ANNALES DE L'I. H. P., SECTION A

BARRY SIMON

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Annales de l'I. H. P., section A, tome 24, n° 1 (1976), p. 91-93.

<http://www.numdam.org/item?id=AIHPA_1976__24_1_91_0>

Ann. Inst. Henri Poincaré, Section A :
Vol. XXIV, n° 1, 1976, p. 91-93. *Physique théorique.*

On the genericity of nonvanishing instability intervals in Hills equation

by

Barry SIMON (*)

Department of Mathematics. Princeton, New Jersey 08540

ABSTRACT. — We prove that for (Baire) almost every C^∞ periodic function V on $\mathbb{R}$, $-d^2/dx^2 + V$ has *all* its instability intervals non-empty.

In the spectral theory of one dimensional Schrödinger operators [*3*] [*10*] with periodic potentials, a natural question occurs involving the presence of gaps in the spectrum. Let $H = -\dfrac{d^2}{dx^2} + V$ on $L^2(\mathbb{R}, dx)$ where $V(x+1) = V(x)$ for all x. Let A^P (resp. A^A) be the operator $-\dfrac{d^2}{dx^2} + V$ on $L^2([0, 1], dx)$ with the boundary condition $f'(1) = f'(0)$; $f(1) = f(0)$ (resp. $f'(1) = -f'(0)$; $f(1) = -f(0)$). Let E_n^P (resp. E_n^A) be the n^{th} eigenvalue, counting multiplicity, of A^P (resp. A^A). Finally define

$$\alpha_n = \begin{cases} E_n^P & n = 1, 3, \dots \\ E_n^A & n = 2, 4, \dots \end{cases}$$

$$\beta_n = \begin{cases} E_n^A & n = 1, 3, \dots \\ E_n^P & n = 2, 4, \dots \end{cases}$$

$$\mu_n = \alpha_{n+1} - \beta_n$$

It is a fundamental result of Lyapunov that

$$\alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \dots \leq \alpha_n < \beta_n \leq \alpha_{n+1} \dots$$

(*) A Sloan Fellow partially supported by USNSF under Grant GP.

and one can show [3] [10] that $\sigma(H) = \bigcup_{n=1}^{\infty} [\alpha_n, \beta_n]$. The numbers $\mu_n \geq 0$ enter naturally as the size of gaps in $\sigma(H)$. In the older literature [9], the equation $-f'' + Vf = Ef$ is called Hill's equation and the intervals $(\beta_n \, \alpha_{n+1})$ (of length μ_n) are called instability intervals.

One has the feeling that for most V's the gap sizes $\mu_n(V)$ are non-zero. This is suggested in part by a variety of deep theorems that show the vanishing of many μ_n's places strong restrictions on V: for example, $\mu_n(V) = 0$ all n implies that V is constant [1] [5]; $\mu_n(V) = 0$ all odd n implies that $V\left(x + \frac{1}{2}\right) = V(x)$ [1] [6]; and $\mu_n(V) = 0$ for all but N values of n forces V to lie on a 2N-dimensional manifold [5] [4]. On the other hand, some argument is necessary to construct an explicit example of a V with each $\mu_n(V) \neq 0$ [7].

The situation is somewhat reminiscent of that concerning nowhere differential functions in C[0, 1]. One's intuition is that somehow most functions in C[0, 1] are nowhere differentiable but some argument is needed to construct an explicit nowhere differentiable function. One's intuition in this case is established by a result that also settles the existence question: a dense G_δ (« Baire almost every ») in C[0, 1] consists of nowhere differentiable functions [2].

In this note we wish to prove a similar result that asserts that, for most V, $\mu_n(V) \neq 0$ for all n. We do not claim that that result is of the depth of the above quoted results but we feel it is of some interest especially since it will be a simple exercise in the perturbation theory of eigenvalues [8] [10] [11].

THEOREM. — Let X denote the vector space of real valued C^∞ functions on $\mathbb{R}$ obeying $V(x + 1) = V(x)$. Place the Frechet topology on X given by the seminorms

$$\| f \|_n = \sup_x | D^n f(x) | .$$

Then the set of V in X with $\mu_n(V) \neq 0$ for *all* n is a dense G_δ in X.

Proof. — Fix n. We will show that $\{ V \mid \mu_n(V) \neq 0 \}$ is a dense open set of X. Thus $\bigcap_n \{ V \mid \mu_n(V) \neq 0 \}$ is a G_δ which is dense by the Baire category theorem.

Suppose that $\mu_n(V) \neq 0$. Suppose n is even (a similar argument works if n is odd). Thus $E^P_{n+1}(V) \neq E^P_n(V)$. Now, the change of $E^P_{n+1}(V + \lambda W)$ as λ changes can be bounded [8] by $\| W \|_{\text{operator}}$ and the W-independent data of the distance of $E^P_{n+1}(V)$ from $E_n(V)$ and $E_{n+2}(V)$. As a result, there is a constant $\varepsilon(V)$ so that $\mu_n(V + W) \neq 0$ if $\| W \|_\infty \leq \varepsilon(V)$. Since $\| - \|_\infty$ is a continuous seminorm, $\{ V \mid \mu_n(V) \neq 0 \}$ is open.

Next suppose $\mu_n(V) = 0$ and again suppose that n is even. Since $E_n = E_{n+1}$, all solutions of $-u'' + Vu = E_n u$ are periodic. Let u_1 be the solution with $u(0) = 0$, $u'(0) = 1$ and u_2 the solution with $u(0) = 1$, $u'(0) = 0$. Since $(u_1(x))^2 \neq (u_2(x))^2$ for x near 0, we can find $W \in X$ with

$$\int W(x) \mid u_1(x) \mid^2 dx \neq \int W(x)(u_2(x))^2 dx .$$

It follows [8] that for λ small $E_n(V + \lambda W) \neq E_{n+1}(V + \lambda W)$ and thus that $\mu_n(V + \lambda W) \neq 0$. We conclude that $\{ V \mid \mu_n(V) \neq 0 \}$ is dense. ■

We conclude by noting that the space $X = C^\infty$ can be replaced by any topological vector space of continuous periodic functions which is a Baire space and which obeys:

a) $\| - \|_\infty$ is a continuous seminorm.

(*b*) If $\rho_1 \neq \rho_2$ as functions in $L^1([0, 1])$, there is W in the space with $\int \rho_1(x)W(x)dx \neq \int \rho_2(x)W(x)dx$.

In particular, we can take the $C^p([0, 1])$ periodic functions with the C^p topology or the periodic entire analytic functions with the compact open topology.

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(Manuscrit reçu le 27 mai 1975)