequalities are associated with work of Lalesco [12], Gheorghiu [6], and Hille and Tamarkin [10] and for $p = 2$ with Schur [29].)

This proof depends less on intricate convex function arguments than do the usual ones [4, 7]. In Section 3, we define (by Eq. (1.4)) the determinant for operators of the form $1 + A$ with $A \in \mathscr{C}_1$, and in Section 4 we prove Lidskii's theorem. In Section 5, we illustrate the usefulness of the definition (1.4) by proving a determinant inequality (essentially found already in [33]). In Section 6, we define $\det_n(1 + A)$ by:

$$\det_n(1 + A) = \det\left[(1 + A)\exp\left(-\sum_{k=1}^{n-1}(-1)^{k+1}A^k/k\right)\right], \qquad (1.15)$$

and show it is defined for $A \in \mathscr{C}_p$ for $p \geqslant n$. Finally, in Section 7, we recover the usual Fredholm theory in abstract form.

We remark that it is an interesting open question to establish the theorem of Lidskii in the Banach space setting (see [8, 14, 27, 28]). Even Weyl's inequality, Eq. (1.14), for $p = 1$ appears to be open in this case. See added note (3).

2. Some Inequalities of Weyl

Our goal here is to prove the inequality (1.14) and some related facts. We first note the following:

Lemma 2.1. *Let A be a compact operator. Let $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be orthonormal sets. Then*

$$(f_n, Ag_n) = \sum_{m=1}^{\infty} \alpha_{nm}\mu_m(A) \qquad (2.1)$$

where (α) is a doubly substochastic matrix, i.e.,

$$\sum_{m=1}^{\infty} |\alpha_{nm}| \leqslant 1; \qquad n = 1, 2, \ldots \qquad (2.2a)$$

$$\sum_{n=1}^{\infty} |\alpha_{nm}| \leqslant 1; \qquad m = 1, 2, \ldots \qquad (2.2b)$$

Proof. By the canonical form (1.7), (2.1) holds with

$$\alpha_{nm} = (f_n, \phi_m)(\psi_m, g_n).$$

Since $\{f_n\}$ and $\{g_n\}$ are orthonormal families, we have by Bessel's inequality and the Schwarz inequality:

$$\sum_{n=1}^{\infty} |\alpha_{nm}| \leqslant \left(\sum_{n=1}^{\infty} |(f_n, \phi_m)|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} |(g_n, \psi_m)|^2\right)^{1/2} \leqslant \|\phi_m\| \|\psi_m\| = 1$$

since the $\{\phi_m\}$ and $\{\psi_m\}$ are normalized. Similarly (2.2a) holds since the $\{\phi_m\}$ and $\{\psi_m\}$ are orthonormal. ∎

The following is so basic to our proof of (1.14), that we overkill it with three proofs, each of which illustrates different aspects of the result.

LEMMA 2.2. *Let* α *be a doubly substochastic matrix (i.e., let* (2.2) *hold). Let* μ_m *be a sequence with* $(\sum_{m=1}^{\infty} |\mu_m|^p)^{1/p} < \infty$ *for some* $1 \leqslant p \leqslant \infty$. *Then the sums*

$$\lambda_n = \sum_{m=1}^{\infty} \alpha_{nm}\mu_m \tag{2.3}$$

are convergent and

$$\left(\sum_{n=1}^{\infty} |\lambda_n|^p\right)^{1/p} \leqslant \left(\sum_{n=1}^{\infty} |\mu_p|^p\right)^{1/p}. \tag{2.4}$$

First Proof. By (2.2a) the result clearly holds if $p = \infty$. By duality and (2.2b), we get the case $p = 1$. The general case now holds by the Reisz–Thorin interpolation theorem on l_p (see, e.g., [24, Sect. IX.4]). ∎

Second Proof. The sum (2.3) is clearly convergent $p = \infty$, so let $p < \infty$. If q is the dual index of p, then:

$$\begin{aligned}
\sum_{n=1}^{\infty} |\lambda_n|^p &\leqslant \sum_{n,m=1}^{\infty} |\lambda_n|^{p-1} |\alpha_{nm}| |\mu_m| \\
&= \sum_{n,m=1}^{\infty} [|\lambda_n|^{p-1} |\alpha_{nm}|^{1/q}][|\alpha_{nm}|^{1/p} |\mu_m|] \\
&\leqslant \left(\sum_{n,m=1}^{\infty} |\alpha_{nm}| |\lambda_n|^p\right)^{1/q} \left(\sum_{n,m=1}^{\infty} |\alpha_{nm}| |\mu_m|^p\right)^{1/p} \\
&\leqslant \left(\sum_{n=1}^{\infty} |\lambda_n|^p\right)^{1/q} \left(\sum_{m=1}^{\infty} |\mu_m|^p\right)^{1/p},
\end{aligned}$$

from which (2.4) follows. In the above we have used Hölder's inequality (on sequences indexed by pairs (n, m)) in the second inequality and (2.2) in the last inequality. ∎

Third Proof (suggested by E. Seiler). Let ϕ be an arbitrary convex function on $[0, \infty)$ with $\phi(0) = 0$ and $\phi(x) \geqslant 0$. It is then automatic that ϕ is monotone. We claim that

$$\sum_{n=1}^{\infty} \phi(|\lambda_n|) \leqslant \sum_{m=1}^{\infty} \phi(|\mu_m|), \tag{2.5}$$

from which (2.4) follows taking $\phi(x) = x^p$. Now, since ϕ is monotone:

$$\begin{aligned}\phi(|\lambda_n|) &\leqslant \phi\left(\sum_m |\alpha_{nm}|\,|\mu_m|\right) \\ &\leqslant \sum_m |\alpha_{nm}|\,\phi(|\mu_m|).\end{aligned} \tag{2.6}$$

The second inequality follows by convexity and (2.2a) writing

$$\sum_{m=1}^{\infty} |\alpha_{nm}|\,|\mu_m| = \sum_{m=0}^{\infty} |\alpha_{nm}|\,|\mu_m|$$

with $\alpha_{n0} = 1 - \sum_{m=1}^{\infty} |\alpha_{nm}|$ and $\mu_0 = 0$. Summing over n, (2.5) follows from (2.2b) and (2.6). ∎

THEOREM 2.3. ([29] *for* $p = 2$, [12] *for* $p = 1$, [36] *for general* p). *For any compact operator* A *and* $1 \leqslant p \leqslant \infty$,

$$\sum_{m=1}^{\infty} |\lambda_m(A)|^p \leqslant \|A\|_p^p. \tag{2.7}$$

More generally, for any orthonormal sets $\{f_n\}$, $\{g_n\}$ *we have*

$$\sum_{m=1}^{\infty} |(f_m, Ag_n)|^p \leqslant \|A\|_p^p.$$

Proof. The general result follows immediately from the last two lemmas. Equation (2.7) then follows from the existence of a Schur 'basis" $\{\eta_n\}_{n=1}^{N(A)}$ (see (5) in Sect. 1) taking $f_n = g_n = \eta_n$. ∎

These are the only inequalities from this section we will need later. However, we wish to make a few remarks about extending the method above by using a few additional "tricks." First we note that, by using the third proof of Lemma 2.2, we can conclude that

$$\sum_{n=1}^{\infty} \phi(|\langle f_n, Ag_n\rangle|) \leqslant \sum_{n=1}^{\infty} \phi(\mu_n(A)) \tag{2.8}$$

for any convex ϕ with $\phi(0) = 0$, $\phi(x) \geqslant 0$. Equation (2.8) is not quite as strong as Weyl's theorem which only requires $t \to \phi(e^t)$ to be convex. For example, the function $\phi(x) = \ln(1 + x)$ is such that (2.8) holds (by Weyl's theorem) but it is not of the form we have treated so far. Second, there is a general principle illustrated by the following:

THEOREM 2.4. *If A is compact and $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are orthonormal sets, then for any N and $p > 1$:*

$$\sum_{n=1}^{N} |(f_n, Ag_n)|^p \leqslant \sum_{n=1}^{N} |\mu_n(A)|^p. \tag{2.9}$$

In particular, for any N eigenvalues $\lambda_1(A),..., \lambda_N(A)$:

$$\sum_{n=1}^{N} |\lambda_n(A)|^p \leqslant \sum_{n=1}^{N} |\mu_n(A)|^p. \tag{2.10}$$

Proof. Let P be the orthogonal projection onto the space spanned by $\{g_n\}_{n=1}^{N}$. Let $B = AP$. Then, by Theorem 2.3:

$$\sum_{n=1}^{N} |(f_n, Ag_n)|^p = \sum_{n=1}^{N} |(f_n, Bg_n)|^p$$

$$\leqslant \sum_{n=1}^{\infty} |\mu_n(B)|^p.$$

Now, since B has rank N, $\mu_n(B) = 0$ for $n \geqslant N + 1$ and by a simple min–max principle argument, $\mu_n(B) \leqslant \mu_n(A)$ for all n and, in particular, for $n = 1,..., N$. Equation (2.9) thus follows. By using a Schur basis, (2.10) follows. ∎

The third principle, following Weyl [36], systematically exploits the antisymmetric tensor products. For example, we have Weyl's original inequality:

$$|\lambda_1(A) \cdots \lambda_N(A)| \leqslant \mu_1(A) \cdots \mu_N(A). \tag{2.11}$$

For $\mu_1(A) \cdots \mu_N(A)$ is the norm of $\wedge^N(A)$ on $\wedge^N \mathscr{H}$ (as the largest eigenvalue of $|\wedge^N(A)| = \wedge^N(|A|)$) and $\lambda_1(A) \cdots \lambda_N(A)$ is an eigenvalue of $\wedge^N(A)$. By combining this idea and the second principle above we can, for example, prove the following theorem of Ostrowski [21]:

THEOREM 2.5. *Let $\lambda_1(A),..., \lambda_N(A)$ be N eigenvalues of a compact operator A. For $k \leqslant N$, let $\sum_k(a_1,..., a_N)$ be the elementary symmetric function given by:*

$$\Sigma_k(a_1, \cdot, a_N) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant N} a_{i_1} \cdots a_{i_k}.$$

Then for any $p \geqslant 1$:

$$\Sigma_k(|\lambda_1|^p,..., |\lambda_N|^p) \leqslant \Sigma_k(\mu_1(A)^p,..., \mu_N(A)^p). \tag{2.12}$$

In particular, for any $r > 0$:

$$\prod_{i=1}^{N} (1 + r \mid \lambda_i(A)|^p) \leqslant \prod_{i=1}^{N} [1 + r\mu_i(A)^p]. \tag{2.13}$$

Proof. As in the construction of a Schur basis, we can find an orthonormal set $e_{ij}, \ldots, e_N$ with

$$Ae_i = \lambda_i e_i + \sum_{j<i} \alpha_{ji} e_j .$$

It follows, that if P is a the orthogonal projection onto the span of $\{e_n\}_{n=1}^{N}$, then the nonzero eigenvalues of $B = AP$ are $\lambda_1, \ldots, \lambda_N$. Thus the nonzero eigenvalues of $\wedge^k(B)$ are $\lambda_{i_1}(A) \cdots \lambda_{i_k}(A)(1 \leqslant i_1 < \cdots < i_k \leqslant N)$. Therefore (2.12) follows from Eq. (2.7) and the method of proof of Theorem 2.4 (2.13) follows from

$$\prod_{i=1}^{N} (1 + a_i) = \sum_{k=1}^{N} \sum_{k} (a_1, \ldots, a_N). \quad \blacksquare$$

Remark. In particular, (2.13) with $r = 1$ is the $\phi(x) = \ln(1 + x)$ result of Weyl mentioned above.

3. Definition and Properties of the Determinant

The basic estimate we need to define $\det(1 + A)$ is

LEMMA 3.1. *For any $A \in \mathscr{C}_1(\mathscr{H})$ we have that $\wedge^k(A) \in \mathscr{C}_1(\wedge^k)\mathscr{H})$ for all k. Moreover*

$$\| \wedge^k(A) \|_1 \leqslant \| A \|_1^k / k! \tag{3.1}$$

Proof. The eigenvalues of $| \wedge^k(A)| = \wedge^k(| A |)$ are

$$(\mu_{i_1}(A) \cdots \mu_{i_k}(A))(i_1 < \cdots < i_k).$$

Thus:

$$\begin{aligned} \| \wedge^k(A) \|_1 &= \sum_{i_1 < \cdot < i_k} \mu_{i_1}(A) \cdots \mu_{i_k}(A) \\ &= \frac{1}{k!} \sum_{i_1 < \cdot < i_k} \mu_{i_1}(A) \cdots \mu_{i_k}(A) \\ &= \frac{1}{k!} \| A \|_1^k . \quad \blacksquare \end{aligned} \tag{3.2}$$

DEFINITION. For $A \in \mathscr{C}_1$, we define $\det(1 + A)$ by

$$\det(1 + A) = \sum_{k=0}^{\infty} \operatorname{Tr}(\Lambda^k(A)). \tag{3.3}$$

THEOREM 3.2. *The sum* (3.3) *converges for any* $A \in \mathscr{C}_1$ *and*

$$|\det(1 + A)| \leqslant \exp(\| A \|_1). \tag{3.4}$$

$$|\det(1 + A)| \leqslant \prod_{n=1}^{\infty} (1 + \mu_n(A)). \tag{3.5}$$

Proof. Equation (3.4) follows from (3.1) and (3.5) from (3.2). ∎

THEOREM 3.3. *Fix* $A_1, \ldots, A_m \in \mathscr{C}_1$. *Then*

$$\det\left(1 + \sum_{i=1}^{m} \lambda_i A_i\right) = F(\lambda_1, \cdot, \lambda_m)$$

is an entire function of m complex variables. More generally, if $F(\lambda)$ *is an analytic function with values in* $\mathscr{C}_1$, *then* $\det(1 + F(\lambda))$ *is analytic where F is analytic.*

Proof. By definition

$$\det\left(1 + \sum_{i=1}^{m} \lambda_i A_i\right) = \sum_{k=0}^{\infty} \operatorname{Tr}\left(\Lambda^k\left(\sum_{i=1}^{m} \lambda_i A_i\right)\right)$$

and by (3.1) the sum converges absolutely and uniformly on compact subsets of C^m. Since each term $\operatorname{Tr}(\Lambda^k(\sum_{i=1}^m \lambda_i A_i))$ is a polynomial, F is an entire function. ∎

THEOREM 3.4. *Fix* $A \in \mathscr{C}_1$. *Then, for any* ϵ, *there is a constant* $C(\epsilon)$ *with*

$$|\det(1 + \lambda A)| \leqslant C(\epsilon) \exp(\epsilon |\lambda|). \tag{3.6}$$

Proof. Since $|1 + x| \leqslant \exp(|x|)$, we have, by (3.5)

$$|\det(1 + \lambda A)| \leqslant \prod_{m=1}^{\infty} (1 + |\lambda| \mu_m(A))$$

$$\leqslant \prod_{m=1}^{M} (1 + |\lambda| \mu_m(A)) \exp\left(\sum_{m=M+1}^{\infty} |\lambda| \mu_m(A)\right).$$

Choose M so that $\sum_{M+1}^{\infty} \mu_m(A) < \epsilon/2$. Now, we can choose $C(\epsilon)$ so that $[\prod_{m=1}^{M} (1 + |\lambda| \mu_m(A))] \leqslant C(\epsilon) \exp((\epsilon/2)|\lambda|)$. ∎

Remark. On account of the inequality [31]:

$$|\det(1 + A + B)| \leqslant \det(1 + |A|) \det(1 + |B|)$$

one can conclude:

$$\left|\det\left(1 + \sum_{i=1}^{m} \lambda_i A_i\right)\right| \leqslant C(\epsilon) \exp\left(\epsilon \sum_{i=1}^{m} |\lambda_i|\right).$$

THEOREM 3.5. *The map* $A \to \det(1 + A)$ from $\mathscr{C}_1$ *to* $\mathbb{C}$ *is continuous, i.e., if* $\| A_n - A \|_1 \to 0$, *then* $\det(1 + A_n) \to \det(1 + A)$.

Remark. By using Cauchy estimates on the analytic function $\det(1 + A + \mu(B - A))$ and the bound (3.4), one can prove [32]:

$$|\det(1 + A) - \det(1 + B)| \leqslant \| A - B \|_1 \exp(\| A \|_1 + \| B \|_1 + 1) \tag{3.7}$$

(see also Theorem 6.5 below).

Proof. Let $C = \sup_n \| A_n \|_1$. Given ϵ, choose M with $\sum_{m \geqslant M+1} C^m/m! < \epsilon/3$. Then by (3.1):

$$|\det(1 + A_n) - \det(1 + A)| \leqslant \frac{2\epsilon}{3} + \sum_{m=1}^{M} \operatorname{Tr}(|\Lambda^m(A_n) - \Lambda^m(A)|).$$

Now, let P_m be the orthogonal projection from $\otimes^m \mathscr{H}$ to $\Lambda^m \mathscr{H}$. Then

$$\begin{aligned}\operatorname{Tr}(|\Lambda^m(A_n) - \Lambda^m(A)|) &= \operatorname{Tr}\left(\left| P_m \left(\bigotimes^m A_n - \bigotimes^m A\right) P_m \right|\right) \\ &\leqslant \operatorname{Tr}\left(\left| \bigotimes^m A_n - \bigotimes^m A \right|\right) \\ &\leqslant m C^{m-1} \| A - A_n \|_1,\end{aligned}$$

so choosing N so that $\| A - A_n \|_1 \leqslant (\epsilon/3)(\sum_{m=1}^{M} m C^{m-1})^{-1}$ we see that for $n > N$:

$$|\det(1 + A_n) - \det(1 + A)| < \epsilon. \quad \blacksquare$$

LEMMA 3.6. *Let A be a finite rank operator with $PAP = A$ for some finite rank orthogonal projection P. Let $\Lambda_P^m(A)$ be the operator $\Lambda^m(PAP)$ as an operator on $\Lambda^m(P\mathscr{H})$. Then, if* $\dim(P\mathscr{H}) = k$:

$$\det(1 + A) = \operatorname{tr}(\Lambda_P^k(1 + A)).$$

Proof. Clearly $\operatorname{tr}(\Lambda^m(A)) = \operatorname{tr}(\Lambda^m(PAP)) = \operatorname{tr}(\Lambda_P^m(A))$, so $\det(1 + A) = \sum_{m=0}^{k} \operatorname{tr}(\Lambda_P^m(A)) = \operatorname{tr}(\Lambda_P^k(1 + A))$. $\blacksquare$

LEMMA 3.7. *Let A and B be finite rank operators. Then*

$$\det(1 + A)\det(1 + B) = \det(1 + A + B + AB). \tag{3.8}$$

Proof. Let P be a finite-dimensional orthogonal projection with ran A, ran A^*, ran B and ran B^* all in ran P. Then, if dim $P = m$:

$$\begin{aligned}\det(1 + A)\det(1 + B) &= \operatorname{tr}(\Lambda_p^m(1 + A))\operatorname{tr}(\Lambda_p^m(1 + B)) \\ &= \operatorname{tr}(\Lambda_p^m(1 + A)\,\Lambda_p^m(1 + B)) \\ &= \det(1 + A + B + AB),\end{aligned}$$

where we have used the fact that $\Lambda^m(P\mathscr{H})$ is one-dimensional so that for operators C, D, on it, $\operatorname{Tr}(C)\operatorname{Tr}(D) = \operatorname{Tr}(CD)$. ∎

THEOREM 3.8. [8]. *For any A, $B \in \mathscr{C}_1$:*

$$\det(1 + A)\det(1 + B) = \det(1 + A + B + AB). \tag{3.9}$$

Proof. Let P_n be a family of finite rank orthogonal projections converging strongly to 1. Then, by Lemma 3.7, (3.9) holds if A, B are replaced by $A_n = P_nAP_n$ and $B_n = P_nBP_n$. As $n \to \infty$, $\| A_n - A \|_1 \to 0$, $\| B_n - B \|_1 \to 0$, and $\| A_nB_n - AB \|_1 \to 0$ so, by Theorem 3.5, the determinants converge. Thus (3.9) holds. ∎

THEOREM 3.9. *Let $A \in \mathscr{C}_1$. Then* $\det(1 + A) \neq 0$ *if and only if* $1 + A$ *is invertible.*

Proof. Suppose $1 + A$ is invertible. Then $B = -A(1 + A)^{-1}$ is in $\mathscr{C}_1$ and $A + B + AB = (1 + A)(1 + B) - 1 = 0$. Then

$$\det(1 + A)\det(1 + B) = \det(1) = 1$$

so $\det(1 + A) \neq 0$. On the other hand, if $1 + A$ is not invertible, then -1 is an eigenvalue of A. Let P be the corresponding spectral projection. Then $1 + A = (1 + AP)[1 + A(1 - P)]$, so it suffices to prove $\det(1 + AP) = 0$. Now, by Lemma 3.6, $\det(1 + AP)$ is the finite-dimensional determinant of an operator with eigenvalue -1 and is thus zero. ∎

THEOREM 3.10. *If $-\mu_0^{-1}$ is an eigenvalue of multiplicity k, then $F(\mu) = \det(1 + \mu A)$ has a zero of order precisely k at $\mu = \mu_0$.*

Proof. Let P be the spectral projection for $-\mu_0^{-1}$. Then

$$\det(1 + \mu A) = \det(1 + \mu AP) \det[1 + \mu A(1 - P)].$$

Now $\det [1 + \mu A(1 - P)] \neq 0$ by Theorem 3.9. Also $B = AP$ is zero on $(1 - P)\mathscr{H}$ and has only spectrum $-\mu_0^{-1}$ on $P\mathscr{H}$. Thus $\operatorname{Tr}(\Lambda^m(AP)) = (-1)^m\binom{k}{m}\mu^{-m}$ so that

$$\det(1 + \mu AP) = \sum_{m=0}^{k} \binom{k}{m} (-\mu/\mu_0)^m = (1 - \mu\mu_0^{-1})^m. \quad \blacksquare$$

4. Lidskii's Theorem

The key to Lidskii's theorem is:

THEOREM 4.1. *Let $F(z)$ be an entire function with zeros at $z_1, z_2, \ldots$ (counting multiplicity) so that*

(1) $F(0) = 1$.

(2) *For any* ϵ, $|F(z)| \leqslant C(\epsilon)\exp(\epsilon |z|)$.

(3) $\sum_{n=1}^{\infty} |z_n|^{-1} < \infty$.

Then

$$F(z) = \prod_{j=1}^{\infty} (1 - zz_j^{-1}). \tag{4.1}$$

Remark. This is not quite the same as Hadamard's theorem. For, in general (2) only implies that $\sum_{n=1}^{\infty} |z_n|^{-1-\epsilon} < \infty$, and $F(z) = e^{az}\prod_{n=1}^{\infty}(1 - zz_n^{-1}) e^{z/z_n}$ with $a = -\sum_{n=1}^{\infty} z_n^{-1}$ (conditional convergence with $|z_1| \leqslant |z_2| \leqslant \cdots$); this is a theorem of Lindelöf [17], see [1]). However, our proof is essentially a piece of a standard proof of Hadamard's theorem (see, e.g., [35]).

Proof. Let $G(z) = \prod_{n=1}^{\infty}(1 - zz_n^{-1})$ which is convergent to an entire function by (3). Since $F(z)/G(z)$ is an entire nonzero function,

$$F(z) = G(z)\, e^{h(z)}.$$

Now for fixed R, let $z_1, \ldots, z_n$ be the zeros of F with $|z_i| < R/2$. Then for $|z| = R$, $|1 - zz_i^{-1}| \geqslant 1$, so $F(z)(\prod_{i=1}^{n} 1 - zz_i^{-1})^{-1} = H_R(z)$ has $\sup_{|z|<R} |H_R(z)| \leqslant C(\epsilon)\, e^{\epsilon R}$. It follows by the Borel–Carathéodory theorem [35, pp. 174–175] that for $|z| \leqslant 1$

$$|\ln H_R(z)| \leqslant 2(\tfrac{1}{2} R - 1)^{-1}[\epsilon R + \ln C(\epsilon)],$$

where we have used the fact that $H_R(0) = 1$. Moreover, if $R \geqslant 4$ and $|z| \leqslant 1$

$$\left| \ln \left[\prod_{i=n+1}^{\infty} (1 - zz_i^{-1})^{-1} \right] \right| \leqslant C |z| \sum_{n+1}^{\infty} |z_i|^{-1},$$

where C is choosen so that $|\ln(1 - z)| \leqslant C|z|$ if $|z| \leqslant \frac{1}{2}$. It follows by taking $R \to \infty$ that $h(z) = \ln H_R(z) + \ln[\prod_{n+1}^{\infty} \cdots]$ obeys $|h(z)| \leqslant 4\epsilon$. Since ϵ is arbitrary, h is identically zero for $|z| \leqslant 1$ and so for all z. ∎

THEOREM 4.2. *For any $A \in \mathscr{C}_1$ and $\mu \in \mathbb{C}$:*

$$\det(1 + \mu A) = \prod_{j=1}^{N(A)} (1 + \mu\lambda_j(A)). \tag{4.2}$$

Proof. Let $F(\mu) = \det(1 + \mu A)$. By Theorems 3.9 and 3.10 the zeros of F are precisely (counting multiplicities) at $-\lambda_j(A)^{-1}$. By Theorems 3.4 and 2.3, $F(\mu)$ obeys the hypothesis of Theorem 4.1. ∎

COROLLARY 4.3. (Lidskii's Theorem [15]). $\operatorname{Tr}(A) = \sum_{j=1}^{N(A)} \lambda_j(A)$ *for* $A \in \mathscr{C}_1$.

Proof. The term linear in μ in the Taylor expansion of $\det(1 + \mu A)$ is $\operatorname{Tr}(A)$. The term linear in μ on the right of (4.2) is $\sum_{j=1}^{N(A)} \lambda_j(A)$. ∎

Remarks. 1. Equation (4.2) of course also implies that

$$\operatorname{Tr}(\Lambda^k(A)) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1}(A) \cdots \lambda_{i_k}(A),$$

but this is just Lidskii's theorem for $\Lambda^k(A)$!

2. From Remark 1 Lidskii's theorem implies (4.2)!

5. DETERMINANT INEQUALITIES

We want to illustrate the use of (1.4) as a tool in proving inequalities on determinants. Seiler and Simon [31] have already used (1.4) to prove:

$$\det(1 + |A + B|) \leqslant \det(1 + |A|) \det(1 + |B|) \tag{5.1}$$

although alternate proofs avoiding (1.4) have been found by Lieb [16] and Kato (unpublished; see [32]). Seiler and Simon [33] have proven a variety of complicated inequalities tailor made for their study of the Yukawa$_2$ quantum field theory. By using their method, we can prove an inequality of some general interest that illustrates the applicability of (1.4):

THEOREM 5.1. *For $A \in \mathscr{C}_1$, define*

$$\det_2(1 + A) = \det(1 + A)\, e^{-\operatorname{Tr}(A)}. \tag{5.2}$$

Then, for any $A, B \in \mathscr{C}_1$:

$$|\det_2(1 + A + B)| \leqslant \exp(1/2 \| B^*B \|_1 + \alpha \| A \|_1) \tag{5.3}$$

where $\alpha = 1 + e^{1/2} = 2.6487\ldots$.

Remark. The point is that $\det_2(1 + A + B)$ extends from $\mathscr{C}_1$ to a continuous function $\mathscr{C}_2$ with

$$|\det_2(1 + A + B)| \leqslant \exp(1/2 \| A + B \|_2^2) \tag{5.4}$$

(see Sect. 6 below). Of course, since $|\det(1 + A)| \leqslant \exp(\| A \|_1)$, we have

$$|\det_2(1 + A + B)| \leqslant \exp[2(\| A \|_1 + \| B \|_1)].$$

However, if $A \in \mathscr{C}_1$ and $B \in \mathscr{C}_2$, we cannot use this to bound $|\det_2(1 + \mu A + B)|$ as $|\mu| \to \infty$ and (5.4) would only give us $|\det_2(1 + \mu A + B)| \leqslant C_1 \exp(C_2 |\mu|^2)$. Equation (5.3) gives $|\det_2(1 + \mu A + B)| \leqslant C_3 \exp(C_4 |\mu|)$.

Proof.

$$\begin{aligned} |\det(1 + A + B)| &= |\det(1 + B) \det[1 + (1 + B)^{-1} A]| \\ &= \Big| \sum_{n=0}^{\infty} \det(1 + B) \operatorname{Tr}(\Lambda^n[(1 + B)^{-1} A]) \Big| \\ &\leqslant \sum_{n=0}^{\infty} \| \det(1 + B)\, \Lambda^n[(1 + B)^{-1}] \| \operatorname{Tr}(|\Lambda^n(A)|) \\ &\leqslant \sum_{n=0}^{\infty} \| \det(1 + B)\, \Lambda^n[(1 + B)^{-1}] \| \, \| A \|_1^n / n! \end{aligned} \tag{5.5}$$

Now, we claim that

$$\| \det(1 + B)\, \Lambda^n[(1 + B)^{-1}] \|^2 \leqslant e^n \exp(2 \operatorname{Re}(\operatorname{Tr}(B)) + \| B^*B \|_1). \tag{5.6}$$

Temporarily deferring the proof of (5.6), we note that (5.2) and (5.5) together with (5.6) imply

$$\begin{aligned} |\det_2(1 + A + B)| &\leqslant \exp(-\operatorname{Re} \operatorname{Tr}(A) - \operatorname{Re} \operatorname{Tr}(B)) \sum_{n=0}^{\infty} (e^{1/2} \| A \|_1)^n / n! \\ &\quad \times \exp(\operatorname{Re} \operatorname{Tr}(B) + 1/2 \| B^*B \|_1) \\ &\leqslant \exp(\alpha \| A \|_1 + 1/2 \| B^*B \|). \end{aligned}$$

We compute

$$\begin{aligned}&\|\det(1+B)\,\Lambda^n[(1+B)^{-1}]\|^2\\ &\qquad=\|\det(1+B+B^*+B^*B)\,\Lambda^n(1+B+B^*+B^*B)\|,\end{aligned}$$

so (5.6) is implied by

$$\|\det(1+C)\,\Lambda^n((1+C)^{-1})\|\leqslant e^n\exp(\operatorname{Tr}(C)) \tag{5.7}$$

for all self-adjoint C with $-1\leqslant C$. Now, let $\lambda_i(C)$ be the eigenvalues of C ordered by $\lambda_1(C)\geqslant\cdots\geqslant-1$. Then:

$$\begin{aligned}\|\det(1+C)\,\Lambda^n((1+C)^{-1})\|&\leqslant\prod_{m=n+1}^{\infty}(1+\lambda_m(C))\\ &\leqslant\exp\left(\sum_{m=n+1}^{\infty}\lambda_m(C)\right)\\ &\leqslant e^n\exp(\operatorname{Tr}(C))\end{aligned}$$

since $-\sum_{m=1}^{n}\lambda_m(C)\leqslant n$. ∎

6. Regularized Determinants

It was realized quite early that Fredholm's original 1903 theory was not applicable to a wide class of integral operators of interest. In 1904 Hilbert [9] showed how to extend the class of operators which could be treated by replacing $K(x, x)$ by zero in all formulas and Carleman [3] later showed that this definition worked for all operators which are now called Hilbert–Schmidt. Contributions to this line of development were made by Lalesco [11] who, in particular, realized when $\operatorname{Tr}(K)$ was finite, Hilbert's determinant "$\det_2$" and Fredholm's determinant, "$\det_1$" were related by

$$\det\nolimits_2(1+A)=\det\nolimits_1(1+A)\exp(-\operatorname{Tr}(A)),$$

and by Hille and Tamarkin [10] and Smithies [34].

In a 1910 paper that has been widely ignored, Poincaré [23], apparently unaware of Hilbert's work, studied integral equations $f=(I+K)g$ where some power of K, say K^n, is an operator to which Fredholm's theory can be applied. By using this theory for K^n, he was able to show that

$$\det\nolimits_n(1+\mu K)\equiv\exp\left(\sum_{j=n}^{\infty}(-1)^{j+1}\mu^jK^j/j\right)$$

is well defined by the series for $|\mu|$ small and defined by analytic continuation an entire function. The interesting feature of Poincaré's work is his ability to reduce the estimates to those of Fredholm except for K^n.

Motivated by the Hilbert–Carleman–Smithies line of development, $\det_n$ has been systematically developed by Gohberg and Krien [7] and Dunford and Schwartz [4]. The theory of $\det_4$ was independently developed by Brascamp [2].

In this section, we wish to establish the main properties of $\det_n(1 + A)$. Unlike most of the treatments discussed above, we avoid the need for any new estimates in defining $\det_n$ by reducing the analysis (following Seiler [30]) to what we have already discussed in defining $\det_1$ (see Lemma 6.1 below). In philosophy (but not techniques), we thus follow Poincaré. Our approach partly follows the appendix of [32]. In particular, we follow the proof of (3.7) in proving that $\det_n(1 + A)$ is Lipschitz on $\mathscr{C}_n$—a continuity statement that appears to be new.

LEMMA 6.1 (essentially in [30, 32]). *For any bounded operator A, define*

$$R_n(A) = \left[(1 + A) \exp\left(\sum_{k=1}^{n-1} (-A)^k/k\right)\right] - 1.$$

Then:

(a) *If $A \in \mathscr{C}_n$, then $R_n(A) \in \mathscr{C}_1$.*

(b) *If $f(z)$ is an analytic function with values in $\mathscr{C}_n$ (analytic as a $\mathscr{C}_n$-valued function), then $R_n(f(z))$ is a function analytic as a $\mathscr{C}_1$-valued function.*

Proof. Let $h(z) = z^{-n}\{[(1 + z) \exp(\sum_{k=1}^{n-1} (-z)^k/k] - 1\}$. By an elementary computation, h is an entire function. Clearly

$$R_n(A) = A^n h(A).$$

Hence, $\| R_n(A)\|_1 \leqslant \| A \|_n^n \| h(A)\|_\infty < \infty$ by Hölder's inequality for operators. Now, let $f(z)$ be a function analytic as a $\mathscr{C}_n$-valued function. Then, it is clearly analytic as a $\mathscr{C}_\infty$-valued function, so $h(f(z))$ is a $B(\mathscr{H})$-valued analytic function. It follows that $\operatorname{Tr}(BR_n(f(z)))$ is analytic for any finite rank B. Moreover, by the above, $\| R_n(f(z))\|_1$ is uniformly bounded on compact subsets of the domain of definition of f. Now under the duality $\langle A, B\rangle \to \operatorname{Tr}(AB)$, $\mathscr{C}_1$ is the dual of $\mathscr{C}_\infty$ and $B(\mathscr{H})$ is the dual of $\mathscr{C}_1$[24, Sect. VI.6]. Since the finite rank operators are dense in $\mathscr{C}_1$, it follows that given any $B \in B(\mathscr{H})$, we can find B_α a net of finite rank operators with $\| B_\alpha \|_{\mathrm{op}} \leqslant \| B \|_{\mathrm{op}}$ so that $\operatorname{Tr}(AB_\alpha) \to \operatorname{Tr}(AB)$ for any $A \in \mathscr{C}_1$. Thus $\operatorname{Tr}(R_n(f(z))\, B_\alpha)$ converges pointwise to $\operatorname{Tr}(R_n(f(z))B)$ so by the Vitali theorem, $\operatorname{Tr}(R_n(f(z))B)$ is analytic for each $B \in \mathscr{L}(\mathscr{H})$. It follows that $R_n(f(z))$ is analytic as a $\mathscr{C}_1$-valued function. ∎

DEFINITION. For $A \in \mathscr{C}_n$, $\det_n(1 + A) = \det(1 + R_n(A))$.

THEOREM 6.2. *Let $\lambda_1(A),\ldots$ be the eigenvalues of $A \in \mathscr{C}_n$. Then*:

$$\det_n(1 + \mu A) = \prod_{m=1}^{\infty} \left[(1 + \mu\lambda_m(A)) \exp\left(\sum_{k=1}^{n-1} \mu^k(-\lambda_m(A))^k/k\right)\right] \tag{6.2}$$

Moreover, for $A \in \mathscr{C}_{n-1}$:

$$\det_n(1 + A) = \det_{n-1}(1 + A) \exp[(-1)^{n-1} \operatorname{Tr}(A^{n-1})/n] \tag{6.3}$$

and, in particular, for $A \in \mathscr{C}_1$:

$$\det_n(1 + A) = \det(1 + A) \exp\left(\sum_{k=1}^{n-1} (-1)^k \operatorname{Tr}(A^k)/k\right). \tag{6.4}$$

Remark. Equation (6.2) is natural from the point of Hademard's theorem which we have been emphasizing. For if we only know that $\sum_{m=1}^{\infty} | \lambda_m(A)|^n < \infty$ and we want a function "det" $(1 + \mu A)$ with zeros precisely at $\mu = -\lambda_m(A)^{-1}$, we need a canonical product of genus $(n - 1)$.

Proof. By the spectral mapping theorem, the eigenvalues (including algebraic multiplicity) of $R_n(\mu A)$ are $(1 + \mu\lambda_m(A))$ $\exp(\sum_{k=1}^{n-1} \mu^k(-\lambda_m(A))^k/k) - 1$, so (6.2) follows from Theorem 4.2, Equation (6.3) follows from

$$(1 + R_n(A)) = (1 + R_{n-1}(A)) \exp((-1)^{n-1}A^{n-1}/n - 1),$$

Theorem 3.8 and the fact that for $A \in \mathscr{C}_1$:

$$\det(e^A) = e^{\operatorname{Tr}(A)}$$

(which follows from Theorem 4.2 and Lidskii's theorem). ∎

COROLLARY 6.3. *Let $A \in \mathscr{C}_n$. Then $(1 + A)$ is invertible if and only if* $\det_n(1 + A) \neq 0$.

For later purposes we note that there exists a constant Γ_n with

$$\left| (1 + z) \exp\left(\sum_{k=1}^{n-1} (-z)^k/k\right)\right| \leqslant \exp(\Gamma_n | z |^n). \tag{6.5}$$

Equation (6.5) is obvious, since it clearly holds for $| z | > \epsilon$ (for any ϵ) and for $| z |$ small since the left side is $1 + 0(z^n)$ for z small. We remark that $\Gamma_1 = 1$, $\Gamma_2 = \frac{1}{2}$, and for any n $\Gamma_n \geqslant 1/n$ (by using z small) $\Gamma_n \leqslant e(2 + \ln n)$ [20], also $\Gamma_4 \leqslant \frac{3}{4}$ [2].

THEOREM 6.4.

$$| \det_n(1 + A)| \leqslant \exp(\Gamma_n \| A \|_n^n). \tag{6.6}$$

Proof. By (6.2) and (6.5):

$$|\det_n(1+A)| \leqslant \exp\left(\Gamma_n \sum_{m-1}^{\infty} |\lambda_m(A)|^n\right)$$
$$\leqslant \exp(\Gamma_n \| A \|_n^n)$$

by Theorem 2.3. ∎

THEOREM 6.5. $\det_n(1+A)$ *is Lipschitz as a function on* $\mathscr{C}_n$ *uniformly on balls, explicitly*:

$$|\det_n(1+A) - \det_n(1+B)| \leqslant \| A - B \|_n \exp[\Gamma_n(\| A \|_n + \| B \|_n + 1)^n].$$

Proof. This clearly follows from (6.6), the lemma following (which is an abstraction of an agreement in [32]), and the fact that if $f(z)$ is an analytic $\mathscr{C}_n$-valued function, $\det_n(1 + f(z))$ is analytic by combining Lemma 6.1 and Theorem 3.3. ∎

LEMMA 6.6. *Let f be a complex-valued function on a complex Banach space X so that*

(a) $f(A + zB)$ *is an entire function of z for all* $A, B, \in X$.

(b) *There is a function G on* $[0, \infty)$ *which is monotone nondecreasing, so that for all* $A \in X$:

$$|f(A)| \leqslant G(\| A \|).$$

Then:

$$|f(A) - f(B)| \leqslant \| A - B \| \, G(\| A \| + \| B \| + 1) \tag{6.7}$$

for all $A, B \in X$.

Proof. Let $g(z) = f(\frac{1}{2}(A + B) + z(A - B))$. Then g is entire in z and

$$\begin{aligned} |f(A) - f(B)| &= |g(\tfrac{1}{2}) - g(-\tfrac{1}{2})| \\ &\leqslant \sup_{-\frac{1}{2}\leqslant t\leqslant \frac{1}{2}} |g'(t)| \\ &\leqslant k^{-1} \sup_{|z|\leqslant k+\frac{1}{2}} |g(z)| \end{aligned} \tag{6.8}$$

for any k. In the last step we use the Cauchy estimate

$$|g'(t)| \leqslant k^{-1} \sup_{|w|=k} |g(t+w)|.$$

Take $k = \| A - B \|^{-1}$. For $| z | \leqslant k + \frac{1}{2}$,

$$\begin{aligned} | g(z) | &\leqslant G(\|(A + B/2) + z(A - B)\|) \\ &\leqslant G(\| A + B/2 \| + (k + \tfrac{1}{2}) \| A - B \|) \\ &\leqslant G(\| A \| + \| B \| + 1). \end{aligned}$$

Thus (6.8) implies (6.7). ∎

Since $\det_n(1 + \mu A)$ is an entire function of μ, it clearly has a convergent power series expansion $\det_n(1 + \mu A) = \sum \alpha_m^{(n)}(A)\, \mu^m/m!$ The form of this series (essentially found by Plemelj [22]($n = 1$) and Smithies [34]($n = 2$)) illuminates the choice of $\det_n(1 + A)$ so we derive the formulas:

Lemma 6.7. *Let $f(z)$ be analytic for z small with $f(z) = \sum_{n=1}^{\infty} (-1)^{n+1} b_n z^n/n$. Let*

$$g(z) \equiv \exp(f(z)) = \sum_{m=0}^{\infty} B_m z^m/m!$$

Then $B_0 = 1$ and B_m is given by the mxm determinant:

$$B_m = \begin{vmatrix} b_1 & m-1 & 0 & \cdots & 0 \\ b_2 & b_1 & m-2 & \cdots & 0 \\ b_3 & b_2 & b_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{m-1} & b_{m-2} & b_{m-2} & \cdots & 1 \\ b_m & b_{m-1} & b_{m-2} & \cdots & b_1 \end{vmatrix} \tag{6.9}$$

Proof. Since $g'(z) = f'(z)\, g(z)$, we find that

$$nB_n = n!(b_1 B_{n-1}/(n-1)! - b_2 B_{n-2}/(n-2)! + \cdots),$$

or

$$B_n = \sum_{k=1}^{n} b_k B_{n-k} (-1)^{k+1} \left[\frac{(n-1)!}{(n-k)!}\right]. \tag{6.10}$$

Now (6.9) clearly holds for B_1 so suppose inductively, that it holds for $B_1, \ldots, B_{m-1}$. Then (6.10) corresponds to the expansion in minors in the first column in (6.9) and so it holds for B_m. ∎

Theorem 6.8. (Plemelj–Smithies formula). *Let $A \in \mathscr{C}_n$. Then*

$$\det_n(1 + \mu A) = \sum_{m=0}^{\infty} \mu^m \alpha_m^{(n)}(A)/m! \tag{6.11}$$

where the series converges for all $\mu \in \mathbb{C}$ and $\alpha_m^{(n)}(A)$ is given by the mxm determinant:

$$\alpha_m^{(n)}(A) = \begin{vmatrix} \sigma_1^{(n)} & m-1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{m-1}^{(n)} & \sigma_{m-2}^{(n)} & \cdots & \cdots & 1 \\ \sigma_m^{(n)} & \sigma_{m-1}^{(n)} & \cdots & \cdots & \sigma_1^{(n)} \end{vmatrix} \tag{6.12}$$

with

$$\begin{aligned} \sigma_k^{(n)} &= \mathrm{Tr}(A^k) \qquad k \geqslant n \\ &= 0 \qquad\qquad k \leqslant n-1. \end{aligned} \tag{6.13}$$

Proof. Since $\det(1 + \mu A)$ is an entire function, it clearly has an expansion (6.11) converging for all μ. The coefficients need only be found for small μ. By Lemma 6.7, (6.12) and (6.13) are equivalent to

$$\det_n(1 + \mu A) = \exp\left(\sum_{m=1}^{\infty} (-1)^{m+1} \sigma_m^{(n)}(A)\, \mu^m\right)$$

for small μ. This follows by using Lidskii's theorem and the product expansion for $\det_n(1 + \mu A)$. ▌

Remark. The beauty of (6.12) is that $\det(1 + A)$ has an expression in terms of $\mathrm{Tr}(A^p)$ and when $\mathrm{Tr}(A)$, $\mathrm{Tr}(A^2)$,..., $\mathrm{Tr}(A^{n-1})$ are set equal to zero in this expression, we just get $\det_n(1 + A)$.

7. Fredholm Theory

The basic result of the Fredholm theory is the ability to write $(1 + \mu A)^{-1}$ as a quotient of explicit entire functions of μ. The "higher minors" of Fredholm will not be discussed here but we note they are essentially the functions $[\wedge^n A(1 + \mu A)^{-1}] \det(1 + \mu A)$: See [32] for methods of estimating these objects. We begin by deriving formulas due to Plemelj [22] and Smithies [34] for the numerator in this quotient and then we discuss Fredholm's original formula.

Theorem 7.1. *If $A \in \mathscr{C}_n$, then*

$$(1 + \mu A)^{-1} \det_n(1 + \mu A) \tag{7.1}$$

is an entire operator-valued function of μ.

Proof. $(1 + \mu A)^{-1}$ is analytic in $\mathbb{C}/\{-\lambda_i(A)^{-1}\}_{i=1}^{N(A)}$. Since the spectral projection at each $\lambda_i(A)$ is finite rank, one can show by an explicit analysis of the Laurent series about $-\lambda_i^{-1}$ [19] that at $\mu = -\lambda_i(A)^{-1}$, $(1 + \mu A)^{-1}$ has a pole of order at most $\dim P_{\lambda_i}$ (alternatively, one can write $(1 + \mu A)^{-1} = (1 - P_i)[1 + \mu A(1 - P_i)]^{-1} + [1 + \mu AP_i]^{-1} P_i$ and note that the first factor is analytic at $\mu = -\lambda_i^{-1}$ and that the second factor has a pole of order at worst $\dim P_{\lambda_i}$ by the analysis of finite-dimensional operators). Thus, since $\det_n(1 + \mu A)$ has a zero of order $\dim P_{\lambda_i}$ at $\mu = -\lambda_i^{-1}$, the product is entire. ∎

DEFINITION. $D_\lambda^{(n)}(A) = -\det_n(1 + \lambda A)[(1 + \lambda A)^{-1} - 1]/\lambda$.

COROLLARY 7.2. *For $A \in \mathscr{C}_n$, $D_\lambda^{(n)}(A)$ is an entire $\mathscr{C}_n$-valued function.*

Proof. $D_\lambda^{(n)}(A) = A[\det_n(1 + \lambda A)(1 + \lambda A)^{-1}]$. ∎

We can now use the method of the last section to find explicit formulas for the coefficients of the power series expansion of $D_\lambda^{(n)}(A)$:

THEOREM 7.3 (Plemelj–Smithies formulas for $D_\lambda^{(n)}$).

$$D_\lambda^{(n)}(A) = \sum_{m=0}^{\infty} \beta_m^{(n)}(A)\, \lambda^m/m! \tag{7.2}$$

where $\beta_m^{(n)}$ is given by the $(m + 1) \times (m + 1)$ determinant:

$$\beta_m^{(n)}(A) = \begin{vmatrix} A & m & 0 & \cdots & 0 \\ A^2 & \sigma_1^{(n)} & m-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A^{m+1} & \sigma_m^{(n)} & \sigma_{m-1}^{(n)} & \cdots & \sigma_1^{(n)} \end{vmatrix} \tag{7.3}$$

where $\sigma_1^{(n)}$ is given by (6.13) *and* (7.3) *is to be interpreted in the sense that $(\psi, \beta_m^{(n)}(A)\phi)$ is of the same form as* (7.3) *with A^j replaced by $(\psi, A^j\phi)$.* ∎

Proof. In terms of the determinants $\alpha_m^{(n)}(A)$, we can evaluate $\beta_m^{(n)}(A)$ by expanding in the first column:

$$\beta_m^{(n)}(A) = A\alpha_m^{(n)} - mA^2\alpha_{m-1}^{(n)} + m(m-1)\, A^3\alpha_{m-2}^{(n)} + \cdots$$

so that

$$\frac{\beta_m^{(n)}(A)}{m!} = A\,\frac{\alpha_m^{(n)}}{m!} - A^2\,\frac{\alpha_{m-1}^{(n)}}{(m-1)!} + \cdots.$$

It follows, that for μ small (where all series converge):

$$\sum_{m=0}^{\infty} \frac{\beta_m^{(n)}(A)}{m!}\,\mu^m = (A - \mu A^2 + \mu A^3 + \cdots)\left(\sum_{m=0}^{\infty} \frac{\alpha_m^{(n)}}{m!}\,\mu^m\right)$$

$$= A(1 + \mu A)^{-1} \det_n(1 + \mu A). \quad ∎$$

COROLLARY 7.4. *If $A \in \mathscr{C}_n$ and $-\mu^{-1}$ is not an eigenvalue of A, then*

$$(1 + \mu A)^{-1} = 1 - \mu[D_\mu^{(n)}(A)/\det_n(1 + \mu A)] \tag{7.4}$$

where $\det_n(1 + \mu A) = \sum_{m=0}^{\infty} \alpha_m^{(n)} \mu^m/m!$ *and* $D_\mu^{(n)}(A) = \sum_{m=0}^{\infty} \beta_m^{(n)}(A) \mu^m/m!$ *with* $\alpha_m^{(n)}, \beta_m^{(n)}$ *given by Eqs.* (6.12) *and* (7.3).

Proof. Equation (7.4) follows by definition of $D_\mu^{(n)}(A)$. ∎

In practical computations, one would like to estimate the error made by dropping the tail of the power series defining $\det_n(1 + \mu A)$ and $D_\mu^{(n)}(A)$. Such estimates follow from Cauchy estimates and bounds on the growth of the functions as $\mu \to \infty$. Our approach here is patterned after that of Dunford and Schwartz [4]. We have already seen that $| \det_n(1 + \mu A)| \leqslant \exp(\Gamma_n | \mu |^n \| A \|_n^n)$. We now prove such estimates on $D_\mu^{(n)}(A)$. We first note the following estimate from the appendix to [32] (see also [4]):

THEOREM 7.5. *For $n = 1$ and $1 \leqslant p \leqslant \infty$:*

$$\| D_\mu^{(1)}(A)\|_p \leqslant \| A \|_p \exp(| \mu | \| A \|_1). \tag{7.5}$$

Proof. It suffices to prove that

$$\|(1 + A)^{-1} \det(1 + A)\|_\infty \leqslant \exp(\| A \|_1) \tag{7.6}$$

and this can be proven for finite rank operators with $(1 + A)$ invertible. Now:

$$\begin{aligned} \|(1 + A)^{-1} \det(1 + A)\| &= \| \, | 1 + A |^{-1} \det(| 1 + A |)\| \\ &= | \mu_1(1 + A)|^{-1} \prod_{j=1}^{N} \mu_j(1 + A) \\ &= \prod_{j=2}^{N} \mu_j(1 + A) \leqslant \prod_{j=2}^{N} (1 + \mu_j(A)) \\ &\leqslant \exp\left(\sum_{j=2}^{N} \mu_j(A)\right) \leqslant \exp(\| A \|_1), \end{aligned}$$

where $N = \operatorname{rank}(A)$ and we have used $\mu_j(1 + A) \leqslant 1 + \mu_j(A)$, which follows min–max characterization of μ_j([4]):

$$\mu_j(B) = \min_{\phi_1, \ldots, \phi_{j-1}} \Big(\max_{\substack{\psi \in [\phi_1, \ldots, \phi_{j-1}]^\perp \\ \|\psi\| = 1}} \| B\psi \|\Big). \quad ∎$$

COROLLARY 7.6.

$$\| \beta_m^{(1)}(A)\|_1 \leqslant e^m \| A \|_1^{m+1}, \tag{7.7}$$

$$| \alpha_m^{(1)} | \leqslant e^m \| A \|_1^m. \tag{7.8}$$

Remark. We emphasize that the power series for $\det(1 + \mu A)$ and $D_\mu^{(1)}(A)$ have $(m!)^{-1}$ in their definition. This $(m!)^{-1}$ control of convergence is an improvement over the celebrated $(m!)^{-1/2}$ bound Fredholm obtains from Hadamard's inequality. In special cases, Fredholm [5] obtains better than $(m!)^{-1/2}$ or even $(m!)^{-1}$; see also Hille–Tamarkin [10].

Proof. By a Cauchy estimate:

$$\| \beta_m^{(1)}(A) \|_1 \leqslant m! \, \| A \|_1 \, R^{-m} \exp(R \, \| A \|_1)$$

for any R. Choosing $R = m \| A \|_1^{-1}$, Eq. (7.7) results. The proof of (7.8) is similar. ∎

Remark. The idea used in these estimates is similar to that by Smithies [34] in his convergence estimates.

COROLLARY 7.7 (essentially in [32]).

$$\| D_\mu^{(1)}(A) - D_\mu^{(1)}(B) \|_1 \leqslant \| A - B \|_1 \{[\| A \|_1 + \| B \|_1 + 1]) \exp[| \mu | (\| A \|_1 + \| B \|_1 + 1)]\}.$$

Proof. Follows from Lemma 6.6. ∎

The basic input in the estimates we prove for $D_\mu^{(n)}$ is a formula which will also be basic to our development of Fredholm's formulas for $\beta_m^{(1)}$, namely: for $A, B \in \mathscr{C}_1$ with $(1 + A)$ invertible

$$\frac{d}{d\mu} \log[\det(1 + A + \mu B)]|_{\mu=0} = \operatorname{Tr}((1 + A)^{-1} B). \tag{7.9}$$

To prove (7.9), we write

$$\begin{aligned} \det(1 + A + \mu B) &= \det(1 + A) \det(1 + \mu(A + 1)^{-1} B) \\ &= \det(1 + A)[1 + \mu \operatorname{Tr}((1 + A)^{-1} B) + 0(\mu^2)], \end{aligned}$$

from which (7.9) follows.

Now let $A, B \in \mathscr{C}_1$. Then

$$\begin{aligned} &\log[\det_n(1 + A + \mu B)] \\ &\quad = \log[\det(1 + A + \mu B)] + \sum_{k=1}^{n-1} (-1)^{k+1} \operatorname{Tr}[(A + \mu B)^k]/k \end{aligned}$$

so that, for $(1 + A)$ invertible

$$\frac{d}{d\mu} \log[\det_n(1 + A + \mu B)]|_{\mu=0} = \operatorname{Tr}((1 + A)^{-1} B) + \sum_{k=1}^{n-1} (-1)^{k+1} \operatorname{Tr}(BA^{k-1}).$$

If $B = (\phi, \cdot)\psi$, then

$$\det_n(1 + A)(\phi, (1 + A)^{-1} \psi) = \frac{d}{d\mu} \det_n(1 + A + \mu B)|_{\mu=0} + \sum_{k=1}^{n-1} (-1)^k \det_n(1 + A)(\phi, A^{k-1}\psi).$$

Now $|\det_n(1 + A)| \leqslant \exp(\Gamma_n \| A \|_n^n)$ so that, using

$$\left| \frac{df}{d\mu} \right|_{\mu=0} \leqslant \sup_{|\mu|=1} |f(\mu)|,$$

for f analytic:

$$\|(1 + A)^{-1} \det_n(1 + A)\| \leqslant C_n \exp(\Gamma_n \| A \|_n^n) \tag{7.10}$$

for C_n sufficiently large. Once we have (7.10) we can take limits to conclude Eq. (7.10) first when $A \in \mathscr{C}_n$ with $(1 + A)$ invertible and then even for $(1 + A)^{-1}$ noninvertible if $(1 + A)^{-1} \det_n(1 + A)$ is interpreted as $\det_n(1 + A) - \lambda D_\lambda^{(n)}(A)$. ∎

Remark. Equation (7.10) appears in Dunford and Schwartz [4] with $C_n = 1$. The estimate in this form is wrong! For take $A = -\mu(\phi \cdot \phi)$ with $\| \phi \| = 1$. Then $\|(1 + A)^{-1}\| = 1 + \mu + O(\mu^2)$, $\det_n(1 + A) = 1 + O(\mu^n)$, and $\exp(\Gamma_n \| A \|_n^n) = 1 + O(\mu^n)$ whence (7.10) with $C_n = 1$ would imply $(1 + \mu) \leqslant 1 + O(\mu^n)$!

As above in Corollaries 7.6 and 7.7 we immediately obtain:

THEOREM 7.8. *For $A \in \mathscr{C}_n$*

(a) *For $p \geqslant n$, $\| D_\lambda^{(n)}(A)\|_p \leqslant C_n \| A \|_p \exp(\Gamma_n |\lambda|^n \| A \|_n^n)$.*

(b) $\| D_\lambda^{(n)}(A) - D_\lambda^{(n)}(B)\|_n \leqslant C_n \| A_n - B \|_n \{(\| A \|_n + \| B \|_n + 1) \exp(\Gamma_n |\lambda|^n (\| A \|_n + \| B_n \| + 1)^n)\}$.

(c) $| \alpha_m^{(n)} | \leqslant (m!)^{1-(1/n)} e^{m\Gamma_n} (\| A \|_n)^m$.

(d) $\| \beta_m^{(n)}(A)\|_n \leqslant C_n (m!)^{1-(1/n)} e^{m\Gamma_n} (\| A \|_n)^{m+1}$.

Remark. To bound the higher Fredholm minors, we would use the fact that they are higher derivatives of $\det(1 + A + \mu B)$(see [32]).

As a final topic in the Fredholm theory, we obtain abstract formula for the coefficients of $D_\lambda^{(i)}(A)$ which agree with Fredholm's formulae [5] for concrete integral operators. Let $\mathscr{H}$ be a Hilbert space and $\otimes_n \mathscr{H}$ its n-fold tensor product. We define the "partial" trace from $\mathscr{C}_1(\otimes_n \mathscr{H})$ to $\mathscr{C}_1(\mathscr{H})$ by (for C compact):

$$\mathrm{Tr}_{\mathscr{H}}[\mathrm{Tr}_{(n-1)}(A)\, C] = \mathrm{Tr}_{\otimes_n \mathscr{H}}(A[C \otimes I \otimes \cdots \otimes I]). \tag{7.11}$$

(7.11) defines an operator in $\mathscr{C}_1(\mathscr{H})$ since, for C compact

$$|\operatorname{Tr}_{\otimes_n \mathscr{H}}(A[C \otimes \cdots \otimes I])| \leqslant \| A \|_1 \| C \|_\infty ,$$

(7.11) can then be shown to hold for any $C \in \mathscr{L}(\mathscr{H})$. Now given B: $\Lambda^n \mathscr{H} \to \Lambda^n \mathscr{H}$, we can extend B to $\otimes_n \mathscr{H}$ by setting B to zero on $(\Lambda^n \mathscr{H})^\perp$ and then form $\operatorname{Tr}_{n-1}(B) \in \mathscr{C}_1(\mathscr{H})$.

THEOREM 7.9. *$D_\mu^{(1)}(A) = \sum_{m=0}^\infty \beta_m^{(1)}(A)\, \mu^m/m!$ Then*

$$\beta_m^{(1)}(A) = \operatorname{Tr}_m(\Lambda^{m+1}(A))\,(m+1)! \tag{7.12}$$

In particular, $\operatorname{Tr}(\beta_m^{(1)}(A)) = \alpha_{m+1}^{(1)}$.

Remark. The last statement is obvious also from the Plemelj–Smithies formulas.

Proof. By (7.10), for C finite rank:

$$\begin{aligned}
&\operatorname{Tr}(CD_\lambda^{(1)}(A)) \\
&\quad = -\operatorname{Tr}\{[(1+\mu A)^{-1}\det(1+\mu A) - \det(1+\mu A)]\, C/\mu\} \\
&\quad = -\left\{\frac{d}{d\lambda}[\det(1+\mu A+\lambda C)]|_{\lambda=0} - \det(1+\mu A)\operatorname{Tr}(C)\right\}\Big/\mu \\
&\quad = -\mu^{-1}\sum_{m=0}^\infty \left\{\left[\frac{d}{d\lambda}\operatorname{Tr}(\Lambda^{m+1}(\mu A+\lambda C))\right|_{\lambda=0} - \mu^m \operatorname{Tr}(C)\operatorname{Tr}(\Lambda^m(A))\right\}.
\end{aligned}$$

Suppose that A is a finite rank operator

$$A = \sum_{i=1}^n \alpha_i(e_i\,,\cdot) f_i$$

with $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$ orthonormal and

$$C = (e_n\,,\ ) f_k$$

Then

$$\begin{aligned}
&\frac{d}{d\lambda}\operatorname{Tr}(\Lambda^{m+1}(\mu A+\lambda C)) \\
&\quad = \frac{d}{d\lambda}\left[\sum_{i_1<\cdots<i_m<i_{m+1}} (e_{i_1}\wedge\cdots\wedge e_{i_{m+1}}, (\mu A+\lambda C)\, e_{i_1}\wedge\cdots\wedge(\mu A+\lambda C) f_{i_{m+1}})\right] \\
&\quad = \mu^m \sum_{i_1<\cdots<i_m} \alpha_{i_1}\cdots\alpha_{i_m}(e_{i_1}\wedge\cdots\wedge e_{i_m}\wedge e_n\,, f_{i_1}\wedge\cdots\wedge f_{i_m}\wedge f_k).
\end{aligned} \tag{7.13}$$

[In (7.13), it isn't necessary to take $i_m < n$, since the terms with $i_m = n$ are 0.]

$$\frac{d}{d\lambda}\operatorname{Tr}\Lambda^{m+1}(\mu A + \lambda C))$$

$$= \mu^m \sum_{i_1<\cdots<i_m} \alpha_{i_1}\cdots\alpha_{i_m}(e_{i_1}\wedge\cdots\wedge e_{i_m}, f_{i_1}\wedge\cdots f_{i_m})(e_n, f_k)$$

$$+ \mu^m \sum_{i_1<\cdots<i_m} \alpha_{i_1}\cdots\alpha_{i_m}\sum_{j=1}^{m}(-1)^{j-m-1}$$

$$\times (e_{i_1}\wedge\cdots\hat{e}_j\wedge\cdots e_{i_m}\wedge\hat{e}_n, f_{i_1}\wedge\cdots f_{i_m})(e_{i_j}, f_k)$$

$$= \mu^m \sum_{i_1<\cdots<i_m} (e_{i_1}\wedge\cdots\wedge e_{i_m}, Ae_{i_1}\wedge\cdots\wedge Ae_{i_m})(e_n, f_k)$$

$$- \mu^m \sum_{i_1<\cdots<i_m}\left[\sum_{j=1}^{n}(e_{i_1}\wedge\cdots e_{i_{j-1}}\wedge e_n\wedge e_{i_{j+1}}\wedge\cdots e_{i_m}, Ae_{i_1}\wedge\cdots\wedge Ae_{i_n})(e_{i_j}, f_k)\right]$$

$$= \mu^m[\operatorname{Tr}(C)\operatorname{Tr}(\Lambda^m(A)) - m\operatorname{Tr}(C\operatorname{Tr}_{m-1}(\Lambda^m(A)))]$$

since

$$(C^*\otimes I\otimes\cdots\otimes I)(e_{i_1}\wedge\cdots\wedge e_{i_m}) = \frac{1}{m}\sum_{j=1}^{m}(e_{i_1}\wedge\cdots\wedge Ce_{i_j}\wedge\cdots\wedge e_{i_m}).$$

Thus, for A and C of the type above:

$$\operatorname{Tr}(CD_\mu^{(1)}(A)) = \sum_{m=0}^{\infty}(m+1)\mu^m\operatorname{Tr}(C\operatorname{Tr}_m(\Lambda^{m+1}(A))). \tag{7.14}$$

Since we can always take α_k or $\alpha_n = 0$, Eq. (7.12) holds for any finite rank operator, so

$$D_\mu^{(1)}(A) = \sum_{m=0}^{\infty}(m+1)\mu^m\operatorname{Tr}_m(\Lambda^{m+1}(A))$$

for A finite rank and, so, by a limiting argument for any A. This proves Eq. (7.12). ∎

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Added Notes. (1) Another systematic presentation of infinite determinants can be found in J. R. Ringrose, "Compact Non-Self-Adjoint Operators," Van Nostrand, 1971. Ringrose proves the Hadamard factorization of the determinant but uses Lidskii's theorem to prove it rather than vice versa.

(2) The determinant inequality (5.1) appears prior to [31] in S. J. Rotfel'd, *Prob. Math. Phys.*, No. 3, 81 (1968).

(3) Rather strong results on the status of Weyl's inequality on a general Banach space will appear in a paper of W. Johnson, B. Maurey, H. König and J. R. Retherford.

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