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by

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Abstract. — We prove that for (Baire) almost every C\infty periodic function V on \( \mathbb{R} \), \(-d^2/dx^2 + V\) has all its instability intervals non-empty.

In the spectral theory of one dimensional Schrödinger operators [3] [10] with periodic potentials, a natural question occurs involving the presence of gaps in the spectrum. Let \( H = -d^2/dx^2 + V \) on \( L^2(\mathbb{R}, dx) \) where \( V(x + 1) = V(x) \) for all x. Let \( A^p \) (resp. \( A^A \)) be the operator \(-d^2/dx^2 + V\) on \( L^2([0, 1], dx) \) with the boundary condition \( f'(1) = f'(0); f(1) = f(0) \) (resp. \( f'(1) = -f'(0); f(1) = -f(0) \)). Let \( E_n^p \) (resp. \( E_n^A \)) be the \( n^{th} \) eigenvalue, counting multiplicity, of \( A^p \) (resp. \( A^A \)). Finally define

\[
\alpha_n = \begin{cases} 
E_n^p & n = 1, 3, \ldots \\
E_n^A & n = 2, 4, \ldots 
\end{cases} \\
\beta_n = \begin{cases} 
E_n^A & n = 1, 3, \ldots \\
E_n^p & n = 2, 4, \ldots 
\end{cases} \\
\mu_n = \alpha_{n+1} - \beta_n
\]

It is a fundamental result of Lyapunov that

\[ \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \ldots \leq \alpha_n < \beta_n \leq \alpha_{n+1} \ldots \]

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and one can show [3] [10] that $\sigma(H) = \bigcup_{n=1}^{\infty} [\alpha_n, \beta_n]$. The numbers $\mu_n \geq 0$ enter naturally as the size of gaps in $\sigma(H)$. In the older literature [9], the equation $-f'' + Vf = Ef$ is called Hill's equation and the intervals $(\beta_n, \alpha_{n+1})$ (of length $\mu_n$) are called instability intervals.

One has the feeling that for most $V$'s the gap sizes $\mu_n(V)$ are non-zero. This is suggested in part by a variety of deep theorems that show the vanishing of many $\mu_n$'s places strong restrictions on $V$: for example, $\mu_4(V) = 0$ all $n$ implies that $V$ is constant [7] [5]; $\mu_n(V) = 0$ all odd $n$ implies that $V(x + \frac{1}{2}) = V(x)$ [7] [6]; and $\mu_n(V) = 0$ for all but $N$ values of $n$ forces $V$ to lie on a $2N$-dimensional manifold [5] [4]. On the other hand, some argument is necessary to construct an explicit example of a $V$ with each $\mu_n(V) \neq 0$ [7].

The situation is somewhat reminiscent of that concerning nowhere differential functions in $C[0, 1]$. One's intuition is that somehow most functions in $C[0, 1]$ are nowhere differentiable but some argument is needed to construct an explicit nowhere differentiable function. One's intuition in this case is established by a result that also settles the existence question: a dense $G_\delta$ (« Baire almost every ») in $C[0, 1]$ consists of nowhere differentiable functions [2].

In this note we wish to prove a similar result that asserts that, for most $V$, $\mu_n(V) \neq 0$ for all $n$. We do not claim that that result is of the depth of the above quoted results but we feel it is of some interest especially since it will be a simple exercise in the perturbation theory of eigenvalues [8] [11].

**Theorem.** — Let $X$ denote the vector space of real valued $C^\infty$ functions on $\mathbb{R}$ obeying $V(x + 1) = V(x)$. Place the Frechet topology on $X$ given by the seminorms

$$ ||f||_n = \sup_x |D^n f(x)|. $$

Then the set of $V$ in $X$ with $\mu_n(V) \neq 0$ for all $n$ is a dense $G_\delta$ in $X$.

**Proof.** — Fix $n$. We will show that $\{ V \mid \mu_n(V) \neq 0 \}$ is a dense open set of $X$. Thus $\bigcap_n \{ V \mid \mu_n(V) \neq 0 \}$ is a $G_\delta$ which is dense by the Baire category theorem.

Suppose that $\mu_n(V) \neq 0$. Suppose $n$ is even (a similar argument works if $n$ is odd). Thus $E_{n+1}^\mu(V) \neq E_n^\mu(V)$. Now, the change of $E_{n+1}^\mu(V + \lambda W)$ as $\lambda$ changes can be bounded [8] by $||W||_{\text{operator}}$ and the $W$-independent data of the distance of $E_{n+1}^\mu(V)$ from $E_n(V)$ and $E_{n+2}(V)$. As a result, there is a constant $d(V)$ so that $\mu_n(V + W) \neq 0$ if $||W||_{\infty} \leq d(V)$. Since $||-||_{\infty}$ is a continuous seminorm, $\{ V \mid \mu_n(V) \neq 0 \}$ is open.

Next suppose $\mu_n(V) = 0$ and again suppose that $n$ is even. Since $E_n = E_{n+1}$,
all solutions of \(-u'' + Vu = E_nu\) are periodic. Let \(u_1\) be the solution with \(u(0) = 0, u'(0) = 1\) and \(u_2\) the solution with \(u(0) = 1, u'(0) = 0\). Since \((u_1(x))^2 \neq (u_2(x))^2\) for \(x\) near 0, we can find \(W \in X\) with

\[
\int W(x) |u_1(x)|^2 \, dx \neq \int W(x)(u_2(x))^2 \, dx.
\]

It follows [8] that for \(\lambda\) small \(E_n(V + \lambda W) \neq E_{n+1}(V + \lambda W)\) and thus that \(\mu_n(V + \lambda W) \neq 0\). We conclude that \(\{ V \mid \mu_n(V) \neq 0 \}\) is dense. 

We conclude by noting that the space \(X = C^\infty\) can be replaced by any topological vector space of continuous periodic functions which is a Baire space and which obeys:

- (a) \(|| - ||_\infty\) is a continuous seminorm.
- (b) If \(\rho_1 \neq \rho_2\) as functions in \(L^1([0, 1])\), there is \(W\) in the space with

\[
\int \rho_1(x)W(x)dx \neq \int \rho_2(x)W(x)dx.
\]

In particular, we can take the \(C^p([0, 1])\) periodic functions with the \(C^p\) topology or the periodic entire analytic functions with the compact open topology.

REFERENCES


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