Annals of Mathematics, **105** (1977), 493-526

Pure states for general $P(\phi)_2$ theories: Construction, regularity and variational equality*

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Abstract

We give a new construction of the $P(\phi)_2$ Euclidean (quantum) field theory and propose a structure analysis of this theory. Among our results are:

(1) For *any* polynomial P bounded from below, we construct two Euclidean states (expectations) $\langle\cdot\rangle_{P,\pm}$, *not* necessarily distinct, which satisfy all Osterwalder-Schrader axioms including clustering and obey the Dobrushin-Lanford-Ruelle (DLR) equations for P.

(2) Equality $\langle\cdot\rangle_{P,+} = \langle\cdot\rangle_{P,-}$ holds if and only if the pressure $\alpha_\infty(\mu)$ corresponding to the polynomial $P(x) - \mu x$ is differentiable at $\mu = 0$, and in this case the state $\langle\cdot\rangle_{P,\pm}$ is independent of a large class of different (in particular classical) boundary conditions.

(3) All $P(\phi)_2$ expectations thus far constructed are locally absolutely continuous with respect to the free field Gaussian expectations with L^p Radon-Nikodym derivatives, for all $p < \infty$.

(4) The strong Gibbs variational equality holds, for all states constructed so far for a given P.

1. Introduction, summary of results

In this paper we discuss a variety of new results for the $P(\phi)_2$ Euclidean (quantum) field theory [46], [43]. Mathematically, this theory is defined as a class of non-Gaussian, generalized stochastic processes over $\mathbf{R}^2$ which are indexed by the positive polynomials on the real line. They are not only mathematically interesting but of some importance to quantum physics, because they yield non-trivial models of relativistic quantum fields in two space-time dimensions.

In order to explain the connection between generalized stochastic processes and relativistic quantum fields, we briefly recall the Euclidean description of relativistic quantum field theory:

A relativistic quantum field theory of one neutral, scalar field ϕ in $d + 1$ space-time dimensions can be described in terms of a sequence

* Research supported in part by NSF Grant MPS 75 11864.

† A. Sloan Foundation Fellows.

$\{W_n(x_1, \cdots, x_n)\}_{n=0}^{\infty}$ of Wightman distributions [47] satisfying the Gårding-Wightman axioms (namely "temperedness," "Poincaré-invariance," "positivity," "spectrum condition," "locality," "clustering"; see [47] for precise statements and consequences). From these axioms it then follows that the Wightman distributions are the boundary values of functions $W_n(z_1, \cdots, z_n)$ holomorphic in a large domain of $\mathbf{C}^{(d+1)n}$ (called the permuted, extended tube [47]). This domain contains the so-called Euclidean points

$$\mathcal{E}_n = \{z_1, \cdots, z_n : z_i = (\vec{x}_i, it_i), z_i \neq z_j, \text{ for } i \neq j\} ,$$

where $\vec{x}_j \in \mathbf{R}^d$ denotes the space—and it_j, $t_j \in \mathbf{R}$, the time—component of the complex vector z_j. On $\mathcal{E}_n$ we define

$$S_n(x_1, \cdots, x_n) = W_n(\vec{x}_1, it_1, \cdots, \vec{x}_n, it_n), x_j = (\vec{x}_j, t_j) .$$

These are called the *Euclidean Green's* or *Schwinger functions.*

The following properties of the Schwinger functions are a consequence of the Gårding-Wightman axioms; (see [34]): The Schwinger functions are *Euclidean invariant*; they have a *positivity property* called OS positivity (for "Osterwalder-Schrader"; see [34] and property (b) below); they are *symmetric* under permutations of their arguments.

These properties are compatible with the existence of a probability measure μ on (the σ-algebra Σ generated by the Borel cylinder sets of) the space $\mathcal{S}'$ of real-valued tempered distributions over $\mathbf{R}^{d+1}$ such that

$$\begin{aligned} S_n(f_1, \cdots, f_n) &\equiv \int S_n(x_1, \cdots, x_n)\textstyle\prod_{i=1}^n f_i(x_i)dx_i \\ &= \int_{\mathcal{S}'} \textstyle\prod_{i=1}^n \phi(f_i)d\mu(\phi) , \end{aligned} \tag{1.0}$$

for arbitrary Schwartz test functions $f_1, \cdots, f_n$. Here $\{\phi(f): f \in \mathcal{S}(\mathbf{R}^{d+1})\}$ are the coordinate functions defined by $\phi(f)[T] = T(f)$, for all $T \in \mathcal{S}'$. Equation (1.0) says that the Schwinger functions are the moments of the measure μ. We should emphasize that those properties of the Schwinger functions that can be derived from the Wightman axioms do *not* imply that a measure (1.0) exists. However, in all models thus far constructed, such a measure does exist. (If the Schwinger functions are the moments of a measure we say they satisfy *Nelson-Symanzik positivity.*)

We recall that the Euclidean group acts in a natural manner on $\mathcal{S}'$ (if β is an element of the Euclidean group, $T \in \mathcal{S}'$, we set $T_\beta(f) = T(f_\beta)$, where $f_\beta(x) = f(\beta^{-1}x)$, for all $f \in \mathcal{S}(\mathbf{R}^{d+1})$). This yields a homomorphism of the σ-algebra Σ onto itself.

By proper Euclidean motions we mean the elements in the connected component of the Euclidean group on $\mathbf{R}^{d+1}$ containing the identity. Let θ be defined by

$$\theta(x_1, \cdots, x_d, x_{d+1}) = (x_1, \cdots, x_d, -x_{d+1}) ,$$

and, for F a Σ-measurable function on $\mathcal{S}'$,

$$(\theta F)(T) = F(T_\theta) .$$

With each open set $\Lambda \subset \mathbf{R}^{d+1}$ we associate the σ-algebra Σ_Λ of sets in $\mathcal{S}'$ generated by the coordinate functions $\{\phi(f): f \in \mathcal{S}(\mathbf{R}^{d+1}), \operatorname{supp} f \subset \Lambda\}$. We set

$$\sum\nolimits_+ = \sum\nolimits_{\Lambda=\{x:\, x_{d+1}>0\}} .$$

We now consider the class of probability measures μ on $(\mathcal{S}', \Sigma)$ with the following four properties:

(a) μ is invariant under all proper Euclidean motions.

(b) $\int \overline{(\theta F)} F \, d\mu \geqq 0$, for all Σ_+-measurable functions F on $\mathcal{S}'$. This property is known as OS *positivity*; (it is a reformulation of Osterwalder-Schrader positivity [34] in a probabilistic context).

(c) There exists a norm $|||\cdot|||$ continuous on $\mathcal{S}(\mathbf{R}^{d+1})$ such that

$$\int \exp(\phi(f)) d\mu$$

is uniformly bounded and continuous in the norm $|||\cdot|||$ on $\{f: f \in \mathcal{S}(\mathbf{R}^{d+1}), |||f||| \leqq 1\}$.

(d) The action of the group of translations of $\mathbf{R}^{d+1}$ is *ergodic*. As a consequence of the Osterwalder-Schrader reconstruction theorem [34] we have

THEOREM 1.0. *The moments of a measure μ on $(\mathcal{S}', \Sigma)$ which has properties* (a)-(d) *are the Schwinger functions of a unique relativistic quantum field theory satisfying all the Wightman axioms. If a measure μ has properties* (a)-(c) *then all its ergodic components have properties* (a)-(d) (i.e., (a)-(c) *are stable under ergodic decompositions*).

Remarks 1. The mathematical structure defined by properties (a)-(d) has been emphasized and studied in [5]. The second part of Theorem 1.0 is due to [5].

2. Property (a) is the Euclidean formulation of *relativistic* (Poincaré) *covariance*; (b) expresses the fact that the inner product on the physical Hilbert space is *positive definite*, and (d) implies *uniqueness of the physical vacuum* (see [47]). Property (c) can be replaced by weaker conditions (see [34]), but it is known to hold in all theories thus far constructed [4], [39].

One way of formulating the Euclidean approach to constructing relativistic quantum fields is that in order to obtain quantum field models it suffices to construct probability measures on $(\mathcal{S}', \Sigma)$ satisfying (a)-(d). This, in turn, is a special problem in the theory of generalized stochastic processes over $\mathbf{R}^{d+1}$.

One can briefly describe our two main results as the construction of *many new* examples of such measures in case $d = 1$ (unifying previous constructions) and the proof of a *very strong regularity* property of these measures and of measures previously constructed. We should also emphasize that we prove a new family of technical estimates which we call *chessboard estimates* (see Section 2) and which subsume many earlier estimates.

It is quite easy to construct one class of measures satisfying (a)-(d), namely the Gaussian process, μ_0, with mean zero and covariance

$$\int \phi(f)\phi(g)d\mu_0 = \left(f, (-\Delta + m_0^2)^{-1}g\right) .$$

The corresponding quantum theory describes non-interacting, relativistic particles of mass m_0, so that this theory is usually called the theory of the *free Euclidean field.* One way of constructing interesting measures is as follows: Suppose that one has a function $F_\Lambda \in L^1(d\mu_0)$, for each bounded open region $\Lambda \subset \mathbf{R}^{d+1}$, such that: (i) $F_\Lambda \geqq 0$, (ii) F_Λ is Σ_Λ-measurable, (iii) for any Euclidean motion, proper or improper, $\alpha(F_\Lambda) = F_{\alpha(\Lambda)}$, (iv) if $\Lambda_1, \cdots, \Lambda_n$ are disjoint and open and $\Lambda\backslash\Lambda_1 \cup \cdots \cup \Lambda_n$ has Lebesgue measure zero, then $F_\Lambda = \prod_{i=1}^n F_{\Lambda_i}$. In analogy with the theory of Markov processes, such families are called *multiplicative functionals.* Given such a functional, and any Λ with $\theta(\Lambda) = \Lambda$, the measure

$$d\nu^{(\Lambda)} = Z_\Lambda^{-1} F_\Lambda d\mu_0; \; Z_\Lambda = \int F_\Lambda d\mu_0 \tag{1.1}$$

is a probability measure satisfying OS positivity (property (b)); for, $d\mu_0$ satisfies OS positivity, and $F_\Lambda = F_{\Lambda_1} \cdot \theta(F_{\Lambda_1})$ with $\Lambda_1 = \{x \in \Lambda \mid x_{d+1} > 0\}$. One can hope to recover (a) by constructing some limit, as $\Lambda \to \mathbf{R}^{d+1}$. Even if this limit does not exist, one could hope to construct a subsequence limit by means of a *compactness argument* (such ideas go back to [7]). This limit is a priori *not* Euclidean invariant, but since the Euclidean group is amenable [15], one could hope to recover (a) by averaging. Such averaging will, in general, destroy OS positivity, so that, given some $\{F_\Lambda\}$, even the existence of the infinite volume limit presents a *non-trivial problem.*

In terms of multiplicative functionals, one can describe the simplification associated with the case $n \equiv d + 1 = 2$. If $n \geqq 3$, no multiplicative functionals are known for μ_0 which are neither Gaussian nor exponentials in the field and it is widely believed that none exist. In two dimensions many exist, e.g. one associated to each semibounded polynomial P (in physics, quantities like the energy tend to be unbounded from above but *not* from below, so a mathematical physicist uses the term "semibounded" to mean bounded from below): namely $F_\Lambda = \exp(-U(\Lambda))$ where

$$U(\Lambda) = \int_\Lambda : P(\phi(x)) : d^n x$$

is a multiplicative functional. Here :—: indicates the necessity of making various "infinite subtractions" known as *Wick ordering* (see e.g., [46], [43]).

The infinite volume problem (i.e., proving (a)-(c) for these functionals) was first solved by Glimm, Jaffe, and Spencer [12], for P fixed and m_0 sufficiently large (equivalently, for m_0 and Q fixed and $P = \lambda Q$, with λ sufficiently small); by Nelson [32] and Guerra, Rosen, and Simon [23], for $P = Q - \mu X$ with Q even; and by Spencer [45], for $P = Q - \mu X^n$, where $n < \deg Q$ is *odd*, and μ is sufficiently large. Among other things, we solve this problem for *arbitrary, semibounded polynomials P*. A typical P for which this is the first construction is: $P(X) = \lambda(X^6 + X^5)$, with λ large.

In solving the infinite volume problem, it is often useful to modify the definition (1.1) by additional terms at the boundary. All that enters about $d\mu_0$ in (1.1) is the measure $d\mu_0 \restriction \Sigma_\Lambda$ which is the Gaussian process on $\mathcal{S}'(\Lambda)$ with covariance $f, g \mapsto (f, (-\Delta + m_0^2)^{-1} g)$. If one takes instead the covariance $f, g \mapsto (f, (-\Delta_\Lambda^D + m_0^2)^{-1} g)$, where Δ_Λ^D is the Laplacian with Dirichlet boundary conditions, one obtains the "Dirichlet field." One also obtains measures of interest if Δ_Λ^N (the Neumann-Laplacian) is used, or if Λ is a rectangle, and Δ_Λ^P (the Laplacian with periodic boundary conditions) is used. One uses a unified symbol $d\mu_{0,\Lambda}^Y$ for all these measures, with $Y = F, D, N, P$ (for free, Dirichlet, Neumann, periodic) to indicate the choice of boundary conditions. An additional complication is that, in defining $U(\Lambda)$, the Wick-ordering, :—:, can be defined with respect to $d\mu_{0,\Lambda}^Y$ for $Y = F, D, N, P$. If the $d\mu_{0,\Lambda}^{F-}$ Wick-ordering is used, the measure $d\nu^{(\Lambda)}$ is called free, half-Dirichlet (HD), half-Neumann, $\cdots$, and if the Wick-ordering and $d\mu_{0,\Lambda}^Y$ involve the same Y, we speak of free, Dirichlet, etc. In the above results, that of [12] uses F states, that of [32] and [23] uses HD and that of [45] uses HP.

Given these various "boundary conditions" one wants a criterion for an infinite volume measure $d\nu$ obeying (a)-(d) to be somehow a $P(\phi)_2$ theory, for a given P. One convenient sense is that it obey the DLR equations of [23], say in the following form: for $\Lambda \subset \mathbf{R}^2$ bounded, the associated measure ν on $\mathcal{S}'(\mathbf{R}^2)$ obeys:

(1.2a) $$d\nu \restriction \Sigma_\Lambda = \exp(-U(\Lambda))\exp(-|\Lambda|\alpha_\infty) g_{\partial\Lambda}(d\mu_0 \restriction \Sigma_\Lambda) \equiv f_\Lambda(d\mu_0 \restriction \Sigma_\Lambda)$$

where $g_{\partial\Lambda} \in L^1(d\mu_0)$ is a function of the fields on $\partial\Lambda$, i.e., Σ_Ω-measurable for any open $\Omega \supset \partial\Lambda$; and α_∞ is the pressure

(1.3) $$\alpha_\infty(P) = \lim_{\Lambda\to\infty} |\Lambda|^{-1}\ln\int\exp(-U(\Lambda)d\mu_0$$

proven to exist by Guerr [18]; see GRS [24] for the equality of pressures with different boundary conditions. The regularity condition which asserts that $d\nu \restriction \Sigma_\Lambda$ is $d\mu_0$ absolutely continuous has been proven already for a variety of states: first by Newman [33] for small coupling states with $Y=F$ and then by Fröhlich [4] for general $Y=F$. Given recent bounds of Glimm-Jaffe [11], the method of [4] could be used to prove local absolute continuity for the states of Nelson [32] and GRS [23]. We will recover all these results (Sections 6, 7) and indeed the stronger property that $g_{\partial\Lambda}$ is in $L^p(d\mu_0)$ for all $p<\infty$ with

(1.2b) $$\|g_{\partial\Lambda}\|_p \leqq C_1\exp(C_2|\partial\Lambda|)$$

for all rectangles Λ. This should be compared with the fact that [4] and [33] only prove $f_\Lambda \in L^1$ (although using estimates of Gross [17], one can show $f_\Lambda \in L^1 \log L$ by the method of [4]).

The local absolute continuity (1.2) should be contrasted with the fact that as measures on $\mathcal{S}'(\mathbf{R}^2)$, the $\langle\ \rangle_{P,\pm}$ are mutually singular for distinct P's. For, we will prove them all ergodic and they are distinct as can be seen by noting the DLR equations in Fröhlich's form [4]: (1.2) implies that for $f\in C_0^\infty$, $\nu(\cdot+f)$ is absolutely continuous with respect to ν with an explicit R.N. derivative which varies as P does. (See Schrader [38], Rosen-Simon [36], Fröhlich [5] and Lenard-Newman [27] for additional discussion of mutual singularity).

Our estimate (1.2) will have an additional consequence. GRS [23] discuss the notions of entropy and the Gibbs variational principle. For any translation invariant measure ν on $\mathcal{S}'(\mathbf{R}^2)$, with

(1.4) $$d\nu \restriction \Sigma_\Lambda = f_\Lambda(d\mu_0 \restriction \Sigma_\Lambda)$$

where Λ is bounded and where for some $p>1$:

(1.5) $$\|f_\Lambda\|_p \leqq C\exp(a(\mathrm{diam}(\Lambda))^2)$$

GRS show that:

(1.6) $$s(f) = \lim_{\Lambda\to\infty} -|\Lambda|^{-1}\int f_\Lambda \log f_\Lambda d\mu_0$$

and

(1.7) $$\rho(f, P) = \lim_{\Lambda\to\infty} |\Lambda|^{-1}\int U(\Lambda) f_\Lambda d\mu_0$$

exist, and

$$s(f) - \rho(f, P) \leqq \alpha_\infty(P) . \tag{1.8}$$

(1.8) is connected with the Gibbs variational principle [37], the Rayleigh-Ritz principle of quantum theory [23] and, we believe, with the variational principles of Donsker-Varadan [2]. Subsequent to [23], GRS proved [24] that f's exist giving one arbitrarily close to equality in (1.8). The estimate (1.2) will allow us to extend this further: first f's exist giving equality in (1.8) and, in fact, all f's obtained from $P(\phi)_2$ states thus far constructed obey equality in (1.8) (Section 6). This represents the translation to Euclidean field theory of a circle of results in the statistical mechanics of lattice systems [37].

We want to close this introduction by saying something about our methods and their antecedents, especially since so many of the methods involve combinations of mild extensions and elaborations of current ideas. One natural alternative method for constructing a state for a polynomial P is as follows: Make the P dependence of U explicit by writing $U(\Lambda; P)$. For another semibounded polynomial Q, suppose one can control the limit

$$d\nu_Q = \lim_{\Lambda\to\infty} \exp(-U(\Lambda; Q))d\mu_0 \Big/ \int \exp(-U(\Lambda; Q))d\mu_0 ,$$

and then the limit

$$d\tilde{\nu}_P = \lim_{\Lambda\to\infty} \exp(-U(\Lambda; P-Q))d\nu_Q \Big/ \int \exp(-U(\Lambda; P-Q))d\nu_Q .$$

Then $d\tilde{\nu}_P$ should be a state associated to P. For P of the form $P(X)=R(X)-\mu X$ with R even, such a procedure was suggested and controlled by Glimm-Jaffe [9] who took $Q(X) = P(X) + \sigma X^2$ with σ large. The limit $d\nu_Q$ is then controlled by the cluster expansion [13] since the Q theory is "equivalent" to weak coupling and $d\tilde{\nu}_P$ by using correlation inequalities of GKS type [16], [25], [23]. The resulting "weak-coupling boundary condition" state can be shown [9] to be identical to the GRS-Nelson half-Dirichlet state. The Glimm-Jaffe construction is useful since free boundary conditions are easier to control technically than the half-Dirichlet boundary conditions [9], [11]. The above construction is limited to P of the form indicated because it relies on GKS inequalities which are restricted to such P's.

We will construct states for arbitrary P by the above method by appealing to FKG inequalities [3] which hold for any P [23]. We will take $Q = P - \mu_\infty X$ for μ_∞ very large so that the Q state can be constructed by Spencer's method [45] of reducing it to a small coupling constant theory. The limit $d\tilde{\nu}$ will be controlled with FKG inequalities.

Our method of proving ergodicity of the states constructed this way relies on a theorem of Simon [40] which reduces the problem to proving clustering for the two point function and a simple extension of a very elegant argument of Guerra [19], [20] for proving such clustering.

We will prove a variety of results asserting that states constructed by different methods are identical, e.g., in the construction discussed above, the state is independent of what value of large μ_∞ is taken. Our methods of proof are a translation to the field theory context of an argument of Lebowitz-Martin-Löf [26] which we now give:

Definition. Let $d\nu_1$, $d\nu_2$ be two probability measures on $\mathcal{S}'(\mathbf{R}^2)$. We say that $\nu_1 \leqq \nu_2$ (FKG) if and only if

$$\int F(\phi(f_1), \cdots, \phi(f_n))d\nu_1 \leqq \int F(\phi(f_1), \cdots, \phi(f_n))d\nu_2$$

for any function F on $\mathbf{R}^n$ which is monotone increasing in each variable separately and for any $f_1, \cdots, f_n \in C_0^\infty(\mathbf{R}^2)$, all positive.

Definition. We say that a probability measure $d\nu$ on $\mathcal{S}'(\mathbf{R}^2)$ obeys exponential bounds if and only if

$$\int \exp(\phi(f))d\nu < \infty$$

for all $f \in \mathcal{S}(\mathbf{R}^2)$, real valued, in such a way that

$$f \longmapsto \int \exp(\phi(f))d\nu < \infty$$

is continuous on $\mathcal{S}(\mathbf{R}^2)$.

THEOREM 1.1. *Let $d\nu_1$ and $d\nu_2$ be probability measures on $\mathcal{S}'(\mathbf{R}^2)$ so that both obey exponential bounds and so that $\nu_1 \leqq \nu_2$(FKG). If*

$$\int \phi(f)d\nu_1 = \int \phi(f)d\nu_2 \tag{1.9}$$

for all $f \in C_0^\infty(\mathbf{R}^2)$, then $\nu_1 = \nu_2$.

Proof. By the exponential bounds, it suffices to prove that

$$\int \phi(f_1)\cdots\phi(f_n)d\nu_1 = \int \phi(f_1)\cdots\phi(f_n)d\nu_2 \qquad \text{for all } f_1, \cdots, f_n \in C_0^\infty$$

and by multilinearity it suffices to prove this equality for $f_1, \cdots, f_n \geqq 0$. Following Simon [40], we introduce random variables $\sigma(f_i)$, $\rho(f_i)$ by:

$$\begin{aligned} \sigma(f) &= \phi(f) \quad \text{if } |\phi(f)| \leqq 1 \,, \\ &= 1 \qquad\;\; \text{if } \phi(f) \geqq +1 \,, \\ &= -1 \quad\;\; \text{if } \phi(f) \leqq -1 \,; \end{aligned}$$

$$\rho(f) = \frac{1}{2}(1 + \sigma(f)) \,.$$

By the exponential bounds:

$$\int \phi(f_1)\cdots\phi(f_n)d\nu_i = \lim_{a\to\infty}\int a^n\sigma(f_1/a)\cdots\sigma(f_n/a)d\nu_i$$

so it suffices to show that

$$\int\rho(f_1)\cdots\rho(f_n)d\nu_1 = \int\rho(f_1)\cdots\rho(f_n)d\nu_2 \tag{1.10}$$

for all $f_1, \cdots, f_n \geqq 0$. Now both

$$\rho(f_1)\cdots\rho(f_n) \text{ and } \frac{1}{2}\sum_{i=1}^n \phi(f_i) - \prod_{i=1}^n \rho(f_i)$$

are monotone functions of the fields $\phi(f_i)$, $f_i \geqq 0$, so that

$$\int\prod_{i=1}^n \rho(f_i)d\nu_1 \leqq \int\prod_{i=1}^n \rho(f_i)d\nu_2, \text{ and}$$
$$-\int\prod_{i=1}^n \rho(f_i)d\nu_1 \leqq -\int\prod_{i=1}^n \rho(f_i)d\nu_2 + \frac{1}{2}\sum_{i=1}^n\int\phi(f_i)d(\nu_2-\nu_1)$$

from which (1.10) follows by using (1.9). ■

Finally, we should like to say something about the general estimates we prove in Section 2. These estimates include as special cases the ϕ and ϕ^j bounds for general states (see e.g., [8], [21], [4], [11]) and estimates needed in the proof of the existence of phase transitions [14]. Moreover, the estimates (1.2) are a simple consequence (§ 7) of them and of checkerboard estimates [23]. The estimates are non-trivial even when $P = 0$ in which case one obtains estimates which are related to checkerboard estimates as the improved linear lower bound [22] is to Nelson's proof [31] of the linear lower bound.

Special cases of these estimates have recently been found independently by Glimm-Jaffe-Spencer [14], Guerra [19], Seiler-Simon [39] and in the periodic case by Park [35], all of whose work has motivated us. In a preliminary version of this paper we closely followed [39] and [35], dealing directly with the Euclidean field theory and OS positivity. In Section 2, we present a proof using "Hamiltonian" methods following [21] and [4]. While these proofs are substantially equivalent, it seems to us the one we give is notationally simpler and more "intuitive."

The basic idea of these estimates is to use the pressure to bound interacting expectations. One of the simplest examples is the following estimate of [19], [39]:

$$\int\exp(\phi(f))d\nu_P \leqq \int[\alpha_\infty(f(x)) - \alpha_\infty(0)]d^2x \tag{1.11}$$

where $d\nu_P$ is a state for the polynomial P and $\alpha_\infty(\mu)$ is the pressure for

$P(X) - \mu X$. Earlier estimates in the spirit of (1.11) can be found in GRS [22], Fröhlich [4], and Simon [42].

These estimates which we call chessboard estimates turn out to be especially powerful when used in conjunction with the checkerboard estimates of [23]:

Let Δ_α, $\alpha \in \mathbf{Z}^2$, be the mesh of squares with centers at $l\alpha$ and side l. Let $\{F_\alpha\}$ be a finite family of functions with F_α Σ_{Δ_α}-measurable. Let

$$p = \left(2/\{1 - e^{-m_0 l}\}\right)^2 .$$

Then the checkerboard estimate asserts that

$$\int \prod_\alpha F_\alpha \, d\mu_0 \leqq \prod_\alpha \|F_\alpha\|_p$$

with $\|\cdot\|_p$ the $L^p(\mathcal{S}', d\mu_0)$ norm.

It is a pleasure to thank F. Guerra and L. Rosen for valuable conversations and for permission to quote an unpublished result (Theorem 6.1) of GRS.

2. Chessboard estimates

By the *mesh of l-squares* we mean the squares of side l with centers at the points $l\alpha$; $\alpha \in \mathbf{Z}^2$. If F is a function of the fields in the l-square Δ_α about $l\alpha$, we define $F^{[\beta]}$ for $\beta \in \mathbf{Z}^2$ as follows: let $(\beta_1, \beta_2) = \beta$. If β_1 and β_2 are even, then $F^{[\beta]}$ is the translation of F to Δ_β. If β_1 (resp. β_2) is odd and β_2(resp. β_1) is even, we reflect F at the line $t = 0$ (resp. $x = 0$) and translate from $(-\alpha_1, \alpha_2)$(resp. $(\alpha_1, -\alpha_2)$) to β. If β_1 and β_2 are both odd we reflect in both lines and translate from $-\alpha$ to β. For fixed a and b, let

$$F^{(a,b)} = \prod_{\substack{0 \leqq \beta_1 \leqq a-1 \\ 0 \leqq \beta_2 \leqq b-1}} F^{[\beta]} . \tag{2.1}$$

The definition of $F^{[\beta]}$ is just so chosen that $F^{(2a,b)} = \tilde{F}^{(a,b)} F^{(a,b)}$ with $\tilde{F}$ the reflection of $F^{(a,b)}$ at the line $t = l(a - 1/2)$ and similarly for $F^{(a,2b)}$. It follows immediately from OS positivity that:

PROPOSITION 2.1. $\int d\mu_0 \, F^{(a,b)} \geqq 0$ *for all even a, b. Moreover*:

$$\gamma_0(F, \Delta_\alpha) = \lim_{n,m\to\infty} \left(\int F^{(2^n, 2^m)} d\mu_0 \right)^{1/2^n \, 1/2^m} \tag{2.2}$$

exists and is finite if $F \in \bigcap_{p<\infty} L^p(d\mu_0)$.

Proof. By OS positivity the quantity on the right of (2.2) is monotone in n, m. Its finiteness for F in a suitable L^p follows from the checkerboard estimates. ∎

When there is no possibility of confusion, we will just refer to $\gamma_0(F)$.

For any semibounded polynomial, P, we define:

$$\gamma(F, \Delta_\alpha, P) = \gamma_0(F\exp(-U(\Delta_\alpha; P)))/\gamma_0(\exp(-U(\Delta_\alpha; P))) \,. \tag{2.3}$$

Remark. If $F = \exp(-U(\Delta_\alpha; Q))$, then $\gamma_0(F) = \exp(+|\Delta_\alpha|\alpha_\infty(Q))$ and $\gamma(F, P) = \exp(|\Delta_\alpha|(\alpha_\infty(P+Q) - \alpha_\infty(P)))$.

Our main goal in this section is to prove the following results which we call "chessboard estimates":

THEOREM 2.2 (free chessboard estimate). *Let $\{F^{(\alpha)}\}_{\alpha\in A}$ with $\#(A)$ finite be a family of functions, each measurable with respect to the fields in a distinct square Δ_α in the l-mesh. Then:*

$$\left|\int \prod\nolimits_{\alpha\in A} F_\alpha\, d\mu_0\right| \leqq \prod\nolimits_{\alpha\in A}\gamma_0(F_\alpha, \Delta_\alpha) \,. \tag{2.4}$$

Definition. Let $\{d\nu_\Lambda\}$ be a family of measures indexed by finite sets $\Lambda_{L,t} = \{(x, s) \mid |x| \leqq L/2,\ |s| \leqq t/2\}$. We say $d\nu_\Lambda \to d\nu_\infty$ by iteration if

$$\lim\nolimits_{L\to\infty}\left(\lim\nolimits_{t\to\infty}\int F d\nu_{\Lambda_{L,t}}\right) = \int F\, d\nu_\infty$$

for any function of the fields in a bounded region in $\bigcap_{p<\infty} L^p(d\mu_0)$. (By a simple limiting argument, it suffices to consider $F = \exp(i\phi(f))$, $f \in C_0^\infty(\mathbf{R}^2)$.)

THEOREM 2.3. *Let F_α be as in Theorem 2.2 and suppose that*

$$\exp(-U(\Lambda; P))\,d\mu_0\Big/\int\exp(-U(\Lambda; P))\,d\mu_0 \longrightarrow d\nu_P$$

by iteration as $\Lambda \to \infty$. Then:

$$\left|\int\prod\nolimits_{\alpha\in A} F_\alpha\, d\nu_P\right| \leqq \prod\nolimits_{\alpha\in A}\gamma(F_\alpha, \Delta_\alpha, P) \,. \tag{2.5}$$

THEOREM 2.4. *Let F_α be as in Theorem 2.2 and $d\nu_P$ as in Theorem 2.3. Suppose that*

$$\exp(-U(\Lambda; Q-P))\,d\nu_P/\text{Norm.} \longrightarrow d\tilde{\nu}_Q$$

by iteration as $\Lambda \to \infty$. Suppose moreover that

$$\alpha_\infty(Q \mid P) = \lim\nolimits_{L\to\infty}\left(\lim\nolimits_{t\to\infty}|\Lambda|^{-1}\log\int\exp(-U(\Lambda; Q-P)\right)d\nu_P$$

exists and that

$$\alpha_\infty(Q \mid P) = \alpha_\infty(Q) - \alpha_\infty(P) \,. \tag{2.6}$$

Then:

$$\left|\int\prod\nolimits_{\alpha\in A} F_\alpha d\tilde{\nu}_Q\right| \leqq \prod\nolimits_{\alpha\in A}\gamma(F_\alpha, \Delta_\alpha, Q) \,. \tag{2.7}$$

Remarks 1. The existence of the limit defining $\alpha_\infty(Q \mid P)$ is general via OS positivity and convexity. Moreover,

$$\alpha_\infty(Q\,|\,P) \leqq \alpha_\infty(Q) - \alpha_\infty(P)$$

by Theorem 2.3. A discussion of the opposite inequality may be found in Section 3.

2. Even without (2.6) a result similar to (2.7) holds but $\gamma(F_\alpha, \Delta_\alpha, Q)$ is replaced by

$$\gamma_0(F_\alpha \exp(-U(\Delta_\alpha; Q)))/\exp[|\Delta_\alpha|(\alpha_\infty(Q\,|\,P) + \alpha_\infty(P))]\,.$$

3. At the close of this section, we briefly describe how to obtain chessboard estimates for periodic boundary conditions.

Proof of Theorem 2.2 (an abstraction of the ϕ-bound proof of [21]). By translation covariance and the fact that $\gamma_0(1) = 1$, we may suppose that $A = \{(\alpha_1, \alpha_2)\,|\,0 \leqq \alpha_1 \leqq a - 1,\ 0 \leqq \alpha_2 \leqq a - 1\}$. Moreover without loss of generality, we may suppose that a is even and that $G = \prod_{\alpha \in A} F_\alpha$ is invariant with respect to reflection about the lines x or $s = (1/2)a - 1/2$; for, let $\widetilde{G}$ be the function obtained by taking the product of G with its reflections about the lines $x = a - 1/2$ and $s = a - 1/2$ and both lines together (four factors). Then, by OS positivity $\left|\int G\, d\mu_0\right| \leqq \left(\int \widetilde{G}\, d\mu_0\right)^{1/4}$ and the right side of (2.4) for $\widetilde{G}$ is the 4th power of the right side of (2.4) for G.

Let J_i be the embedding [23] of the Fock space fields at time $s = l(i-1/2)$. Let B_i be the operator

$$B_i = J_{i+1}^*\left(\textstyle\prod_{k=0}^{a-1} F_{(k,i)}\right)J_i\,, \qquad i = 0, \cdots, a-1\,.$$

Then, by the Markov property:

$$\left|\int \textstyle\prod_{\alpha \in A} F_\alpha d\mu_0\right| = |(\Omega_0, B_{a-1}\cdots B_0\Omega_0)| \leqq \textstyle\prod_{i=0}^{a-1} \|B_i\|\,. \tag{2.8a}$$

Moreover, by a general argument and the supposed symmetry of G, each operator B_i "couples" to the vaccum, i.e.,

$$\|B_i\| = \lim_{n\to\infty}\left(\Omega_0, (B_i^* B_i)^{2^n}\Omega_0\right)^{1/2^{n+1}}\,. \tag{2.8b}$$

This general argument is due independently to McBryan [29] and Seiler-Simon [39] and may be found in their papers. Now:

$$\left(\Omega_0, (B_i^* B_i)^{2^n}\Omega_0\right) = \int \textstyle\prod_{\alpha \in A_n} H_\alpha d\mu_0 \tag{2.9}$$

where $A_n = \{(\alpha_1, \alpha_2)\,|\,0 \leqq \alpha_1 \leqq a - 1,\ 0 \leqq \alpha_2 \leqq 2^{n+1} - 1\}$ where each $H_{(\alpha_1,\alpha_2)}$ is a translate or translate and reflection of $F_{(k,i)}$. (2.9) follows from the Markov property. Repeating the argument leading to (2.8), we see that

$$\lim_{n\to\infty}\left(\Omega_0, (B_i^* B_i)^{2^n}\Omega_0\right)^{1/2^{n+1}} \leqq \textstyle\prod_{k=0}^{a-1} \gamma_0(F_{(k,i)})\,; \tag{2.10}$$

(2.8) and (2.10) imply (2.4). ∎

Before the proof of Theorem 2.3, it is useful to note the following corollary of Theorem 2.2.

COROLLARY 2.5. *Let $\{\Delta_\alpha\}_{\alpha\in A}$ be a family of l-squares and let Λ be an L-square with $\bigcup_{\alpha\in A}\Delta_\alpha\subset\Lambda$. Let F_α be Δ_α-measurable. Then*:

$$\gamma\left(\prod\nolimits_{\alpha\in A}F_\alpha,\ \Lambda,\ P\right)\leqq\prod\nolimits_{\alpha\in A}\gamma(F_\alpha,\ \Delta_\alpha,\ P)\ . \tag{2.11}$$

Proof. Since $\gamma(1,\Delta_\alpha,P)=1$, we can, by setting some F_α equal to 1, suppose that $\bigcup_{\alpha\in A}\Delta_\alpha=\Lambda$. Let $G=\left(\prod_{\alpha\in A}F_\alpha\right)\exp(-U(\Lambda))$. Then, by the free chessboard estimates:

$$\left|\int G^{(a,b)}d\mu_0\right|\leqq\left[\prod\nolimits_{\alpha\in A}\gamma_0\left(F_\alpha\exp(-U(\Delta_\alpha)),\ \Delta_\alpha\right)\right]^{ab}$$

so that (2.11) follows, since

$$\gamma_0\left(\exp(-U(\Lambda))\right)=\exp\left(|\Lambda|\alpha_\infty(P)\right)=\prod\nolimits_{\alpha\in A}\gamma_0\left(\exp(-U(\Delta_\alpha))\right)\ .\qquad\blacksquare$$

Proof of Theorem 2.3 (abstraction of the method of [4]). On account of Corollary 2.5, it suffices to prove the theorem for a single square, i.e., $\#(A)=1$, since any finite union of l-squares is contained in some L-square. As in the proof of Theorem 2.2, we can, without loss of generality, suppose that the factor F is in a square centered about (0, 0), symmetric under reflection about the line $x=0$. For simplicity of notation, suppose F is in a unit square. Fix l. Then in terms of the usual Hamiltonian H_l (infspec $H_l\equiv E_l$)

$$\begin{aligned}\left|\int F\,d\nu_{l,t}\right|&=\left|(\Omega_0,\ e^{-(t-1/2)H_l}A_le^{-(t-1/2)H_l}\Omega_0)\right|/(\Omega_0,\ e^{-tH_l}\Omega_0)\\&\leqq\|A_l\|(\Omega_0,\ e^{-(t-1)H_l}\Omega_0)/(\Omega_0,\ e^{-tH_l}\Omega_0)\end{aligned} \tag{2.12}$$

for a suitable operator $A_l=J_l^*\,F\exp\left(-U[(-l/2,\ l/2)\times(-1/2,\ 1/2)]\right)J_0$. By a general convexity argument [39] or the spectral theorem, the ratio in (2.12) converges to $\exp(E_l)$ as $t\to\infty$. Since A_l couples to the vacuum,

$$\|A_l\|=\lim\nolimits_{n\to\infty}(\Omega_0,\ (A_l^*A_l)^{2^n}\Omega_0)^{2^{-n-1}}\ ,$$

so, applying Nelson's symmetry and the free chessboard estimate:

$$\|A_l\|\leqq\exp(-E_{l-1})\gamma_0\left(F\exp(-U(\Delta))\right)\ . \tag{2.13}$$

(To prove (2.13), we use Nelson's symmetry to bound $\left(\Omega_0,\ (A_l^*A_l)^{2^n}\Omega_0\right)$ by $\left(\Omega_0,\ \exp(-2^{n+1}H_{l-1})\Omega_0\right)\|B_n\|$ where B_n is the norm of an operator associated to a strip of size $1\times2^{n+1}$ with $Fe^{-U(\Delta)}$ or its reflection in each unit square. $\|B_n\|$ is controlled by using the free chessboard estimate.) Since $E_l-E_{l-1}\to-\alpha_\infty(P)$ by general convexity arguments [39], taking first $t\to\infty$ and then $l\to\infty$ in (2.12) and using (2.13) we find that

$$\left|\int F\,d\nu_P\right| \leqq \gamma_0(F\exp(-U(\Delta)))/\exp(\alpha_\infty(P)) = \gamma(F, \Delta, P)\;. \qquad \blacksquare$$

Proof of Theorem 2.4. This is essentially identical to the passage from Theorem 2.2 to Theorem 2.3. We emphasize that we did not use the existence of a vacuum Ω_l for H_l in that proof. The net result is that

$$\begin{aligned}\left|\int F\,d\tilde{\nu}_Q\right| &\leqq \gamma(F\exp(-U(\Delta;\, Q-P), \Delta, P))/\exp(\alpha_\infty(Q\,|\,P)) \\ &= \gamma_0(F\exp(-U(\Delta;\, Q)))/\exp(\alpha_\infty(Q\,|\,P) + \alpha_\infty(P))\end{aligned}$$

which is $\gamma(F, \Delta, Q)$ given the hypothesis on $\alpha_\infty(Q\,|\,P)$. $\blacksquare$

As a general consequence of chessboard estimates, we recover the following result of Guerra [19] and Seiler-Simon [39] (see also [4]):

COROLLARY 2.6. *For a measure $d\nu_P$ or $d\tilde{\nu}_P$ of Theorem* 2.3 *or* 2.4:

$$\int \exp\left(-\int f(x) : Q(\phi(x)) : dx\right) d\nu_P \leqq \exp\left(\int [\alpha_\infty(P + f(x)Q) - \alpha_\infty(P)]dx\right) \tag{2.14}$$

for any f bounded with polynomial falloff at ∞.

Remark. In (2.14), by $\alpha_\infty(P + f(x)Q)$ we mean the pressure for the interaction $P + \lambda Q$ with $\lambda = f(x)$.

Proof. By a limiting argument, we need only consider f with compact support, piecewise constant on some set of l-squares. For such an f, (2.14) is precisely the chessboard estimate translated into α_∞-language. $\blacksquare$

Rather than formally state chessboard estimates for (half)-Dirichlet and periodic states, we settle for making a series of remarks:

1. Chessboard estimates hold for the (half)-Dirichlet limits of [23], [32] since they can be realized by the Glimm-Jaffe weak coupling construction [9] and we prove (2.6) in that case in Section 3.

2. Chessboard estimates hold for periodic states for P if $\alpha_\infty(P + \mu X)$ is differentiable in μ at $\mu = 0$. This follows from our general result (Section 5) that for such P, the periodic state is realizable via construction in the spirit of Theorem 2.4. This includes all periodic states thus far controlled.

3. A chessboard-like estimate but with γ_0 replaced by a "periodic" pressure (which is presumably equal to γ_0!) holds by abstracting an argument of Park [35]. (2.8) is replaced by:

$$\begin{aligned}\left|\int \prod\nolimits_{\alpha\in A} F_\alpha d\mu_{0,P}\right| &= \left|\text{“Tr”}\left(\prod\nolimits_{i=1}^k B_i\right)\right| \\ &\leqq \left|\prod\nolimits_{i=1}^k \text{“Tr”}\left((B_i)^k\right)^{1/k}\right|\end{aligned}$$

for a suitable unnormalized trace, "Tr" (see [24], [35]).

4. It is easy to extend chessboard estimates to Y_2 and $(\phi^4)_3$; see e.g., the discussion of exponential bounds in [39].

3. $\alpha_{\infty,\mathrm{WC}} = \alpha_{\infty,\mathrm{LE}} = \alpha_\infty$

Theorem 2.4 raises a natural question whose answer will be important in several other places in this paper. Let $d\nu_P$ be a translation invariant DLR state for P. Define

$$\alpha(Q\,|\,\nu_P) = \alpha_\infty(P) + \lim_{\Lambda\to\infty} |\Lambda|^{-1} \ln \int e^{-U(\Lambda;Q-P)} d\nu_P$$

where the limit will exist going through rectangles if ν_P obeys OS positivity. We conjecture that $\alpha(Q\,|\,\nu_P) = \alpha_\infty(Q)$ for any ν_P and Q but we have no idea of how to prove a result of this generality. Instead, we will consider two cases of interest: (i) $\alpha_{\infty,\mathrm{WC}}(Q)$ which is defined by taking $P = Q + \sigma X^2$ with σ large and using the weak coupling cluster expansion [13] to define $d\nu_P$; (ii) $\alpha_{\infty,\mathrm{LE}}(Q)$ which is defined by taking $P = Q \pm \mu_\infty X$ with μ_∞ large and using the large external field cluster expansion of Spencer to define ν_P. A priori these pressures depend on σ and μ_∞. In this section we prove:

THEOREM 3.1. *If ν_P is defined by the cluster expansion* [13], *then $\alpha_\infty(Q, \nu_P) = \alpha_\infty(Q)$ for any semibounded Q.*

COROLLARY 3.2. $\alpha_{\infty,\mathrm{WC}}(Q) = \alpha_{\infty,\mathrm{LE}}(Q) = \alpha_\infty(Q)$.

Proof of Corollary. The WC equality is immediate. The LE equality follows by using the covariance of α_∞ under translations of the field and the fact that after translation by a fixed amount (fixed, once μ_∞ is chosen) the Spencer state is defined by a weak coupling cluster expansion. ■

LEMMA 3.3. *Suppose that (P, m_0) is a pair for which the cluster expansion is applicable. For $\Lambda \subset \Lambda'$, rectangles, let $Z^{D;\Lambda}_{\Lambda'}$ be the partition function defined with interaction in Λ', with a Gaussian measure with a Dirichlet boundary condition on $\partial\Lambda$ and with the interaction Wick ordered relative to this Dirichlet Gaussian measure. Then:*

$$\lim_{\Lambda\to\infty} |\Lambda|^{-1} \lim_{\Lambda'\to\infty} \{\ln[Z^{D;\Lambda}_{\Lambda'}/Z_{\Lambda'}]\} = 0\,. \tag{3.1}$$

Proof. Let G_0 be the free Green's function and G^D_Λ the Green's function with Dirichlet data on $\partial\Lambda$. Let $C(s) = sG_0 + (1-s)G^D_\Lambda$ and $\dot{C} = G_0 - G^D_\Lambda$. Let $d\mu_{0,s,\Lambda}$ be the Gaussian measure with covariance $C(s)$ and $U_s(\Lambda')$ the interaction in Λ' with $C(s)$ Wick ordering. Define

$$Z^\Lambda_{\Lambda'}(s) = \int \exp(-U_s(\Lambda'))d\mu_{0,s,\Lambda} \tag{3.2}$$

so that $Z^\Lambda_{\Lambda'}(1) = Z_{\Lambda'}$, $Z^\Lambda_{\Lambda'}(0) = Z^{D;\Lambda}_{\Lambda'}$. We first claim that:

$$\frac{d}{ds} Z^\Lambda_{\Lambda'}(s) = \int_{x,y\in\Lambda'} dx\,dy\,\dot{C}(x,y)F(x,y,s)Z^\Lambda_{\Lambda'}(s)\,, \tag{3.3a}$$

(3.3b) $$F(x, y, s) = [Z^{\Lambda}_{\Lambda'}(s)]^{-1}\int :P'(\phi(x)):_s :P'(\phi(y)):_s \exp(-U_s(\Lambda'))d\mu_{0,s,\Lambda'} \,.$$

(3.3) follows from a standard integration by parts formula (e.g., (1.8) of [13]); there is a cancellation of the $(d/ds)U_s(\Lambda')$ term and the P'' term as is well-known when matched Wick ordering is used. (See e.g., Cooper-Rosen [1].) From (3.3) one immediately obtains:

(3.4) $$\ln[Z^{\Lambda}_{\Lambda'}(1)/Z^{\Lambda}_{\Lambda'}(0)] = \int_0^1 ds \int_{x,y\in\Lambda'} \dot{C}(x, y)F(x, y, s)dx\,dy \,.$$

By the cluster expansion [13], $\lim_{\Lambda\to\infty} F(x, y, s)$ exists and independently of Λ, s:

(3.5a) $$\left|\int dx\,dy\, H(x, y)F(x, y, s)\right| \leqq |||H||| \,,$$

(3.5b) $$|||H||| = \sum_{\alpha,\beta} \|H\chi_\alpha\chi_\beta\|_2 \,,$$

(3.5c) $$\|H\chi_\alpha\chi_\beta\|_2^2 = \int \chi_\alpha(x)\chi_\beta(y)\,|H(x, y)|^2 \,,$$

where χ_α is the characteristic function of the unit square about $\alpha\in\mathbf{Z}^2$. Now, by an elementary estimate [23], $\dot{C}(x,y)\leqq G_0(\mathrm{dist}(x,\partial\Lambda))$ and $\dot{C}(x,y)\leqq G_0(x-y)$, so

(3.6) $$|\dot{C}(x, y)|^2 \leqq G_0(\mathrm{dist}(x, \partial\Lambda))^{2/3}G_0(\mathrm{dist}(y, \partial\Lambda))^{2/3}G_0(x-y)^{2/3}$$

by symmetry. By (3.6),

(3.7) $$\|\dot{C}\chi_\alpha\chi_\beta\|_2 \leqq C\exp[-D[\mathrm{dist}(\alpha, \partial\Lambda) + \mathrm{dist}(\beta, \partial\Lambda) + |\alpha-\beta|]]$$

for suitable positive C, D. (3.4), (3.5) and (3.7) imply that

$$|\lim_{\Lambda'\to\infty}[\ln[Z^{\Lambda}_{\Lambda'}(1)/Z^{\Lambda}_{\Lambda'}(0)]]| \leqq C\,|\partial\Lambda| \,,$$

proving the lemma. ∎

Remark. We expect that (3.5) will hold in any $P(\phi)_2$ state and not just in case the state is defined by the cluster expansion. This would allow one to extend Theorem 3.1 to general states obeying chessboard estimates.

Proof of Theorem 3.1. By a chessboard estimate:

$$\ln\left[\int \exp(-U(\Lambda; Q-P))d\nu_P\right] \leqq |\Lambda|\,[\alpha_\infty(Q) - \alpha_\infty(P)]$$

so that

(3.8) $$\alpha_\infty(Q, \nu_P) \leqq \alpha_\infty(Q) \,.$$

On the other hand, by conditioning [23], [24], for $\Lambda\subset\Lambda'$:

$$\begin{aligned}\int &\exp(-U(\Lambda; Q-P))\exp(-U(\Lambda'; P))d\mu_0 \\ &\geqq \int \exp(-U_D(\Lambda; Q-P))\exp(-U_D(\Lambda'; P))d\mu_{0,D,\Lambda} \\ &= \exp[|\Lambda|[\alpha_{D,\Lambda}(Q) - \alpha_{D,\Lambda}(P)]]\,Z^{D;\Lambda}_{\Lambda'}\end{aligned}$$

where $Z_{\Lambda'}^{D;\Lambda}$ is given by Lemma 3.3 and we have used the decoupling nature of Dirichlet boundary conditions and

$$\alpha_{D,\Lambda}(Q) = |\Lambda|^{-1} \ln \int \exp(-U_D(\Lambda; Q)) d\mu_{0,D,\Lambda} .$$

This and the definition of ν_P yield

$$\ln \int \exp(-U(\Lambda; Q-P)) d\nu_P \geqq |\Lambda| [\alpha_{D,\Lambda}(Q) - \alpha_{D,\Lambda}(P)] + \lim_{\Lambda' \to \infty} [\ln(Z_{\Lambda'}^{D;\Lambda}/Z_{\Lambda'})] .$$

Using Lemma 3.3 we conclude that:

$$\alpha_\infty(Q, \nu_P) \geqq \alpha_\infty(P) + \alpha_{D,\infty}(Q) - \alpha_{D,\infty}(P) . \tag{3.9}$$

Since $\alpha_{D,\infty}(R) = \alpha_\infty(R)$ for any R [24], (3.8) and (3.9) complete the proof. ■

4. Construction and properties of $\langle\ \rangle_{P,\pm}$

Fix a semibounded polynomial P and consider the family of polynomials $P(X) - \mu X$, $-\mu_\infty \leqq \mu \leqq \mu_\infty$ where μ_∞ is chosen so large that $Q_\pm(X) = P(X) \mp \mu_\infty X$ are polynomials for which infinite volume states $d\nu_\pm^\infty$ can be constructed by Spencer's method.

THEOREM 4.1. *The measures*

$$\exp((\mu \mp \mu_\infty)\phi(\chi_\Lambda)) d\nu_\pm^\infty / N_\pm$$

have limits in the following senses: (i) *as* $\Lambda \to \infty$ *by inclusion in the sense of characteristic functions and moments,* (ii) *by iteration. The resulting limits* $d\nu_{P-\mu X,\pm}$ *obey all the* OS *axioms except* (*a priori*) *clustering. Moreover the Schwinger functions and Schwinger generating function for* $d\nu_{P-\mu X,+}$ (*resp.* $d\nu_{P-\mu X,-}$) *are continuous from the right* (*resp. the left*) *in* μ, *and for* $\mu' < \mu$:

$$d\nu_{P-\mu' X,\pm} \leqq d\nu_{P-\mu X,\pm} \qquad \text{(FKG)} , \tag{4.1}$$

$$d\nu_{P-\mu X,-} \leqq d\nu_{P-\mu X,+} \qquad \text{(FKG)} . \tag{4.2}$$

Remark. We will prove shortly that clustering also holds. A priori, the states $d\nu_{P,+}$, $d\nu_{P,-}$ depend on μ_∞, but we will prove this is not so. In Section 7 we prove $\nu_{P,\pm}$ are DLR states for P.

Proof. Let $f \geqq 0$; then by the FKG inequalities for $\nu_\pm^\infty$:

$$\int \exp(\pm\phi(f)) \exp((\mu \mp \mu_\infty)\phi(\chi_\Lambda)) d\nu_\pm^\infty / N_\pm$$

is monotone decreasing jointly as Λ is increased and μ is decreased (all top signs) (resp. increased for all bottom signs). The existence of the required limits for such f's follows and then for arbitrary f by standard "Vitali" methods (see e.g., [5]). Moreover for $f \geqq 0$ the interchange of the $\Lambda \to \infty$ and $\mu \downarrow \mu_0$ (resp. $\mu \uparrow \mu_0$) limits is allowed in the Schwinger generating functions by monotonicity. The interchange for general f and for the Schwinger functions is again by general methods. The OS axioms follow as for half-Dirichlet states [5], [43]. (4.1) is obvious from the FKG inequalities for $\nu_\pm^\infty$. (4.2) follows by putting in cutoffs in the definition for $\nu_\pm^\infty$ using, say, periodic boundary conditions for their definition. ■

Let $\alpha_\infty(\mu) = \alpha_\infty(P - \mu X)$. Since $\alpha_\infty(\mu)$ is convex in μ, the derivatives from the right, $D^+\alpha(\mu)$ and left $D^-\alpha(\mu)$ exist for all μ, and $D^+\alpha(\mu)$ (resp. $D^-\alpha$) is continuous from the right (resp. left). Moreover, for all but countably many μ, $(D^+\alpha)(\mu) = (D^-\alpha)(\mu)$ and these are precisely the μ for which $\alpha_\infty(\mu)$ is differentiable.

We use $\langle\cdot\rangle_{P,\pm}$ for $\int\cdot d\nu_{P,\pm}$ and $\langle\phi(0)\rangle_{P,\pm}$ for the number with

$$\langle\phi(f)\rangle_{P,\pm} = \langle\phi(0)\rangle_{P,\pm}\int f(x)d^2x\ .$$

THEOREM 4.2.

$$\langle\phi(0)\rangle_{P-\mu X,\pm} = D^\pm\alpha_\infty(\mu)\ . \tag{4.3}$$

Proof. Since both sides of (4.3) are continuous from the right (resp. left) in the $+$ (resp. $-$) case, it suffices to prove (4.3) for those μ with $\alpha_\infty(\mu)$ differentiable. For such a μ, chessboard estimates in the form of Corollary 2.6 (which hold for $\langle\ \rangle_{P,\pm}$ by Theorem 2.4, Theorem 3.1 and Theorem 4.1) imply for $a > 0$:

$$\langle\exp(\pm a\phi(\chi_\Lambda))\rangle_{\mu,\pm} \leqq \exp[|\Lambda|(\alpha_\infty(\mu \pm a) - \alpha_\infty(\mu))]\ ,$$

so subtracting 1, dividing by $|a||\Lambda|$ and taking $|a| \to 0$:

$$\pm\langle\phi(0)\rangle_{\mu,\pm} \leqq \pm D^\pm\alpha_\infty(\mu)\ .$$

At points with $D^+\alpha_\infty = D^-\alpha_\infty$, this implies (4.3). ■

COROLLARY 4.3. (i) $\langle\ \rangle_{P-\mu X,\pm}$ *is independent of the choice of* μ_∞.

(ii) $\langle\ \rangle_{P,+} = \langle\ \rangle_{P,-}$ *if and only if* $\alpha_\infty(\mu)$ *is differentiable at* $\mu = 0$.

(iii) $\alpha_\infty(\mu)$ *is strictly convex in* μ.

Proof. (i) If $\mu'_\infty > \mu_\infty$ the associated states $\langle\ \rangle'_{P,\pm}$ obey

$$\pm\langle\ \rangle_{P,\pm} \leqq \pm\langle\ \rangle'_{P,\pm} \qquad \text{(FKG)}\ .$$

By Theorem 4.2, the one point functions are the same so by Theorem 1.1, $\langle\ \rangle_{P,\pm} = \langle\ \rangle'_{P,\pm}$.

(ii) Same as the proof of (i), given (4.2).

(iii) By (4.1) and the proof of (i), if $\langle\phi(0)\rangle_{\mu,+} = \langle\phi(0)\rangle_{\mu',+}$, then $\langle\ \rangle_{\mu',+} = \langle\ \rangle_{\mu,+}$. This is impossible for $\mu \neq \mu'$ since the states obey DLR equations (see

Sections 6, 7). It follows by Theorem 4.2 that $D^{\pm}\alpha(\mu)$ are strictly monotone. ∎

THEOREM 4.4. *$\langle\ \rangle_{P,\pm}$ obey clustering (equivalently the associated Wightman theories have a unique vacuum).*

Proof. Without loss of generality, consider the case $\langle\ \rangle_{P,+}$. By a theorem of Simon [40], it suffices to prove that $\langle\phi(x)\phi(y)\rangle_T = \langle\phi(x)\phi(y)\rangle_+ - \langle\phi(x)\rangle_+\langle\phi(y)\rangle_+$ goes to zero as $|x-y|\to\infty$. Since $\langle\phi(x)\phi(y)\rangle_T$ is monotone decreasing as $|x-y|$ increases, it suffices to show that

$$\overline{\lim}_{\Lambda\to\infty}\langle\phi(\chi_\Lambda)\phi(\chi_\Lambda)\rangle_T/|\Lambda|^2 \leqq 0\ . \tag{4.4}$$

Let c be a constant to be fixed later and let $\psi(x) = \phi(x) + c$. Since $\langle\psi\psi\rangle_T = \langle\phi\phi\rangle_T$, (4.4) follows from:

$$\overline{\lim}_{\Lambda\to\infty}\langle\psi(\chi_\Lambda)\psi(\chi_\Lambda)\rangle_+^{1/2}/|\Lambda| \leqq \langle\psi(0)\rangle_+ = \langle\phi(0)\rangle_+ + c\ . \tag{4.5}$$

Now, by Hölder's inequality $\langle\psi(\chi_\Lambda)^2\rangle_+^n \leqq \langle\psi(\chi_\Lambda)^{2n}\rangle_+$, so for $a>0$:

$$\begin{aligned}\exp[a<\psi(\chi_\Lambda)^2>_+^{1/2}] &\leqq 2\cosh[a\langle\psi(\chi_\Lambda)^2\rangle_+^{1/2}]\\ &\leqq 2\langle\cosh a\psi(\chi_\Lambda)\rangle_+\\ &\leqq \exp[|\Lambda|(ac+\alpha_\infty(a)-\alpha_\infty(0))]\\ &\quad + \exp[|\Lambda|(-ac+\alpha_\infty(-a)-a_\infty(0))]\end{aligned}$$

where we have used chessboard estimates in the last step. Choose c large and ε small so that for $0<a<\varepsilon$:

$$ac+\alpha_\infty(a)-\alpha_\infty(0) \geqq -ac+\alpha_\infty(-a)-\alpha_\infty(0)$$

(e.g., $c = -(1/2)(D^+\alpha)(0)-(1/2)(D^-\alpha)(0)+1$ will do). Then:

$$\exp[a\langle\psi(\chi_\Lambda)^2\rangle_+^{1/2}] \leqq 2\exp(|\Lambda|(ac+\alpha_\infty(a)-\alpha_\infty(0)))\ .$$

Taking logs, dividing by $|\Lambda|$ and taking $|\Lambda|$ to ∞:

$$|\Lambda|^{-1}\langle\psi(\chi_\Lambda)^2\rangle_+^{1/2} \leqq c + a^{-1}(\alpha_\infty(a)-\alpha_\infty(0))\ .$$

Taking $a\downarrow 0$ using Theorem 4.2, we obtain (4.5) and thus unique vacuums. ∎

Remark. The first result of the above type is that differentiability of $\alpha_\infty(\mu)$ at $\mu=0$ implies unique vacuum for the ϕ^4 HD theory [41], [44]. Simon [41] related uniqueness of the vacuum in $\phi^4-\mu\phi$ to the fact that $\alpha_\infty(\mu)$ was C^2 for $\mu\neq 0$. The critical observation that C^1 suffices is due to Guerra [19], [20]. Our proof is a mild extension of his, exploiting one additional trick.

5. Consequences of differentiability of the pressure

We have already seen (Cor. 4.3 (ii)) that $\langle\cdot\rangle_{P,+} = \langle\cdot\rangle_{P,-}$ if and only if $\alpha_\infty(P-\mu X)$ is differentiable at $\mu=0$. On account of the one-sided continuity of $\langle\cdot\rangle_{P-\mu X,\pm}$, we have:

THEOREM 5.1. *If $\alpha_\infty(P - \mu X)$ is differentiable at $\mu = 0$, then the Schwinger functions and generating functional for $\langle\cdot\rangle_{P-\mu X,+}$ and $\langle\cdot\rangle_{P-\mu X,-}$ are continuous at $\mu = 0$.*

Our main goal in this section is to give a more significant consequence of differentiability of α_∞.

THEOREM 5.2. *If $\alpha_\infty(P - \mu X)$ is differentiable at $\mu = 0$, then all the classical boundary conditions measures (free, Dirichlet, half-Dirichlet, periodic, Neumann) for P converge as $\Lambda \nearrow \mathbf{R}^2$ and to the same limit $\langle\cdot\rangle_{P,+} = \langle\cdot\rangle_{P,-}$.*

Unfortunately the proof is somewhat technical, requiring some fine tuning of various elements of the cluster expansion. We defer these technicalities to an appendix, giving here the proof for $Y=P$ and the basic strategy of the general proof. We will not pause to explain the importance of Theorem 5.2 since it answers a natural question that also comes up at various technical points. We emphasize that $\alpha_\infty(P - \mu X)$ is differentiable for "most μ's." We note that earlier results of a much weaker kind occur in [24] and that for the special case of $P(X) = X^4 - \mu_0 X$, $\mu_0 \neq 0$, most of the results in Theorem 5.2 have been proved earlier in [6] using GHS inequalities and the Lee-Yang theorem. It is also clear that our method can be modified to accomodate other B.C.—e.g., free boundary conditions with a constant external field turned on outside the region of interaction (such boundary conditions occur in forthcoming work of Glimm, Jaffe and Spencer). Before turning to some aspects of the proof of Theorem 5.2, we prove a corollary due to Guerra (note added in proof to [20]):

COROLLARY 5.3 (Guerra [20]). *If Q is even and $\mu > 0$, then the* HD *state for Q is identical to the $\langle\cdot\rangle_{Q-\mu X,-}$ state.*

Proof. By monotonicity in μ and Λ, the HD state is continuous in μ from the left so equality for most μ's implies equality for all $\mu > 0$. But this equality follows from Theorem 5.2. ∎

Proof of Theorem 5.2 *for* $Y = P$. Let $\langle\cdot\rangle_\infty$ be any limit point for the periodic states (such limit points exist by the ϕ-bounds for periodic states). Then, it suffices to prove

$$\langle\cdot\rangle_{P,-} \leqq \langle\cdot\rangle_\infty \leqq \langle\cdot\rangle_{P,+} \tag{5.1}$$

in FKG senses. For (5.1), the equality of $\langle\cdot\rangle_{P,+}$ with $\langle\cdot\rangle_{P,-}$ and Theorem 1.1 imply that $\langle\cdot\rangle_\infty = \langle\cdot\rangle_{P,+}$. Since $\langle\cdot\rangle_\infty$ is an arbitrary limit point, the limit exists and equals $\langle\cdot\rangle_{P,+}$.

Now (5.1) follows in the same way that (4.2) did; namely by putting in enough cutoffs. Explicitly, let $d\nu_{P,\Lambda,\Lambda'}$ for $\Lambda\subset\Lambda'$ be the measure with periodic boundary conditions in Λ' and interaction P in Λ and $P-\mu_\infty X$ in $\Lambda'\backslash\Lambda$. Then clearly, by the usual FKG inequalities,

$$d\nu_{P,\Lambda',\Lambda'} \leqq d\nu_{P,\Lambda,\Lambda'}$$

from which $\langle\cdot\rangle_\infty \leqq \langle\cdot\rangle_{P,+}$ follows by noting that $\langle\cdot\rangle_\infty = \lim_{n\to\infty} d\nu_{P,\Lambda_n,\Lambda_n}$ for suitable Λ_n and that $\langle\cdot\rangle_{P,+} = \lim_{\Lambda\nearrow\mathbf{R}^2}(\lim_{\Lambda'\nearrow\mathbf{R}^2} d\nu_{P,\Lambda,\Lambda'})$. ∎

The strategy for proving Theorem 5.2 is similar to the special case just proven. Given some boundary condition Y_0, we will seek another boundary condition Y_+ so that:

(i) $d\nu_{Y_0,\Lambda',\Lambda'} \leqq d\nu_{Y_+,\Lambda,\Lambda'}$ (FKG sense)

where $d\nu_{Y_+,\Lambda,\Lambda'}$ is defined as above, with interaction P in Λ and $P-\mu_\infty X$ in $\Lambda'\setminus\Lambda$.

(ii) For fixed Λ, $\lim_{\Lambda'\nearrow\mathbf{R}^2}(d\nu_{Y_+,\Lambda,\Lambda'} - d\nu_{P,\Lambda,\Lambda'}) = 0$.

Now (ii) will assure us that $\langle\cdot\rangle_{P,+} = \lim_{\Lambda\nearrow\mathbf{R}^2}(\lim_{\Lambda'\nearrow\mathbf{R}^2} d\nu_{Y_+,\Lambda,\Lambda'})$ so that (i) implies that any limit point of $d\nu_{Y_0,\Lambda',\Lambda'}$ is less than $\langle\cdot\rangle_{P,+}$ in FKG sense. A similar Y_- construction will then lead to the existence of the limit $d\nu_{Y_0,\Lambda',\Lambda'}$ and its equality to $\langle\cdot\rangle_P$, as in the above case, $Y=P$.

To prove (i) it suffices in practice to prove the inequality when $P=0$. The proof of (ii) will depend on the use of a cluster expansion. Following Spencer [45] we will work with a translated field so that the freedom to choose Y_+ different from Y_0 will be useful to obtain a boundary condition which is simple in terms of the translated field.

The details in the proof of Theorem 5.2 appear in the appendix.

6. Ultraregularity: General consequences and the strong Gibb's principle

Fix a bare mass, m_0 and a polynomial, P.

Definition. A probability measure $d\nu$ on $\mathcal{S}'(\mathbf{R}^2)$ is called *ultraregular* (for (P, m_0)) if and only if:

(1) For every bounded open Λ in $\mathbf{R}^2$, $d\nu\restriction\Sigma_\Lambda$ is absolutely continuous with respect to $d\mu_0$ with Radon-Nikodym derivatives $f_\Lambda\in\bigcap_{p<\infty}L^p(\mathcal{S}', d\mu_0)$ obeying for any $p<\infty$

$$\|f_\Lambda\|_p \leqq C_p\exp\big(a_p d(\Lambda)^2\big)$$

where $d(\Lambda)$ = diameter of Λ.

(2) $f_\Lambda = e^{-U(\Lambda)}e^{-\alpha_\infty|\Lambda|}g_{\partial\Lambda}$ where $g_{\partial\Lambda}$ is measurable with respect to $\Sigma_{\partial\Lambda}$.

(3) $g_{\partial\Lambda}\in L^p(\mathcal{S}', d\mu_0)$ for all $p<\infty$ and $g_{\partial\Lambda}>0$ almost everywhere.

(4) Let $g_{l,t}$ denote $g_{\partial\Lambda}$ for Λ the rectangle of sides l and t centered at the origin with sides parallel to the coordinate axes. There exists $T(p)$ for

$p < 4/3$ so that for $l, t > T(p)$,

$$\|g_{l,t}\|_p \leqq 1 .$$

(5) There exists $T(p)$ for any $p < \infty$ and b_p so that for $l, t \geqq T(p)$,

$$\|g_{l,t}\|_p \leqq \exp[b_p(2l + 2t)] .$$

Remarks 1. Condition (1) is a slightly strengthened version of the temperedness condition of [23] and (2) is just the DLR equation of [23]. By results of [23], it is enough to prove (1) and (2) when Λ is a rectangle. As we shall see below, it actually suffices to prove (2) for rectangles and (4).

2. By (1) and properties of $d\mu_0$, $\phi(f)$ can be extended by continuity to distributions f of compact support in the Sobolev space, $\mathcal{H}_{-1} = \{f \,|\, (f, (-\Delta + 1)^{-1}f) < \infty\}$. Thus, the sigma algebra $\Sigma_{\partial\Lambda}$ of (2) can be defined either as an intersection of Σ_R, R open with $\partial\Lambda \subset R$ or the sigma algebra generated by $\{\phi(f) \,|\, \operatorname{supp} f \subset \partial\Lambda, f \in \mathcal{H}_{-1}\}$.

The point of the above definition is that we shall prove $\langle\ \rangle_{P,\pm}$ and various other states are ultraregular. In the bulk of this section we will derive properties of ultraregular states, of which the most important is the strong Gibbs equality:

$$s(f) - \rho(f, P) = \alpha_\infty(P) . \tag{6.1}$$

Let us first consider an example which illustrates the special role played by $p < 4/3$ and our normalization of $g_{\partial\Lambda}$.

Example. For $P(\phi)_1$ processes, f_Λ can be written down exactly (see e.g., [23], equation (VII. 18)). f_Λ obeys (2) with

$$g_{[a,b]} = \Omega\big(q(a)\big)\Omega\big(q(b)\big) \tag{6.2}$$

where Λ is an interval $[a, b]$, Ω is a "vacuum" for the corresponding quantum process, normalized to $\|\Omega\|_2 = 1$ and $q(\cdot)$ is the $d\mu_0$ "field." Since $\Omega \in \bigcap L^p(R, d\tilde{\mu}_0)$ (where $d\tilde{\mu}_0$ is a suitable Gaussian measure on R), it is easy to see that $\|g_{[a,b]}\|_p$ is bounded for any $p < \infty$ but it is only less than 1 if $p < 2$ and $|b - a|$ is sufficiently large (how large is p-dependent).

(6.2) shows the naturalness of our choice of normalization of g, and $p < 4/3$ replaces $p < 2$ in our proofs since $4/3$ is the dual index of $4 = 2^2$. It would be interesting to do the detailed computation in the linear or quadratic model to see whether one should expect condition (4) for $p < 2$ in the $P(\phi)_2$ case (as $l \to \infty$).

The following result is a specialization of an unpublished result of Guerra, Rosen and Simon:

THEOREM 6.1. *For ν to be ultraregular, it suffices to prove that* (2) *holds* *for rectangles and that* (4) *holds.*

Proof. The key observation is that for $\Lambda \subset \Lambda'$

$$g_{\partial\Lambda} = E_{\partial\Lambda}(e^{-U(\Lambda'\backslash\Lambda)}g_{\partial\Lambda'})e^{-\alpha_\infty|\Lambda'\backslash\Lambda|} \tag{6.3}$$

where $E_{\partial\Lambda}$ is the conditional expectation with respect to $d\mu_0$ onto the $\Sigma_{\partial\Lambda}$ measurable functions. By Hölder's inequality for such expectations:

$$||g_{\partial\Lambda}||_p \leqq e^{-\alpha_\infty|\Lambda'\backslash\Lambda|}||e^{-U(\Lambda'\backslash\Lambda)}||_q||E_{\partial\Lambda}(g_{\partial\Lambda'})||_r$$

with $p^{-1} = q^{-1} + r^{-1}$. Thus, for fixed $r > p$,

$$||g_{\partial\Lambda}||_p \leqq e^{+c|\Lambda'\backslash\Lambda|}||E_{\partial\Lambda}(g_{\partial\Lambda'})||_r \tag{6.4}$$

where we have used the estimate ([23], Lemma III.13))

$$||e^{-U(\Lambda)}||_q \leqq \exp\bigl(q^{-1}\alpha_\infty(q)|\Lambda|\bigr)$$

where $\alpha_\infty(q)$ is the pressure for qP. By Theorem III.3 of [23],

$$||E_{\partial\Lambda}(g_{\partial\Lambda'})||_r \leqq ||g_{\partial\Lambda'}||_s$$

so long as $(r-1)(s-1)^{-1} \leqq e(d)$, where $d = \text{dist}(\partial\Lambda, \partial\Lambda')$ and where $e(d)$ is a universal function which goes to infinity as $d \to \infty$. Thus, for any p, we put $r = 2p$ and $s = 5/4 \in (1, 4/3)$ and conclude, using the hypothesis (4), that for d sufficiently large, and $l', t' \geqq T(5/4)$, $||g_{\partial\Lambda}||_p \leqq \exp[c(|(l', t')\backslash\Lambda|)]$ so long as $d(\partial(l', t'), \partial\Lambda) > d$. Taking Λ to be an l by t rectangle and Λ' to be the rectangle of side $l + 2d$ by $t + 2d$, we find

$$||g_{\partial\Lambda}||_p \leqq \exp[cd(2l + 2t + 4d)] \leqq \exp[(2cd(2l + 2t))]$$

so long as $l, t \geqq d$. This proves (5).

The first part of condition (3) follows from (6.4) and the second from the following argument: $g_{\partial\Lambda} \geqq 0$, and if some function $f \geqq 0$ is $\Sigma_{\partial\Lambda}$ measurable and $\int fg_{\partial\Lambda}d\mu_0 = 0$, then $\int fe^{-U(\Lambda'\backslash\Lambda)}g_{\partial\Lambda'}d\mu_0 = 0$ so $fg_{\partial\Lambda'} = 0$ almost everywhere. Now $E_{\partial\Lambda}E_{\partial\Lambda'}$ is the second quantization of a strict contraction (by Lemma III. 4 of [23]) so long as $d(\partial\Lambda, \partial\Lambda')$ is sufficiently large and thus it is positivity improving (Theorem I.16 of [43]) so $E_{\partial\Lambda}(g_{\partial\Lambda'}) > 0$ almost everywhere. Thus $f = 0$. We conclude that $g_{\partial\Lambda} > 0$ almost everywhere. This proves the second part of (3). (1) follows from (5) as in [23]. ∎

While ultraregularity is stated for rectangles, it contains some information about circles.

THEOREM 6.2. *Let* $\Lambda(r)$ *be the circle of radius* r *centered about* 0. *Let* ν *be an ultraregular state. Then*

$$f_{\Lambda(r)} = e^{-\alpha_\infty|\Lambda(r)|}e^{-U(\Lambda(r))}g_{\partial\Lambda(r)}$$

with $\overline{\lim}_{r\to\infty} |\Lambda(r)|^{-1} \ln ||g_{\partial\Lambda(r)}||_p \leqq 0$ *for any* $p < \infty$.

Proof. For each r, let $\Lambda'(r)$ be the square of side $3r$. By using hypercontractivity, as in the last proof, we find that by taking r large enough, for any fixed $\varepsilon > 0$:

$$\| g_{\partial\Lambda(r)} \|_p \leqq \exp\left(|\Lambda'(r) \backslash \Lambda(r)| [(1+\varepsilon)^{-1}\alpha_\infty(1+\varepsilon) - \alpha_\infty] \right)$$

so that

$$\overline{\lim}_{r\to\infty} |\Lambda(r)|^{-1} \ln \| g_{\partial\Lambda(r)} \|_p \leqq [(1+\varepsilon)^{-1}\alpha_\infty(1+\varepsilon) - \alpha_\infty](9\pi^{-1} - 1) .$$

Since α_∞ is continuous, the $\overline{\lim}$ is $\leqq 0$. ■

THEOREM 6.3. *Let $d\nu$ be a translation invariant measure on $\mathcal{S}'(\mathbf{R}^2)$ so that $d\nu \upharpoonright \Sigma_\Lambda = e^{-\alpha_\infty|\Lambda|} e^{-U(\Lambda)} g_{\partial\Lambda} d\mu_0$. Suppose that $d\nu$ is a weakly tempered state and that there is a sequence of sets $\Lambda_n \to \infty$ in the Fisher sense so that $\overline{\lim}_{n\to\infty} |\Lambda_n|^{-1} \ln \| g_{\partial\Lambda_n} \|_p \leqq 0$ for all $p < \infty$. Then*

$$s(\nu) - \rho(\nu, P) = \alpha_\infty(P) . \tag{6.1}$$

In particular, (6.1) *holds if ν is ultraregular.*

Proof. By the general Gibbs inequality

$$s(\nu) - \rho(\nu, P) \leqq \alpha_\infty(P) .$$

On the other hand, by definition of $g_{\partial\Lambda}$ and s:

$$\begin{aligned} s(\nu) &= \lim_{n\to\infty} \int d\nu(\alpha_\infty + |\Lambda_n|^{-1}[U(\Lambda_n) - \ln g_{\partial\Lambda}]) \\ &= \alpha_\infty + \rho(\nu, P) - \lim \int d\nu [\ln g_{\partial\Lambda}] |\Lambda_n|^{-1} \end{aligned}$$

so we need only prove that

$$\lim \int d\nu |\Lambda_n|^{-1} \ln g_{\partial\Lambda} \leqq 0 . \tag{6.5}$$

But, by Jensen's and Hölder's inequality, for p arbitrary

$$\begin{aligned} |\Lambda_n|^{-1} \int d\nu \ln g_{\partial\Lambda} &\leqq |\Lambda_n|^{-1} \ln \int d\nu\, g_{\partial\Lambda} \\ &\leqq |\Lambda_n|^{-1} \ln \left[\left(\int d\mu_0 g_{\partial\Lambda}^{2p'} \right)^{1/p'} \left(\int d\mu_0 e^{-pU(\Lambda)} \right)^{1/p} e^{-\alpha_\infty|\Lambda|} \right] \\ &\leqq [p^{-1}\alpha_\infty(p) - \alpha_\infty(1)] \end{aligned}$$

as $\Lambda_n \to \infty$ using the hypothesis. Taking $p \to 1$ and using the continuity of α_∞, we see that (6.5) follows. ■

One final remark: A proof of ultraregularity provides alternative proofs of quasiinvariance of $d\nu$ under translation of the field,

$$\phi(f) \longrightarrow \phi(f) + \int f(x)g(x)dx \qquad (\phi(x) \longrightarrow \phi(x) + g(x),\ g \in C_0^\infty)$$

and of integration by parts in the $d\nu$ theory; for a restricted class of states, these are results of Fröhlich [4] and Glimm-Jaffe [10]. For, picking $\Lambda \supset \operatorname{supp} g$, the form of f_Λ immediately proves the quasi-invariance and gives the explicit formula of Fröhlich for the Radon-Nikodym derivative. This quasi-invariance allows one to define operators $\pi(g)$ on $L^2(\mathcal{S}', d\nu)$ and Fröhlich's formula yields an explicit formula for $\pi(g)1$ which implies the integration by parts formula. We emphasize that this connection between the three properties of DLR, quasi-invariance and integration by parts is essentially already in [4], [10].

7. Ultraregularity in $P(\phi)_2$ and regularity in Y_2

The key to proving ultraregularity in $P(\phi)_2$ is to combine chessboard and checkerboard estimates:

LEMMA 7.1. *Let Δ be an l-square. Let $q \geqq 4(1 - e^{-m_0 l})^{-2}$. Then for any function F of the fields in Δ:*

$$\gamma_0(F, \Delta) \leqq \|F\|_q .$$

Proof. The bound holds for each $\left(\int F^{(a,b)} d\mu_0\right)^{1/ab}$ by the checkerboard estimates (Theorem III.12 of [23]). ∎

THEOREM 7.2. *Measures obeying the hypotheses of Theorems* 2.3 *or* 2.4 *are ultraregular. In particular, the measures $d\nu_{P,\pm}$ of Section* 4, *the cluster expansion states, and the* (*Dirichlet and*) *half-Dirichlet states for $P = Q - \mu X$* (*Q even*) *are ultraregular and obey the strong Gibbs equality* (6.1).

Remarks 1. We emphasize that, by construction and Theorem 5.2, the cluster expansion states (and "most" Dirichlet or half-Dirichlet states) are $d\nu_{P,\pm}$ so there is some redundancy in the statement of this theorem.

2. By similar arguments with the estimates described at the end of Section 2, limit points of periodic states are ultraregular.

3. In the assertion of the Gibbs equality, the reader can see that a critical role is played by Theorem 3.1.

Proof. By Lemma 7.1 and chessboard estimates (Theorems 2.3, 2.4), if F is Δ-measurable, Δ an l-square and $q \geqq 4(1 - e^{-m_0 l})^{-2}$, then:

$$\begin{aligned} \left|\int F\, d\nu_P\right| &\leqq \gamma_0(Fe^{-U(\Delta)})e^{-|\Delta|\alpha_\infty} \\ &\leqq \|Fe^{-U(\Delta)}\|_q\, e^{-|\Delta|\alpha_\infty} . \end{aligned} \tag{7.1}$$

By duality theory for the L^p spaces, this show that

$$d\nu_P \restriction \Sigma_\Delta = e^{-U(\Delta)}e^{-|\Delta|\alpha_\infty} g_\Delta$$

with $\|g_\Delta\|_p \leqq 1$ where $p = (1 - q^{-1})^{-1}$. Since q can be taken arbitrarily close to 4 as $l \to \infty$ we have proved condition (4) in the definition of ultraregularity. All that remains, by Theorem 6.1, is to show that g_Δ is $\partial\Delta$ measurable.

For any $F \in \bigcap_{p<\infty} L^p(\Delta, d\mu_0)$, it is not hard to prove, using the method of Section 2, that for $Fe^{U(\Delta)} \in L^p(d\nu_P)$

$$\int Fe^{U(\Delta)} d\nu_P = \lim_{\Lambda \supset \Delta} \int Fe^{U(\Delta)} d\nu_{P,\Lambda}$$

for suitable "finite volume" $d\nu_{P,\Lambda}$. (For the states of Theorem 2.3 these are the usual finite volume states; for the states of Theorem 2.4, we use the DLR equations when proven for Theorem 2.3 to write $d\nu_{P,\Lambda}$ as the usual finite volume state times a factor $g_{\partial\Lambda}$.) If F is orthogonal in $L^2(\Delta, d\mu_0)$ to $L^2(\partial\Delta, d\mu_0)$, then $\int Fe^{U(\Delta)} d\nu_{P,\Lambda} = 0$ by the Markov property for $d\mu_0$, so for such F, $\int g_\Delta F d\mu_0 = 0$. Thus

$$g_\Delta \in L^2(\partial\Delta, d\mu_0)^{\perp\perp} = L^2(\partial\Delta, d\mu_0)$$

is $\partial\Delta$-measurable. ∎

We wish to end with a brief result expressing the applicability of our ideas to the Yukawa$_2$ theory:

THEOREM 7.3. *Let $d\nu$ be the infinite volume Y_2 Euclidean measure obtained from the theory of Magnen-Sénéor* [28] *and Cooper-Rosen* [1] *by restricting to purely Bose expectations. Then for Λ finite:*

$$d\nu \restriction \Sigma_\Lambda = f_\Lambda d\mu_0$$

where $f_\Lambda \in L^p(d\mu_0)$ for all $p < 4/3$.

Proof. One proves an estimate of the form (7.1) by abstracting the argument of Seiler-Simon [39] or our method in Section 2 and using checkerboard estimates. ∎

Appendix

Proof of Theorem 5.2. In this section, we complete the proof of Theorem 5.2 following the strategy described in Section 5. In the first part of this appendix, we always *match Wick ordering* to the Gaussian measure appearing with the interaction—this measure may be one with classical boundary conditions or an interpolation arising in the details of the cluster expansion. In this way we will prove Theorem 5.2 for the cases $Y = D, N, F$. In a brief finale to the appendix we describe how to handle a variety of perturbations of these basic boundary conditions including the half-D (N, P) states.

The reason for the length of this appendix is that we rely extensively on the cluster expansion which requires separate estimates on Green's functions and differences of Green's functions in each special case. To keep the size manageable we will suppose the reader to be familiar with the Erice lectures of Glimm, Jaffe and Spencer [13] and we will use freely various technical devices taken from [1]. We describe the proof of strategy steps (i) and (ii) for the case of $+$ boundary conditions; the $-$ case is identical except for obvious notational changes.

Given a polynomial P, we pick $\mu_\infty > 0$ so large that the cluster expansion of [45] converges. For such a μ_∞, we pick, following Spencer [45], ϕ_c a real number so that the term linear in X in the polynomial

$$Q_1(X) \equiv P(X - \phi_c) - \mu_\infty(X - \phi_c)$$

vanishes. If $P(X) = a_{2m}X^{2m} + \cdots$, then clearly for μ_∞ large, $\phi_c \approx -(\mu_\infty/2ma_{2m})^{1/2m-1}$. Let $\tilde{a}_2$ be the term quadratic in X in $Q_1(X)$ so that for $\mu_\infty \gg 1$,

$$\tilde{a}_2 \approx a_{2m}\phi_c{}^{2m-2}\alpha' \approx a_{2m}^{1/2m-1}\alpha\mu_\infty^{2m-2/2m-1}$$

where $\alpha' = m(2m-1)$ and $\alpha = \alpha'(2m)^{-2m+2/2m-1}$. We define a polynomial $Q(X)$ by

$$Q(X) = P(X - \phi_c) - \mu_\infty(X - \phi_c) - \tilde{a}_2X^2 - m_0^2\phi_c X\,. \tag{A1}$$

The point of this change of variables is the following: Let $d\nu_{Y_+,\Lambda,\Lambda'}$ be the measure described in Section 5. Then in terms of a variable $\tilde{\phi} = \phi + \phi_c$:

$$d\nu_{Y_+,\Lambda,\Lambda'} = Z^{-1}\exp\big(-U(\Lambda'; Q(\tilde{\phi})\big)\exp\big(-\mu_\infty\phi(\chi_\Lambda)\big)d\mu_0{}^{(Y_+,a_2)}(\tilde{\phi}) \tag{A2}$$

where

$$U(\Lambda; S(\tilde{\phi})) = \int_\Lambda : S(\tilde{\phi}(x)) \colon d^2x$$

($::$ with respect to $d\mu_0^{Y_+}(\phi)\big)$ and where $d\mu_0^{(Y_+,a_2)}$ is the Gaussian process on $\mathcal{S}'(\Lambda')$ with *mean* 0 and covariance $(-\Delta_\Lambda^{\tilde{Y}} + m_0^2 + 2\tilde{a}_2\chi_{\Lambda'})^{-1}$. Here $\tilde{Y}$ is obtained from Y_+ by replacing ϕ by $\tilde{\phi} - \phi_c$ (we will be more explicit below—we note for now, that if $Y = P$ or N, we will take $Y = P$ (resp. N) in which case $\tilde{Y}$ is P (resp. N). That is these boundary conditions are stable under a shift of the field, see [24]). Henceforth for notational convenience, we will suppose $\Lambda = \emptyset$ and then will use Λ systematically in place of the above Λ'.

Next for $0 \leqq \tau \leqq 1$ define a covariance C_Λ^τ by

$$C_\Lambda^\tau(x, y) = \tau C_\Lambda^{(Y_+,\tilde{a}_2)}(x, y) + (1 - \tau)C_\Lambda^P(x, y) \tag{A3}$$

and $\langle\ \rangle_{\tau,\Lambda}$ by (A2) with $d\mu_0^{(Y_+,\tilde{a}_2)}$ replaced by the Gaussian measure with mean zero and covariance C_Λ^τ(and changing the meaning of Wick ordering in $U(\Lambda; Q)\big)$. In (A3), C_Λ^P is the periodic Green's function with mass $(m_0^2+2\tilde{a}_2)^{1/2}$.

Now, for an arbitrary expectation $\langle\cdot\rangle$, let

$$\langle A; B\rangle \equiv \langle AB\rangle - \langle A\rangle\langle B\rangle$$

and set

$$k_\Lambda(x, y) = (C_\Lambda^{(Y+,a_2)} - C_\Lambda^P)(x, y)$$

and

$$U'(y) = \frac{\delta}{\delta\tilde{\phi}(y)} U(\Lambda; Q(\tilde{\phi}))$$

(which is τ dependent because of the Wick ordering!). To establish (ii) of the basic strategy, it suffices to prove that

$$\text{(A4)} \qquad \langle R\rangle_{\tau=1,\Lambda} - \langle R\rangle_{\tau=0,\Lambda} \longrightarrow 0$$

for R a product of Wick polynomials localized in some bounded open region $X_0 \subset \mathbf{R}^2$. By the fundamental theorem of calculus and integration by parts on function space [13]:

$$\begin{aligned}\langle R\rangle_{\tau=1,\Lambda} - \langle R\rangle_{\tau=0,\Lambda} &= \int_0^1 d\tau \, \frac{d}{d\tau}\langle R\rangle_{\tau,\Lambda} \\ &= \int_0^1 d\tau \int_{\Lambda\times\Lambda} d^2x d^2y \, k_\Lambda(x, y)\{\textstyle\sum_{j=1}^3 G_j(x, y)\}\end{aligned}$$

with

$$\text{(A5a)} \qquad G_1(x, y) = \left\langle \frac{\delta^2}{\delta\tilde{\phi}(x)\delta\tilde{\phi}(y)} R\right\rangle_{\tau,\Lambda},$$

$$\text{(A5b)} \qquad G_2(x, y) = -\left\langle\left(\frac{\delta}{\delta\tilde{\phi}(x)} R\right) U'(y)\right\rangle_{\tau,\Lambda},$$

$$\text{(A5c)} \qquad G_3(x, y) = \langle R;\ U'(x) U'(y)\rangle_{\tau,\Lambda}.$$

Note that a term $-\langle R; \delta^2 U/\delta\tilde{\phi}(x)\delta\tilde{\phi}(y)\rangle$ does *not* occur because we have matched Wick ordering to $d\mu_0$(see, e.g., [1]). We are now able to abstract the basic estimates needed to establish (A4):

THEOREM A1. *To establish* (A4), *it suffices to prove that uniformly in* $\tau \in [0, 1]$ *and* Λ *sufficiently large*:

(1) $\|k_\Lambda(x, y)\|_p \leqq C_1|\Lambda|$ and $|k_\Lambda(x, y)| \leqq C_2 \exp[-C_3\{\mathrm{dist}(x, \partial\Lambda) + \mathrm{dist}(y, \partial\Lambda) + \mathrm{dist}(x, y)\}]$

for $\mathrm{dist}(x, \partial\Lambda)$, $\mathrm{dist}(y, \partial\Lambda) \geqq 1$, *and* $p < \infty$.

(2) $G_1(x, y) \in L^1(X_0 \times X_0)$,

(3) $G_2(x, y) \in L^1(X_0 \times \Lambda)$,

(4) $G_3(x, y)$ *is locally integrable and* $\|G_3\chi_{B\times B}(x, y)\|_2 \leqq C_4 \exp[-C_5\, \mathrm{dist}(X_0, B)]$ *for any* $B \subset \Lambda$ *with* $\mathrm{dist}(X_0, B) \geqq 1$.

Proof. We show that for $j = 1, 2, 3$, $\int d^2x\, d^2y\, k_\Lambda\, G_j \to 0$ uniformly in τ as $\Lambda \nearrow \mathbf{R}^2$ (so that, in particular, $\mathrm{dist}(\partial\Lambda, X_0) \to \infty$). For $j = 1$ this follows directly from (1), (2) and for $j = 2, 3$ it follows from (1), (3), (4) by breaking the region Λ into squares and using exponential falloff to get convergence of the sum. ■

Dirichlet boundary conditions. Here, we will choose Y_+ so that $\tilde{Y}$ is also D. Thus Y_+ can be described in the language of [24] as the field obtained by setting the boundary field $\phi_{\partial\Lambda}$ to $-\phi_c$ (so that $\tilde{\phi}_{\partial\Lambda} = \phi_{\partial\Lambda} + \phi_c$ is zero). To check (i) of the basic strategy one can follow one of two paths: (a) pass to a lattice approximation and note that since $\phi_c < 0$, to arrange for a positive expectation for $\phi_{\partial\Lambda}$ we must include a term of the form $\exp(\alpha\phi)$ for $\alpha > 0$ in Y_+ which leads to (i); (b) realize D boundary conditions via the method of [4] by adding a mass term external to Λ and taking this term to infinity. In one case, for D, we have $Z^{-1}\exp(-(1/2)M{:}\,\phi^2{:})$ and for Y_+ we have $Z^{-1}\exp(-(1/2)M{:}\,(\phi+\phi_c)^2{:})$. In the latter case, the additional term $\exp(-M\phi_c\phi)$ leads to an FKG inequality in the right direction since $\phi_c < 0$. We leave the details to the reader and concentrate on checking the hypotheses of Theorem A1 with $\tilde{Y} = D$. We note that by taking μ_∞ sufficiently large (so that $-\phi_c$ is large), we can carry through (i) for D boundary conditions with an arbitrarily large (but fixed) external field on the boundary.

Notice that since D boundary conditions decouple Λ and $\mathbf{R}^2\backslash\Lambda$ we can, without loss assume that the covariance is $(-\Delta_\Lambda^D + m_0^2 + 2\tilde{a}_2)^{-1}$ rather than $(-\Delta_\Lambda^D + m_0^2 + 2\tilde{a}_2\chi_\Lambda)^{-1}$.

Estimate (1) follows easily by the Cooper-Rosen method [1] of writing

$$k_\Lambda(x, y) = \int_{\partial\Lambda} P_{\partial\Lambda}(x, z) k_\Lambda(z, y) dz$$

where $P_{\partial\Lambda}$ is the Poisson kernel for $-\Delta + m_0^2 + 2\tilde{a}_2$ in Λ or alternatively, since Λ is a rectangle, by the method of images, see e.g., [24].

Estimates (2), (3), (4) follow in a standard manner once one establishes the convergence of the cluster expansion (see e.g., §4 [13]). This convergence also allows us to take Λ other than a rectangle in $\langle\cdot\rangle_{\tilde{Y},\Lambda} \to \langle\ \rangle_{P,\infty}$. The combinatorics and estimates in the expansion are identical to those in [45] with the exception of *two technical* estimates on covariances, namely Corollary 9.6 $\left(\text{which bounds } \int R\, d\phi_{C(s)}\right)$ and Proposition 8.1 (which bounds L^p norms of $\partial^\gamma C$) of [13]. Corollary 9.6 is easy once one notes that since D-covariances are involved and since $C_\Lambda^{\tilde{Y}}(s)$, the interpolated covariance of the cluster expansion, is less than C_Λ^P in the sense of quadratic forms (see e.g., [24]), the estimate need only be proved for C_Λ^P ("conditioning"). This can be found in [45] and is quite easy. Proposition 8.1 follows from a simple modification of the iterated Poisson method of Cooper-Rosen [1].

Neumann boundary conditions. It is easily seen by passing to the lattice approximation [24], that N boundary conditions are left invariant by translations of the field. Thus we can take $Y_+ = N$ so that (i) is trivial (i.e., identical to the periodic case) and still have $\tilde{Y} = N$.

Estimate (1) of Theorem A1 follows by the method of images [24]. As in the Dirichlet case, to prove (2), (3), (4) we must prove convergence of the cluster expansion. Corollary 9.6 follows as above if we note that $C_\Lambda^\tau(s) \leqq C_\Lambda^N$ in operator sense and then bound C_Λ^N by a method of images argument.

It is in proving Proposition 8.1 that the Cooper-Rosen method is especially useful. For, on account of the Neumann boundary conditions, stopping time arguments as in [45] do not have an immediate applicability although a modification probably exists. The Cooper-Rosen method of repeated Poisson iteration yields Proposition 8.1. The estimates are slightly worse than in the free or periodic case since C_Λ^N is more singular than C_0 as $x, y \to \partial\Lambda$ but, by the method of images, $|C_\Lambda^N - C_\Lambda^P| \leqq d_\Lambda C_0$ pointwise where d_Λ is bounded as $\Lambda \nearrow \mathbf{R}^2$ so estimates are only worse than those in [1] by an overall constant.

Free boundary conditions. These are technically the most subtle ones since they don't decouple and they aren't left invariant by translation of the field. We pick Y_+ so that $\tilde{Y}$ has mean zero and covariance

$$C_\Lambda^F \equiv (-\Delta + m_0^2 + 2a_2\chi_\Lambda)^{-1} .$$

Thus Y_+ corresponds to the free boundary conditions covariance but with $\phi \approx -\phi_c$ in $\mathbf{R}^2\backslash\Lambda$. As in the case of D boundary conditions, $\phi_c < 0$ implies that step (i) goes through.

For step (ii), we need only check estimate (1), and Corollary 9.6, Proposition 8.1 in the cluster expansion for $C_\Lambda^\tau = \tau C_\Lambda^F + (1 - \tau)C_\Lambda^P$.

To prove estimate (1), let $\tilde{C}_0$ be the free covariance with mass m_0. Then by a simple Feynman-Kac formula: $C_\Lambda^D \leqq C_\Lambda^F \leqq \tilde{C}_0$. Since C_Λ^P can be written down in terms of images, estimate (1) follows. We remark that we have been crude in this estimate obtaining only $|C_\Lambda^F| \leqq 0\,[\exp(-m_0|x - y|)]$ for $|x - y|$ large rather than

$$|C_\Lambda^\tau| \leqq 0\,[\exp(-(m_0^2 + 2\tilde{a}_2)^{1/2}|x - y|)]$$

for $x - y$ large. This improved estimate can be obtained from the Cooper-Rosen machine but is not needed at this point.

Corollary 9.6 follows if we note that $C_\Lambda^\tau(s) \leqq C_\Lambda^N$ in the sense of operators. Proposition 8.1 follows from the Cooper-Rosen method as in the NBC case if we note that $|C_\Lambda^F - C_\Lambda^P| \leqq d_\Lambda \tilde{C}_0$.

There are some changes in the combinatorics of [13] needed on account of the slower decay of C_Λ^F near $\partial\Lambda$.

Half (*and multiplicative*) *boundary conditions.* Finally we want to consider extending Theorem 5.2 to a class of boundary conditions which will include the various half-D (P, N) states and a variety of other choices of change in Wick ordering. It is clear that this idea can be extended to prove that for the polynomials with $\alpha_\infty(P - \mu X)$ differentiable at $\mu = 0$, one has $\langle\cdot\rangle_{P+\lambda Q,\pm} \to \langle\cdot\rangle_{P,\pm}$ as $\lambda \downarrow 0$ for any Q with $P + \lambda Q$ semibounded for all small $\lambda > 0$.

We recall from [31], [32]:

Definition. A family $\{G(\Lambda)\colon \Lambda \subset \mathbf{R}^2, \text{Borel}\}$ with values in functions on $\mathcal{S}'$ is called an *additive functional* if and only if for arbitrary Λ_1, Λ_2 with $\Lambda_1 \cap \Lambda_2 = \varnothing$:

$$G(\Lambda_1 \cup \Lambda_2) = G(\Lambda_1) + G(\Lambda_2), \text{ and}$$

for each Λ, $G(\Lambda)$ is measurable with respect to the σ-algebra generated by $\{\phi(f)\,|\,\operatorname{supp} f \subset \Omega\}$ for any open $\Omega \supset \Lambda$.

Let $d\nu_{Y,\Lambda}$ be a classical boundary conditions state for P.

Definition. Suppose that for each Λ we are given an additive functional G_Λ so that $e^{-F_\Lambda} \in L^1(d\nu_\Lambda)$ where $F_\Lambda = G_\Lambda(\mathbf{R}^2)$. Then the measures

$$Z^{-1}e^{-F_\Lambda}d\nu_{Y,\Lambda} \equiv d\tilde{\nu}_{Y,\Lambda}$$

are called a *multiplicative* (*boundary conditions*) *perturbation* of $d\nu_Y$.

To show that $d\tilde{\nu}_{Y,\Lambda}$ converges to $\langle\cdot\rangle_{P,\pm}$ as $\Lambda \nearrow \mathbf{R}^2$, it clearly suffices that the measures $d\tilde{\nu}_{Y,\Lambda}$ obey FKG inequalities and that in the large μ_∞ region we prove

$$\lim_{\Lambda\to\infty}\int R[d\nu_{Y,\Lambda} - d\tilde{\nu}_{Y,\Lambda}] = 0 \tag{A6}$$

for local polynomials R. To prove (A6) in this large μ_∞ region, we define

$$\langle R\rangle_{\tau,\Lambda} \equiv \langle e^{-\tau F_\Lambda}\rangle_{Y,\Lambda}^{-1}\langle Re^{-\tau F_\Lambda}\rangle_{Y,\Lambda}$$

Then

$$\begin{aligned}\langle R\rangle_{1,\Lambda} - \langle R\rangle_{0,\Lambda} &= \int_0^1 d\tau\,\frac{d}{d\tau}\langle R\rangle_{\tau,\Lambda}\\ &= -\int_0^1 d\tau\langle R;\,F_\Lambda\rangle_{\tau,\Lambda}\end{aligned}$$

so that (A6) follows if we can prove

$$\langle R;\,F_\Lambda\rangle_{\tau,\Lambda} \longrightarrow 0 \tag{A7}$$

as $\Lambda \nearrow \mathbf{R}^2$, uniformly in $\tau \in [0, 1]$.

The following result is so obvious that it does not require a formal proof.

THEOREM A2. *Let $d\tilde{\nu}_{Y,\Lambda}$ be a multiplicative perturbation of $d\nu_{Y,\Lambda}$ so that for any unit square, Δ,*

$$\text{(A8)} \qquad |\langle R; G_\Lambda(\Delta)\rangle_{\tau,\Lambda}| \leqq C_\Lambda \operatorname{dist}(\Delta, X_0)^{-(2+\varepsilon)}\ , \quad C_\Lambda \downarrow 0 \text{ as } \Lambda \nearrow \mathbf{R}^2$$

where X_0 is a fixed square containing supp R. *Then* (A7) *and thus* (A6) *hold. In particular, if* (i) *$d\tilde{\nu}_{Y,\Lambda}$ obey* FKG *inequalities,* (ii) (A8) *holds for the $P(X) \mp \mu_\infty X$ states,* (iii) *$d\nu_{Y,\Lambda} - d\nu_{P,\Lambda} \to 0$ as $\Lambda \nearrow \mathbf{R}^2$ for $P(X) \mp \mu_\infty X$, then for polynomials P with $\alpha_\infty(P - \mu X)$ differentiable at $\mu = 0$, $d\tilde{\nu}_{Y,\Lambda} \to \langle \cdot \rangle_{P,+} = \langle\ \rangle_{P,-}$ where $d\tilde{\nu}_{Y,\Lambda}$ is the state with (Y, e^{-F_Λ}) boundary conditions and polynomial P.*

We conclude with a general result that, in particular, implies Theorem 5.2 for the HP, HD, HN states:

PROPOSITION A3. *Let $\{V_i\}_{i=1}^k$ be polynomials of degree less than $2m = \deg P$. Let $\{g_{\Lambda,i}(x)\}$ be a family of functions satisfying* supp $g_{\Lambda,i} \subset \Lambda$ *and*:

$$\begin{array}{ll}\text{(A9a)} & \\ \text{(A9b)} & \end{array} \qquad |g_{\Lambda,i}(x)| \leqq \begin{cases} C_1[\operatorname{dist}(x, \partial\Lambda)]^{-\delta_1}; & d(x, \partial\Lambda) < 1 \\ C_2[\operatorname{dist}(x, \partial\Lambda)]^{-\delta_2}; & d(x, \partial\Lambda) \geqq 1 \end{cases}$$

where $\delta_1 < 1/2m$, $\delta_2 > 2$. Let

$$F_\Lambda \equiv \sum_{i=1}^k \int :V_i(\phi(x)):\, g_{\Lambda,i}(x) d^2x\ .$$

Then $\{e^{-F_\Lambda}\}$ satisfies the hypotheses of Theorem A2 *for $Y =$* F, P, N, D.

Proof. GRS [24] (Section VIII) prove a linear lower bound under the hypotheses of this proposition for $\int e^{-U(\Lambda)} e^{-F_\Lambda} d\mu_{0,\Lambda}^Y$ which replaces Theorem 9.5 of [13]. The remaining estimates needed to prove the convergence of the cluster expansion for $\langle \cdot \rangle_{\tau,\Lambda}$ have already been proved above. The cluster expansion leads to exponential falloff of $\langle A; B\rangle_{\tau,\Lambda}$ and so to (A8). ■

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(Received May 19, 1976)