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The Scattering of Classical Waves from Inhomogeneous Media

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Abstract. By extending Kato's theory of two Hilbert space scattering, we are able to formulate both optical and accoustical scattering from inhomogeneous media as strictly elliptic problems. We use this formulation to present simple proofs of the existence and completeness of scattering states.

§ 1. Introduction

In this note, we present basic existence and completeness theorems for the scattering of acoustical and optical waves by inhomogeneous media. With the exception of some technical results in the appendices, which may be new, we make no claim that our basic results are new – the basic results have already been obtained by Schulenberger and Wilcox [24–26]. However, we feel that our presentation has some virtues of conceptual and technical simplicity. The usual treatments use some combination of the following technical methods and theorems: the two Hilbert space scattering formalism (Wilcox [30], Kato [15]), the theory of uniformly propagative systems (Wilcox [30]), coerciveness estimates (Schulenberger-Wilcox [25]), local compactness (Birman [4]), and Birman's subtle extension of the usual trace class criteria of Birman, de Branges and Kato (Birman [4], Belopol'skii-Birman [2]). Of these ideas, we only use the two Hilbert space theory, which, while not really intrinsic to the problem at hand (see the remarks at the end of § 2), is fairly natural.

Our main points are three in number. The first is that one can improve the existing techniques for proving that differences of powers of resolvents are trace class – these technical improvements are presented in two appendices, one of which extends Stinespring's famous trace class criterion [29]. Secondly, we extend Kato's two Hilbert space theory [15] to include the case under discussion. When Kato specializes his theory to hyperbolic equations, he supposes that both equations are on the same Hilbert space – the two Hilbert spaces come when one rewrites the second order equations as first order equations. We allow two different base spaces. This simple extension of Kato's work is presented in § 2. Our final point is connected with the second. Since one must rewrite the acoustic equations as first order in time, it is natural, at first sight, to make them first order in space also. Moreover, Maxwell's equations are first order although they can, of course, be rewritten as a second order wave equation. Our point is that it is better to keep all equations as second order in the space variables. In the acoustical case, which we discuss in § 3, this keeps everything strictly elliptic whereas rewriting them as first order equations introduces a spurious static mode (i.e., the "dynamic equations" include a constraint equation). In the optical case, Maxwell's equations include constraints – in our treatment of completeness, which appears in § 4, we are able to deal nevertheless with second order strictly elliptic equations. For the static modes completely decouple from the dynamic modes. Thus, we can give the static modes a dynamics without effecting the dynamic modes!

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§ 2. Scattering for Abstract Wave Equations

In this section, we describe some abstract results comparing two wave equations

$$\ddot{u}_i(t) = -A_i u_i(t) \tag{1}$$
$$u_i(0) = u_i^{(0)}, \quad \dot{u}_i(0) = v_i^{(0)}$$

where A_1 and A_2 are non-negative self-adjoint operators on *different* Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$. Kato, in [15], considered scattering theory in the case $\mathscr{H}_1 = \mathscr{H}_2$. We are interested in more general situations: in applications, $\mathscr{H}_1$ and $\mathscr{H}_2$ are setwise equal but have distinct, equivalent inner products. Since the generalization of Kato's results are straight-forward, we only sketch the proofs. We follow Kato's notation and refer the reader to [15] for details.

In terms of the non-negative square root, B_i, of A_i, one can write down the solution of (1) as:

$$u_i(t) = \cos(B_i t)\, u_i^{(0)} + B_i^{-1} \sin(B_i t)\, v_i^{(0)} \tag{2}$$

where $B_i^{-1} \sin(B_i t)$ is defined to be tI on $N(B_i)$, the null space of B_i. If $u_i^{(0)} \in D(B_i^2)$, $v_i^{(0)} \in D(B_i)$, then $u_i(t)$ is twice continuously differentiable as a $\mathscr{H}_i$-valued function and satisfies (1). In order to treat the scattering theory for (1), it is convenient to reformulate it as a first order equation in t. Let $v_i = \dot{u}_i$ and $\phi_i = \langle u_i, v_i \rangle$, where we write column vectors as rows to facilitate notation. Then (1) becomes

$$\frac{d}{dt}\phi_i = -iH_i \phi_i, \quad \phi_i(0) = \langle u_i^{(0)}, v_i^{(0)} \rangle \tag{3}$$

where

$$H_i = i \begin{pmatrix} 0 & I \\ -B_i^2 & 0 \end{pmatrix}.$$

We define $[D(B_i)]$ to be the closure of the domain of B_i, $D(B_i)$, in the pseudo-norm $\|B_i u\|_{\mathscr{K}_i}$, where we identify elements whose difference lies in $N(B_i)$ and set

$$\mathscr{H}_i = [D(B_i)] \times \mathscr{K}_i.$$

Then $\mathscr{H}_i$ is a Hilbert space and H_i is self-adjoint on $[D(B_i^2)] \times D(B_i)$. If $U_i(t) \equiv e^{-itH_i}$, then the first component of $\phi_i(t) \equiv U_i(t)\,\phi(0)$ is just (2). The details of this construction are standard (see e.g. [15] or [22]), so we omit them.

Let $[R(B_i)]$ denote the closure of the range of B_i in the $\mathscr{K}_i$-norm and let F_i be the orthogonal projection from $\mathscr{K}_i$ to $[R(B_i)]$. Let $\tilde{B}_i$ be the unitary operator from $[D(B_i)]$ to $[R(B_i)]$ obtained by continuously extending B_i from $D(B_i)/N(B_i)$ to $[D(B_i)]$.

Given any unitary V from $\mathscr{K}_1$ onto $\mathscr{K}_2$, define an operator J: $\mathscr{H}_1 \to \mathscr{H}_2$ by

$$J: \langle u, v\rangle \to \langle \tilde{B}_2^{-1} F_2 V \tilde{B}_1 u, Vv\rangle$$

J is a contraction and J^*: $\mathscr{H}_2 \to \mathscr{H}_1$ is given by:

$$J^*: \langle w, x\rangle \to \langle \tilde{B}_1^{-1} F_1 V^{-1} \tilde{B}_2 w, V^{-1} x\rangle$$

by an elementary computation. If the limits exist, we define wave operators from $\mathscr{H}_1$ to $\mathscr{H}_2$ by:

$$\Omega^{\pm}(H_2, H_1, J) \equiv \operatorname*{s-lim}_{t \to \mp\infty} U_2(-t)\, J U_1(t) P_{ac}(H_1)$$

where $P_{ac}(H)$ is the projection onto the absolutely continuous part of the spectrum of H. As usual, we say that $\Omega^{\pm}$ are complete if $\operatorname{Ran}\Omega^{\pm} = \operatorname{Ran}P_{ac}(H_2)$. The operators $\Omega^{\pm}(H_2, H_1, J)$ are somewhat artificial and are chosen to make the proof of the next theorem easy. We will later discuss more physical wave operators. If it exists, we define $\Omega^{\pm}(V^{-1}B_2 V, B_1)$ by:

$$\Omega^{\pm}(V^{-1}B_2 V, B_1) = \operatorname*{s-lim}_{t \to \mp\infty} e^{iV^{-1}B_2 Vt} e^{-iB_1 t} P_{ac}(B_1).$$

Notice that this operator maps $\mathscr{K}_1$ into *itself.*

Theorem 2.1. *Suppose that the wave operators $\Omega^{\pm}(V^{-1}B_2 V, B_1)$ exist (resp. exist and are complete). Then the wave operators $\Omega^{\pm}(H_2, H_1, J)$ exist (resp. exist and are complete), and are partial isometries.*

Proof. The proof follows closely Kato's proof for the case $\mathscr{K}_1 = \mathscr{K}_2$ and V = identity, so we merely provide a sketch. The proof relies on the factorization

$$\frac{d^2}{dt^2} + B^2 = \left(\frac{d}{dt} - iB\right)\left(\frac{d}{dt} + iB\right)$$

so that if u obeys $\ddot{u} = -B^2 u$, then $f_{\pm} = \dot{u} \pm iBu$ obey $\frac{d}{dt} f_{\pm} = \pm iBf_{\pm}$.

To accomodate the null space $N(B_i)$ we decompose $\mathscr{H}_i$ as

$$\mathscr{H}_i = [D(B_i)] \times [R(B_i)] \oplus \{0\} \times N(B_i).$$

(We caution the reader that $\times$ denotes Cartesian product and not tensor product. The above formula looks more reasonable if one uses the isomorphism $S_1 \times S_2 \simeq S_1 \oplus S_2$ for vector spaces.) We make precise the $f_\pm$ decomposition above by defining

$$T_i\colon \mathscr{H}_i \to [R(B_i)] \times [R(B_i)] \oplus \{0\} \times N(B_i)$$

by

$$T_i = \frac{1}{\sqrt{2}} \begin{pmatrix} B_i & i \\ B_i & -i \end{pmatrix} \oplus I$$

(where the i factor in the matrix is $\sqrt{-1}$ and not $i=1$ or 2!). Then by the calculation above:

$$T_i U_i(t) T_i^{-1} = \begin{pmatrix} e^{-iB_i t} & 0 \\ 0 & e^{iB_i t} \end{pmatrix} \oplus I. \tag{4}$$

We propose to use the formula:

$$\begin{aligned} &U_2(-t) J U_1(t) P_{ac}(H_1) \phi \\ &\quad = T_2^{-1}(T_2 U_2(-t) T_2^{-1})(T_2 J T_1^{-1})(T_1 U_1(t) T_1^{-1}) T_1 P_{ac}(H_1) \phi. \end{aligned} \tag{5}$$

Now by Lemma 8.1 of [15], $T_1 P_{ac}(H_1) \phi = \langle u, v \rangle \oplus \langle \{0\}, 0 \rangle$ where $u, v \in P_{ac}(B_1)$. Thus, by (4),

$$T_1 U_1(t) T_1^{-1} T_1 P_{ac}(H_1) \phi = \langle e^{-iB_1 t} u, e^{iB_1 t} v \rangle \oplus \langle \{0\}, 0 \rangle.$$

On vectors of the form $s = \langle w, z \rangle \oplus \langle \{0\}, 0 \rangle$, one easily checks that

$$T_2 J T_1^{-1} s = \begin{pmatrix} F_2 V & 0 \\ 0 & F_2 V \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} + \frac{(I-F_2) V}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ -i & i \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix}$$

so that, using (4) again, T_2 times the right hand side of (5) consists of components of the form

$$e^{-itB_2} F_2 V e^{itB_1} \theta$$

or

$$e^{-itB_2} (I - F_2) V e^{itB_1} \theta$$

where θ is u, v or a linear combination of them. If $\Omega^\pm(V^{-1} B_2 V, B_1)$ exist, then all these terms have limits in $\mathscr{H}_2$. Since T_2^{-1} is an isometry from $\mathscr{H}_2 \oplus \mathscr{H}_2$ onto $\{\phi \in \mathscr{H}_2 \mid \phi = \langle u, v \rangle,\ v \perp N(B_2)\}$, $\Omega^\pm(H_2, H_1, J)$ exists.

We next claim that if $\Omega^\pm(V^{-1} B_2 V, B_1)$ exist, then J^* is a $(U_1, +)$ asymptotic left inverse to J, i.e. for a dense set of ϕ:

$$\|(1 - J^* J) U_1(t) P_{ac}(H_1) \phi\|_{\mathscr{H}_1} \to 0. \tag{6}$$

Since J^* is a contraction, (6) implies that $\|J U_1(t) P_{ac}(H_1) \phi\|_{\mathscr{H}_2} \to \|P_{ac}(H_1) \phi\|_{\mathscr{H}_1}$ so that $\Omega^\pm(H_2, H_1, J)$ are partial isometries with initial space $P_{ac}(H_1)$. Moreover, by

Theorem 6.3 of [15], (6) implies that $\Omega^{\pm}(H_2, H_1, J)$ are complete if and only if $\Omega^{\pm}(H_1, H_2, J^*)$ exist. Under these circumstances one can conclude completeness knowing the completeness of $\Omega^{\pm}(VB_2V^{-1}, B_1)$ for that completeness implies the existence of $\Omega^{\pm}(B_1, VB_2V^{-1})$, and so of $\Omega^{\pm}(V^{-1}B_1V, B_2)$, and so of $\Omega^{\pm}(H_1, H_2, J^*)$ by the above argument.

We have thus reduced the proof of the rest of the theorem to the proof of (6). Letting ϕ_1 denote the first component of ϕ in the $\mathscr{H}_1$ representation, one computes that the left side of (6) is equal to

$$\|(I-F_1V^{-1}F_2V)\tilde{B}_1(U_1(t)P_{\mathrm{ac}}(H_1)\phi)_1\|_{\mathscr{K}_1}$$

so that it suffices to prove that

$$\|(I-F_1(V^{-1}F_2V))e^{-itB_1}\theta\|_{\mathscr{K}_1}\to 0 \tag{7}$$

for a dense set of θ in $P_{\mathrm{ac}}(B_1)$. Since $F_2'=V^{-1}F_2V$ is just the projection onto the closure of $\mathrm{Ran}(V^{-1}B_2V)$ and since $\Omega^{\pm}(V^{-1}B_2V, B_1)$ are assumed to exist, (7) follows by an argument on page 359 of Kato's paper [15]. □

In practice the existence or existence and completeness of $\Omega^{\pm}(V^{-1}B_2V, B_1)$ is established by proving existence or existence and completeness of $\Omega^{\pm}(V^{-1}A_2V, A_1)$ together with an appropriate invariance principle. Thus Theorem 2.1 reduces the existence question for the complicated object $\Omega^{\pm}(H_2, H_1, J)$ to a question involving objects on a single Hilbert space which in typical applications are just Schrödinger-type operators (but with first derivative terms). These can be treated with standard methods as we shall see. As pointed out by Kato [15], the identification operator J is chosen just so that Theorem 2.1 will be easy. In practice, there are more natural identification operators around and it is natural to ask about the wave operators with J replaced by these more natural wave operators. Fortunately, one can say quite a bit even on the abstract level.

Let us specialize our abstract formalism by adding an assumption that holds in the applications in § 3 and § 4. Suppose that $\mathscr{K}_1$ and $\mathscr{K}_2$ are the same set with equivalent inner products and that $Q(A_1)$, the quadratic form domain of A_1 as an operator on $\mathscr{K}_1$ is equal to $Q(A_2)$ and that:

$$0\leqq c_1(u, A_1u)_{\mathscr{K}_1}\leqq(u, A_2u)_{\mathscr{K}_2}\leqq c_2(u, A_1u)_{\mathscr{K}_1} \tag{8}$$

that is:

$$c_1\|B_1u\|^2_{\mathscr{K}_1}\leqq\|B_2u\|^2_{\mathscr{K}_2}\leqq c_2\|B_1u\|_{\mathscr{K}_1}. \tag{8'}$$

(8) implies that $[D(B_1)]$ and $[D(B_2)]$ are equal as sets and have equivalent inner products and thus the same is true of the spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ constructed above. Under these circumstances, it is natural to use the bounded identification operator $I\colon \mathscr{H}_1\to\mathscr{H}_2$ and ask about $\Omega^{\pm}(H_2, H_1, I)$.

Theorem 2.2. *Let A_i, B_i, $\mathscr{K}_i$, $\mathscr{H}_i$, H_i $(i=1,2)$ be as described in this section. Suppose that the wave operators $\Omega^{\pm}(V^{-1}B_2V, B_1)$ exist (resp. exist and are complete). Suppose that $\mathscr{K}_1$ and $\mathscr{K}_2$ are set-wise equal with equivalent norms and that* (8) *holds. Suppose that $\mathscr{D}$ is a dense subset of Ran $P_{\mathrm{ac}}(A_1)$ in $D(A_1)\cap D(A_1^{-1})$ left invariant by*

B_1, B_1^{-1} and $e^{\pm iB_1t}$ so that for any $w\in\mathscr{D}$, $Vw\in D(A_2)\cap D(A_1)$ and:

$$(I+A_1)(V-I)\,e^{\pm itB_1}\,w\to 0, \tag{9}$$

$$(A_1-V^{-1}A_2\,V)\,e^{\pm itB_1}\,w\to 0 \tag{10}$$

as $t\to\pm\infty$. Then

$$\underset{t\to\mp\infty}{\text{s-lim}}\; e^{iH_2t}\,e^{-iH_1t}\,P_{\text{ac}}(H_1)$$

exist (resp. exist and are complete).

Remarks. 1. Typically A_1 is second order homogeneous constant coefficient operator and $\mathscr{D}$ is the set of functions whose Fourier transforms have compact support in $\mathbb{R}^n\setminus\{0\}$.

2. We will have to work a little harder than we would have to if (9), (10) had B_1, B_2 in place of A_1, A_2. In applications, it is useful to have A_i since they are differential operators rather than only pseudo differential operators.

3. Interestingly enough, there is a sense in which the problem is no longer a two Hilbert space problem. For on $\mathscr{H}_1$, e^{iH_2t} is a group of bounded operators – the wave operators really make no mention of a second Hilbert space.

Proof. By Theorem 2.1, the fact that I and J are bounded and the argument on page 348 of Kato [15], we need only prove that

$$\|(I-J)\,U_1(t)\,P_{\text{ac}}(H_1)\,\phi\|_{\mathscr{H}_2}\to 0 \tag{11}$$

as $t\to\pm\infty$. Now, by the definition of $\|\cdot\|_{\mathscr{H}_2}$:

$$\begin{aligned}\|(I-J)\,U_1(t)\,\phi\|_{\mathscr{H}_2}^2 &= \|B_2(I-B_2^{-1}F_2VB_1)(U_1(t)\,\phi)_1\|_{\mathscr{H}_2}^2\\ &\quad+\|(I-V)(U_1(t)\,\phi)_2\|_{\mathscr{H}_2}^2\end{aligned}$$

where $(U_1(t)\,\phi)_1$ is given by (2) and $(U_1(t)\,\phi)_2=\dfrac{d}{dt}(U_1(t)\,\phi)_1$. Now, since $\mathscr{D}$ is invariant under B_1 and B_1^{-1}, $(U_1(t)\,\phi)_j$ is a sum of vectors of the form $e^{\pm itB_1}\,w$ with $w\in\mathscr{D}$. Thus (11) follows if we prove that:

$$\|(I-V)\,e^{itB_1}\,w\|_{\mathscr{H}_2}\to 0 \tag{12}$$

and

$$\|(B_2-F_2VB_1)\,e^{itB_1}\,w\|_{\mathscr{H}_2}\to 0 \tag{13}$$

as $t\to\pm\infty$. (12) follows from (9) and the fact that $I+A_1$ is invertible. To deal with (13), we write it as

$$\begin{aligned}&\|(V^{-1}B_2-V^{-1}F_2VB_1)\,e^{itB_1}\,w\|_{\mathscr{H}_1}\\ &\quad\leqq\|(V^{-1}B_2\,V-V^{-1}F_2VB_1)\,e^{itB_1}\,w\|_{\mathscr{H}_1}+\|V^{-1}B_2(I-V)\,e^{itB_1}\,w\|_{\mathscr{H}_1}.\end{aligned} \tag{14}$$

The last term in (14) goes to zero by using (8) to bound it by $c_2\,\|B_1(I-V)\,e^{itB_1}\,w\|_{\mathscr{H}_1}$ and the fact that $B_1(I+A_1)^{-1}$ is bounded which permits us to use the hypothesis (9). Thus, we need only show that the first term on the right of (14) goes to zero. But this term is bounded by

$$\|(V^{-1}B_2V-B_1)e^{iB_1t}w\|_{\mathscr{H}_1}+\|(I-V^{-1}F_2V)B_1e^{iB_1t}w\|_{\mathscr{H}_1}. \tag{15}$$

Now $I-V^{-1}F_2V$ is the spectral projection onto the kernel of $B_2'\equiv V^{-1}B_2V$ on $\mathscr{H}_1$. Since $B_1w\in P_{ac}(B_1)$ and $\Omega^{\pm}(B_2', B_1)\equiv\Omega^{\pm}$ exist we have

$$\begin{aligned}\lim_{t\to\pm\infty}\|P_{\{0\}}(B_2')e^{iB_1t}B_1w\| &= \|P_{\{0\}}(B_2')\Omega^{\pm}(B_1w)\|\\ &= \|P_{\{0\}}(B_2')B_2'\Omega^{\pm}w\|=0.\end{aligned}$$

So the second term in (15) goes to zero.

To prove that the first term in (15) goes to zero, we need (10) and a different argument due to Kato [15]. We first note that by (10),

$$\begin{aligned}\overline{\lim_{t\to+\infty}}\|B_2'e^{-iB_2't}e^{+iB_1t}w\|_{\mathscr{H}_1}^2 &= \overline{\lim_{t\to+\infty}}(e^{iB_1t}w, V^{-1}A_2Ve^{iB_1t})_{\mathscr{H}_1}\\ &= \overline{\lim_{t\to\infty}}(e^{iB_1t}w, A_1e^{iB_1t}w)_{\mathscr{H}_1}\\ &= \|B_1w\|_{\mathscr{H}_1}^2=\|\Omega^-B_1w\|_{\mathscr{H}_1}^2=\|B_2'\Omega^-w\|_{\mathscr{H}_1}^2.\end{aligned}$$

Since $\langle u, B_2'e^{-iB_2't}e^{iB_1t}w\rangle\to\langle u, B_2'\Omega^-w\rangle$ for a dense set of u's, it follows that

$$e^{-iB_2't}B_2'e^{iB_1t}w\to B_2'\Omega^-w=\Omega^-B_1w$$

in norm as $t\to+\infty$. Since

$$e^{-iB_2't}B_1e^{iB_1t}w=e^{-iB_2't}e^{iB_1t}B_1w\to\Omega^-B_1w$$

we have that

$$\|(B_2'-B_1)e^{iB_1t}w\|=\|e^{-iB_2't}(B_2'-B_1)e^{iB_1t}w\|\to 0$$

as $t\to+\infty$. Thus the first term in (15) goes to zero. □

To summarize, the basic estimates which one has to prove are:

(I) The inner products on $\mathscr{H}_1$ and $\mathscr{H}_2$ are equivalent. The form estimate (8) for A_1, A_2.

(II) The estimates (9) and (10).

(III) The existence and completeness of the wave operators $\Omega^{\pm}(V^{-1}B_2V, B_1)$ on $\mathscr{H}_1$.

Of course given the theorem of Birman [3], DeBranges [7] and Kato [14], (III) follows from:

(III') $(V^{-1}A_2V+z)^{-n}-(A_1+z)^{-n}$ is trace class for some $z>0$ and some integer n.

§ 3. Acoustical Scattering

We want to construct a scattering theory for the pair of equations

$$u_{tt}=c_0^2\Delta u, \tag{16}$$

$$u_{tt}=c(x)^2\rho(x)\nabla\cdot(\rho(x)^{-1}\nabla u) \tag{17}$$

with $x \in \mathbb{R}^n$. In these equations, $u(t, x)$ represents the difference between the pressure at x at time t and an equilibrium pressure. (16) governs the propagation of pressure waves in an homogeneous medium with propagation speed c_0. (17) governs the propagation in a medium with a speed of sound $c(x)$ and density $\rho(x)$ which vary with x. We assume throughout that

$$0 < c_3 \leqq c(x) \leqq c_4 < \infty, \tag{18a}$$

$$0 < \rho_3 \leqq \rho(x) \leqq \rho_4 < \infty, \tag{18b}$$

and that $\rho(x) \to \rho_0$, $c(x) \to c_0$ as $|x| \to \infty$. We will later add assumptions on the rate of convergence and on the smoothness of ρ and c.

For this problem, it will be easy to verify the program (I)–(III) stated at the end of Section 2 once we properly formulate the spaces. Take $\mathcal{H}_1 = \mathcal{H}_2 = L^2(\mathbb{R}^n)$ with the inner products

$$(u, v)_{\mathcal{H}_1} \equiv (c_0^2 \rho_0)^{-1} (u, v)_{L^2(\mathbb{R}^n)},$$
$$(u, v)_{\mathcal{H}_2} \equiv (u, (c(x)^2 \rho(x))^{-1} v)_{L^2(\mathbb{R}^n)}.$$

The operator $A_1 = -c_0^2 \Delta$ on $\mathcal{H}_1$ has quadratic form

$$q_1(u, u) \equiv (u, A_1 u)_{\mathcal{H}_1}$$
$$= \rho_0^{-1} (\nabla u, \nabla u)_{L^2(\mathbb{R}^n)}$$

with form domain $H^1(\mathbb{R}^n) = \{f \in L^2 \mid \nabla f \in L^2\}$, with ∇f being the distributional gradient. If one defines a quadratic form q_2 with domain H^1 and

$$q_2(u, v) \equiv (\nabla u, (\rho(x)^{-1}) \nabla v)_{L^2(\mathbb{R}^n)}$$

then q_2 is the quadratic form on $\mathcal{H}_2$ of a operator, A_2 which is formally $c(x)^2 \rho(x) \nabla \cdot (\rho(x)^{-1} \nabla \cdot)$. There is clearly a natural choice for V, a unitary operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Namely:

$$V\colon\ u(x) \to (c(x)^2 \rho(x) / c_0^2 \rho_0)^{1/2} u(x).$$

Now, we must check (I), (II) and (III′).

(I) By (18):

$$c_4^{-2} \rho_4^{-1} c_0^2 \rho_0 \|u\|_{\mathcal{H}_1}^2 \leqq \|u\|_{\mathcal{H}_2}^2 \leqq c_3^{-2} \rho_3^{-1} c_0^2 \rho_0 \|u\|_{\mathcal{H}_1}^2$$

and

$$\rho_4^{-1} \rho_0 q_1(u, u) \leqq q_2(u, u) \leqq \rho_3^{-1} \rho_0 q_1(u, u)$$

so the basic estimate (8) holds.

The above calculations show the advantage of the formalism with $\mathcal{H}_1 \neq \mathcal{H}_2$. We are able to arrange all inner products so that neither ρ nor c appears inside a gradient so that inequalities like (7) become easy.

(II) To check (9) and (10), we use the fact that A_1 and A_2 are differential operators. Thus the terms on the left side of (9) and (10) can be written as a sum of terms of the form

$$f(x)\, e^{\pm itB_1}\, P(D)\, w \tag{19}$$

where $P(D)$ is a differential operator and $f(x)$ is a product of terms of the form $\rho(x) - \rho_0, c(x)-c_0, \rho(x), \rho_0, c(x), c_0$ or their inverses or square roots or their derivatives up to order two – moreover at least one factor of $\rho(x)-\rho_0$ or $c(x)-c$ or their derivatives occurs.

Let $\mathscr{D}$ be the set of functions in $\mathscr{S}(\mathbb{R}^n)$ whose Fourier transforms have compact support in $\mathbb{R}^n \setminus \{0\}$. Then $e^{\pm itB_1} P(D)\, w$ is a solution of the free wave equation so that, as is well-known, $\|e^{\pm itB_1} P(D)\, w\|_\infty \leqq C t^{-(n-1)/2}$ so, if $f \in L^2$, then (19) goes to zero. Actually, more is true; by stationary phase methods (see e.g. Hörmander [12]), so long as $f(x)$ is polynomially bounded

$$\|(19)\|^2 \leqq O(t^{-n}) + \int_{at \leqq |x| \leqq bt} |f(x)|^2\, t^{-(n-1)}\, d^n x$$

so that $\|(19)\|$ goes to zero so long as $\int_{|x-y| \leqq 1} |f(x)|^2\, dx \to 0$ as $|y| \to \infty$. Thus

Lemma 3.1. *Suppose that* (18) *holds and that* $\rho(x)$, $c(x)$ *are* C^2 *and that* $\rho(x) \to \rho_0$, $c(x) \to c_0$, $D^\alpha \rho(x)$, $D^\alpha c(x) \to 0$ *for all derivatives of order* 1 *and* 2. *Then the estimates* (9) *and* (10) *hold.*

(III′) We must show that $(V^{-1}A_2 V + z)^{-k} - (A_1 + z)^{-k}$ is trace class for some k. By the results of Appendix 1, (see Theorem A.2) it suffices to prove that $(V^{-1}A_2 V - A_1)(A_1 + z)^{-k-1}$ is trace class for some k and that $V^{-1}A_2 V$ and A_1 have the same form domain. But since

$$(V^{-1}A_1 V)\, w = -(c(x)^2 \rho(x))^{1/2} (\nabla \cdot \rho(x)^{-1} \nabla)(c(x)^2 \rho(x))^{1/2}\, w$$

is easily seen that $V^{-1}A_2 V - A_1$ is of the form $f(x)\Delta + g(x)\cdot\nabla + h(x)$ with f, g, h bounded if ρ, c are C^2 with bounded derivative. It follows that they have the same form domain since $0 < \alpha \leqq f(x) \leqq \beta < \infty$. The relative trace class result follows from the Theorem in Appendix 2 if we take $k > \frac{1}{2}n$ so long as f, g, h obey $\int (1+x^2)^m |f(x)|^2\, dx < \infty$ for some $m > n/2$.

We have thus proven:

Theorem 3.2. *Suppose that* (18) *holds and that* $\rho(x)$ *and* $c(x)$ *are* C^2 *and that the functions* $\rho(x)-\rho_0$, $c(x)-c_0$, $D^\alpha c$, $D^\alpha \rho$ $(1 \leqq \sum \alpha_i \leqq 2)$ *obey (F stands generically for these functions):*

$$F(x) \to 0 \quad \text{as } |x| \to \infty,$$

$$\int (1+x^2)^m |F(x)|^2\, dx < \infty \qquad \text{some } m > n/2. \tag{20}$$

Then the scattering operator for the system (16), (17) *exists and is complete.*

Remarks. 1. Results of this sort were first proven by Schulenberger and Wilcox [24, 25, 26].

2. Deift [8] has used rather different methods to eliminate the smoothness hypothesis on c, ρ.

The hypothesis (20) requires $F(x)$ to more or less have $|x|^{-n-\varepsilon}$ fall off. While this includes all cases of physical interest, it is natural from a mathematical point of view to ask if $|x|^{-n-\varepsilon}$ is really necessary. The answer is no. Kato [13] suggested using the machinery of Agmon [1] and Kuroda [18, 19] (see also Reed-Simon [23]) to extend Kato's theory of two Hilbert space scattering. Following this suggestion:

Theorem 3.3. *The conclusions of Theorem* 3.2 *remain true if* (20) *is replaced by:*

$$|F(x)| \leqq C(1+|x|^2)^{-1/2-\varepsilon} \tag{21}$$

for some $\varepsilon > 0$.

Proof. Steps (I) and (II) follow as above. The existence and completeness of the wave operators $\Omega^{\pm}(V^{-1}A_2 V, A_1)$ follow from the Agmon-Kuroda estimates (the easiest way of realizing this is to use Lavine's local smoothness theory [19A]; see Reed-Simon [23]). Moreover, results of Hörmander [12] imply that, if Cook's method is used, the rate of convergence of the derivative of the wave operators is $t^{-1-\varepsilon}$ so that there is an invariance principle for $\Omega^{\pm}(V^{-1}A_2 V, A_1)$ by a general theorem of Chandler and Gibson [6]. □

§ 4. Optical Scattering

In this section, we will consider scattering of electromagnetic waves by an inhomogeneous medium which may even be non-isotropic. The basic equations are, of course, Maxwell's equations:

$$\begin{aligned} &\nabla \cdot (\varepsilon(x) E) = 0 \qquad \nabla \cdot (\mu(x) H) = 0 \\ &\nabla \times E = -\mu(x) \frac{\partial H}{\partial t} \qquad \nabla \times H = \varepsilon(x) \frac{\partial E}{\partial t}. \end{aligned} \tag{22}$$

Here E and H are three-vector valued functions on $\mathbb{R}^3$ and $\varepsilon(x)$ and $\mu(x)$ are three-by-three-matrix valued functions which we suppose are C^2 throughout. Moreover, we suppose that

$$0 < \varepsilon_3 I \leqq \varepsilon(x) \leqq \varepsilon_4 I; \quad 0 < \mu_3 I \leqq \mu(x) \leqq \mu_4 I; \quad \text{all } x. \tag{23}$$

(23) is a basic restriction on the dielectric and magnetic susceptibilities required by positivity of the energy

$$\mathscr{E}(E, H) = \int [(E(x), \varepsilon(x) E(x)) + (H(x), \mu(x) H(x))]\, dx. \tag{24}$$

Moreover, we suppose there are ε_0, μ_0 so that $\varepsilon(x) - \varepsilon_0$, $\mu(x) - \mu_0$ go to zero at ∞. We will specify the rate later. Of course, in most cases of physical interest, ε_0 and μ_0 are multiples of the identity but there is no simplification in assuming this so we do not.

We will see below that the proof of (I) and (II) of our program in § 2 for this situation is as effortless as in the acoustic case but completeness will require an additional trick and extra work. In most of the section we work with E; this will mean that the convergence of the wave operators is in a norm not as natural as the energy norm $(\mathscr{E}(E,H))^{1/2}$ – at the conclusion of the section, we explain how to use magnetic vector potentials to remedy this defect.

Rewriting (22) as a second order equation in E, we obtain

$$\ddot{E} = -\varepsilon^{-1}\,\nabla\times(\mu^{-1}(\nabla\times E)). \tag{25}$$

We thus define $\mathscr{H}_1$ and $\mathscr{H}_2$ to be $L^2(\mathbb{R}^3)^3$ with inner products

$$(E,F)_1 = \int (E(x), \varepsilon_0\, F(x))\,dx,$$
$$(E,F)_2 = \int (E(x), \varepsilon(x)\, F(x))\,dx$$

and quadratic forms on $\mathscr{H}_1$ and $\mathscr{H}_2$ with form domain $\{E\in L^2(\mathbb{R}^3)^3 \mid \nabla\times E\in L^2\}$ by:

$$q_1(E,F) = \int (\nabla\times E, \mu_0^{-1}(\nabla\times F))\,dx,$$
$$q_2(E,F) = \int (\nabla\times E, \mu(x)^{-1}(\nabla\times F))\,dx.$$

(In order to check that (25) is indeed $\ddot{E} = -A_2 E$, we need to use $A\cdot(B\times C) = -(B\times A)\cdot C$ with $B=\nabla$.) We define $V\colon \mathscr{H}_1 \to \mathscr{H}_2$ by

$$(VE)(x) = \varepsilon(x)^{-1/2}\,\varepsilon_0^{1/2}\,E(x).$$

The formal structure of the above is very similar to the acoustic case, the only change being that A_1 and A_2 now have non-zero kernels. Since we have been careful to allow this possibility in § 2, this in itself is no problem.

(I) As in the acoustic case, this follows from (23).

(II) As in the acoustic case, (9) and (10) follows from the fact that $E(t) = e^{\pm itB_1} P_{\mathrm{ac}}(B_1)\, E_0$ obeys a free wave equation so that $\|E(t)\|_\infty \leqq C t^{-1}$. Since the free wave equation has some non-zero modes let us be more explicit. We can write $\widehat{A_1 E}(k) = \tilde{A}_1(k)\,\hat{E}(k)$ where $\tilde{A}_1(k)$ has one zero eigenvalue and two non-zero eigenvalues, $\lambda_1(k)$ and $\lambda_2(k)$, which are homogeneous of order two. Let $e_1(k)$ and $e_2(k)$ be the corresponding eigenvectors in $\mathbb{R}^3$. Then $(P_{\mathrm{ac}}(B_1)\,E)\hat{}$ has the form $f(k)\,e_1(k) + g(k)\,e_2(k)$ so that

$$(e^{\pm itB_1}\, P_{\mathrm{ac}}(B_1)E)\hat{}(k) = e^{\pm i\sqrt{\lambda_1}t}\, f(k)\, e_1(k) + e^{\pm i\sqrt{\lambda_2}t}\, g(k)\, e_2(k)$$

and thus, if f and g have compact support in $\mathbb{R}^3 \setminus \{0\}$, we will have $\|E(t)\|_\infty \leqq C t^{-1}$ by the usual method (see e.g. Reed-Simon [23]).

(III) As in the proof of (II) we could prove existence of $\Omega^\pm(V^{-1}A_2 V, A_1)$ directly and also the applicability of the invariance principle of Chandler and Gibson. Since we wish to obtain completeness at the same time we will follow a different route.

We define operators $\tilde{A}_i$ on $\mathscr{H}_i$ by the quadratic forms

$$\tilde{q}_i(E,F) = q_i(E,F) + \int (\overline{(\nabla\cdot\gamma_i E)}, (\nabla\cdot\gamma_i F))\,dx$$

where $\gamma_i = \varepsilon_0$ for $i=1$ *and* $\gamma_i = \varepsilon$ *for* $i=2$. Now, in the $\mathscr{H}_i$ inner product

$$\mathrm{Ker}(A_i)^\perp = \{E \mid \nabla\times E = 0\}^\perp = \{E \mid \nabla\cdot\gamma_i\, E = 0\}.$$

It follows that $\tilde{A}_i$ leaves Ran $P_{ac}(A_i)$ invariant and that under the decomposition $\mathscr{K}_i = N(A_i)^\perp \oplus N(A_i)$, we have $\tilde{A}_i = A_i \oplus \tilde{A}_i \restriction N(A_i)$. In particular,

$$\tilde{B}_i \restriction \mathrm{Ker}(A_i)^\perp = B_i. \tag{26}$$

Next, we claim that $(V^{-1}\tilde{A}_2 V+1)^{-2} - (\tilde{A}_1+1)^{-2}$ is trace class if (23) holds, ε_i and μ_i are C^2 with bounded derivatives, and if $\varepsilon(x)-\varepsilon_0$ and $\mu(x)-\mu_0$ go to zero sufficiently rapidly as $|x|\to\infty$. By Theorem A.2, it suffices to prove that $(V^{-1}\tilde{A}_2 V - \tilde{A}_1)(\tilde{A}_1+1)^{-3}$ is trace class and $Q(V^{-1}\tilde{A}_2 V) = Q(\tilde{A}_1+1)$. The latter equality is a standard result that all second order strictly elliptic operators with asymptotically constant coefficients define the same Sobolev space; see e.g. Gilkey [10]. (We note in passing that the equality on Q's is the "easy" estimate $c_3(V^{-1}\tilde{A}_2 V+1) \leqq \tilde{A}_1+1 \leqq c_4(V^{-1}\tilde{A}_2 V+1)$ and not the subtler estimate $c_1\tilde{A}_2 \leqq \tilde{A}_1 \leqq c_4 \tilde{A}_2$ which is true if $\varepsilon-\varepsilon_0$ has compact support but is much harder to prove.) The trace class estimate follows from the results of Appendix 2 if $\varepsilon-\varepsilon_0$, $\mu-\mu_0$, $D^\alpha \varepsilon$, $D^\alpha \mu$ all obey (20) so long as we prove that $(-\Delta+1)^{+3}(\tilde{A}_1+1)^{-3}$ is bounded. But this follows if we prove that for all k and a in $\mathbb{R}^3$:

$$(k\times a, \mu^{-1}(k\times a)) + |k\cdot\varepsilon_0 a|^2 \geqq \mathrm{const}\,|k|^2|a|^2. \tag{27}$$

Since the left hand side is homogeneous of degree 2 in k and a, we need only check (27) for k and a in the unit sphere. Thus (27) follows immediately so long as the left hand side is not zero for $k \neq 0 \neq a$. If the left hand side is zero then $k\times a=0$ and $k\cdot\varepsilon_0 a=0$. But $k\times a=0$ implies that k is parallel to a whence $k\cdot\varepsilon_0 a\neq 0$ unless k or a is zero. This completes the proof that $(V^{-1}\tilde{A}_2 V+1)^{-2}-(\tilde{A}_1+1)^{-2}$ is trace class. We remark that (27) where ε_0 is replaced by $\varepsilon(x)$ is used in the proof of the strict ellipticity of $V^{-1}\tilde{A}_2 V$.

By the Birman-deBranges-Kato theorem, $\Omega^\pm(V^{-1}\tilde{B}_2 V, \tilde{B}_1)$ and $\Omega^\pm(\tilde{B}_1, V^{-1}\tilde{B}_2 V)$ exist. We want to use this to prove that $\Omega^\pm(V^{-1}B_2 V, B_1)$ and $\Omega^\pm(B_1, V^{-1}B_2 V)$ exist. Let W be multiplication by $\varepsilon_0^{-1/2}\varepsilon^{-1/2}\varepsilon_0$ so that VW, which is multiplication by $\varepsilon^{-1}\varepsilon_0$, takes $\mathrm{Ker}(B_1)^\perp = \{E \mid V\cdot\varepsilon_0 E=0\}$ into $\mathrm{Ker}(B_2)^\perp = \{E \mid V\cdot\varepsilon E=0\}$. By hypothesis, $W-1$ obeys (20), so for a dense set of vectors, E_0,

$$\lim_{t\to\pm\infty} (W-1)\, e^{\pm i\tilde{B}_1 t} E_0 = 0$$

so that s-lim$(W-1)\,e^{\pm i\tilde{B}_1 t}=0$. Since $\Omega^\pm(\tilde{B}_1, V^{-1}\tilde{B}_2 V)$ exist,

$$\begin{aligned}\text{s-lim}\,(W^{-1}-1)\,e^{\pm iV^{-1}\tilde{B}_2 V t} &= \text{s-lim}_{t\to\infty}\,((W^{-1}-1)\,e^{\pm i\tilde{B}_1 t})\,\Omega^\pm(\tilde{B}_1, V^{-1}\tilde{B}_2 V)\\ &=0.\end{aligned}$$

Thus, the strong limits of

$$e^{+it\tilde{B}_1}\, W^{-1}\, e^{-itV^{-1}\tilde{B}_2 V} P_{ac}(V^{-1}B_2 V) \tag{28a}$$

and

$$e^{itV^{-1}\tilde{B}_2 V}\, W e^{-it\tilde{B}_1} P_{ac}(B_1) \tag{28b}$$

exists as $t \to \pm\infty$. But, by (26)

$$e^{-it\tilde{B}_1} P_{ac}(B_1) = P_{ac}(B_1) e^{-itB_1} P_{ac}(B_1)$$

and

$$e^{itV\tilde{B}_2 V^{-1}} W P_{ac}(B_1) = e^{itVB_2V^{-1}} W P_{ac}(B_1),$$

so the $\tilde{}$ can be dropped in (28). Now we can again replace W and W^{-1} by 1 and so conclude that $\Omega^{\pm}(V^{-1}B_2 V, B_1)$ exist and are complete.

We have thus proven:

Theorem 4.1. *If ε, μ obey* (23) *and if $\varepsilon - \varepsilon_0$, $\mu - \mu_0$, $D^\alpha \varepsilon$, $D^\alpha \mu$ all obey* (20) *for $1 \leqq |\alpha| \leqq 2$, then suitable wave operators for the equations* (22) *exist and are complete.*

Here "suitable" means that the natural identification is used and convergence is in the topology of $\mathscr{H}_1$, i.e. $B_1 E$ and $\dot{E}$ converge; equivalently $\nabla \times \mathrm{E}$ and $\nabla \times H$ converge. Actually, one can arrange for the wave operators to converge in the energy norm given by the square root of (24). For the map of $\{A \in C_0^\infty(\mathbb{R}^3)^3 \mid \nabla \cdot \varepsilon A = 0\}$ to $L^2(\mathbb{R}^3)^3$ given by $H = \mu^{-1}(\nabla \times A)$ is onto a dense subset of the set of H obeying $\nabla \cdot (\mu H) = 0$. If we take $E = \dot{A}$ and solve $\ddot{A} = -\varepsilon^{-1} \nabla \times \mu^{-1}(\nabla \times A)$ with initial condition $\dot{A}$ obeying $\nabla \cdot \varepsilon \dot{A} = 0$, then $H = \mu^{-1}(\nabla \times A)$, $E = \dot{A}$ run through a dense set of solutions of Maxwell's equations. The wave operators for A are identical on A's to those constructed for E and the square of the norm of convergence for the A's which is $\int (\nabla \times A, \mu^{-1}(\nabla \times A)) + (\varepsilon \dot{A}, \dot{A})$ is just (24)!

Appendix 1. Trace Class Properties of Differences of Resolvent Powers

In this appendix, we prove a general result about trace class conditions on $(H_0 + V + E)^{-k} - (H_0 + E)^{-k}$ and discuss applications to scattering theory. Let I_p be the trace ideal of operators with $|A|^p \in \mathscr{I}_1$, the trace class, see [9, 11, 22] and let $\mathscr{I}(\mathscr{H})$ denote the bounded operators in $\mathscr{H}$.

Theorem A.1. *Let k be a non-negative integer. Suppose that H_0 is a positive self-adjoint operator and that either*

(a) *V is a symmetric relatively form bounded form with relative bound smaller than* 1 *(i.e. $\|(H_0 + E)^{-1/2} V (H_0 + E)^{-1/2}\| < 1$ for E large) and $(H_0 + 1)^{-1/2} V (H_0 + 1)^{-k-1/2}$ is trace class*

or

(b) *V is a symmetric relatively bounded operator with relative bound smaller than* 1 *(i.e. $\|V(H_0 + E)^{-1}\| < 1$ for E large) and $(H_0 + 1)^{-1} V (H_0 + 1)^{-k}$ is trace class.*

Then $(H_0 + V + E)^{-k} - (H_0 + E)^{-k}$ is trace class for all $-E \in \rho(H) \cap \rho(H_0)$.

Remarks. 1. For $k = 1$, this result is well-known, see e.g. Kuroda [17] for (b) and Simon [28] for (a).

2. In the proof and in Remark 3 below, we need the following complex interpolation result, see e.g. Kunze [16], Goh'berg-Krein [11], Calderon [5], Reed-Simon [22]: Let D be a dense subspace of $\mathscr{H}$ and let $a(z; \phi, \psi)$ be defined for $\phi, \psi \in D$, and $z \in \mathbb{C}$ with $0 \leqq \mathrm{Re}\, z \leqq 1$, so that:

(i) $a(\cdot;\phi,\psi)$ is analytic in $0<\operatorname{Re} z<1$ and continuous in the closure for each fixed ϕ,ψ and $a(z;\cdot,\cdot)$ is a sesquelinear form on D for each z fixed,

(ii) For y real, there are bounded operators $A(iy)$ and $A(1+iy)$ so that $a(z;\phi,\psi)=(\phi,A(z)\psi)$ for $z=iy,\ 1+iy$,

(iii) For some p_0 and p_1, $A(iy)\in\mathscr{I}_{p_0}$ and $A(1+iy)\in\mathscr{I}_{p_1}$ for all y and $\sup_y(\|A(iy)\|_{p_0},\|A(1+iy)\|_{p_1})<\infty$.

Then, for each z, there is a bounded operator $A(z)$ with $a(z;\phi,\psi)=(\phi,A(z)\psi)$ so that $A(t+iy)\in\mathscr{I}_{p_t}$ with $p_t^{-1}=t\,p_1^{-1}+(1-t)\,p_0^{-1}$. Moreover, this result remains true if $\mathscr{I}_\infty$ is replaced by $\mathscr{L}(\mathscr{H})$.

3. In applications the following proposition is useful:

Proposition. *If $V(H_0+1)^{-\alpha}$ is trace class, then $(H_0+1)^{-\beta}V(H_0+1)^{-\alpha+\beta}$ is trace class for any β with $0\leqq\beta\leqq\alpha$.*

Proof. Let $F(z)=(H_0+1)^{-\alpha z}V(H_0+1)^{-\alpha(1-z)}$ and interpolate. □

Proof of Theorem A.1. We give the proof of (a); that for (b) is similar. By interpolating between

$$(H_0+1)^{-1/2}V(H_0+1)^{-1/2}\in\mathscr{L}(\mathscr{H}),$$

and

$$(H_0+1)^{-1/2}V(H_0+1)^{-1/2-k}\in\mathscr{I}_1,$$

we conclude that

$$(H_0+1)^{-1/2}V(H_0+1)^{-1/2-\alpha}\in\mathscr{I}_{k/\alpha}\qquad 0<\alpha\leqq k. \tag{29}$$

Choose E so large that $\|(H_0+E)^{-1/2}V(H_0+E)^{-1/2}\|=\beta<1$. We claim that

$$(H_0+E)^{-1/2}V(H+E)^{-k}(H_0+E)^{-1/2}\in\mathscr{I}_1. \tag{30}$$

To prove (30), we expand

$$(H+E)^{-1}=(H_0+E)^{-1/2}\cdot\left[\sum_{j=0}^{\infty}(-1)^j((H_0+E)^{-1/2}V(H_0+E)^{-1/2})^j\right](H_0+E)^{-1/2}$$

so that the operator in (30) has an expansion with terms of the form:

$$(H_0+E)^{-1/2}V(H_0+E)^{-n_1-1}V\ldots V(H_0+E)^{-n_l-1/2} \tag{31}$$

where $n_1+\cdots+n_l=k$. Using (29) and $\|(H_0+E)^{-1/2}V(H_0+E)^{-1/2}\|\leqq\beta$ we see that

$$\|(31)\|_1\leqq c\,\beta^{l-k}$$

so that, summing the series, (30) follows. Since $(H_0+E)^{-1/2}(H+E)^{+1/2}$ and $(H+E)^{-1/2}(H_0+E)^{1/2}$ are bounded we conclude that $(H_0+E)^{-1/2}V(H+E)^{-k-1/2}$

and $(H_0+E)^{-k-1/2}V(H+E)^{-1/2}$ are in $\mathscr{I}_1$ and so by interpolation,

$$(H_0+E)^{-\alpha}V(H+E)^{-k-1+\alpha}\in\mathscr{I}_1 \qquad \tfrac{1}{2}\leqq\alpha\leqq k+\tfrac{1}{2} \tag{31}$$

and (32) then holds for all $E\in\rho(H)\cap\rho(H_0)$.

The result of the theorem follows from (32) and the expansion

$$(H_0+E)^{-k}-(H+E)^{-k}=\sum_{j=1}^{k}(H_0+E)^{-j}V(H+E)^{-k-1+j}. \quad \square$$

Application (Schrödinger Operators). Let $H_0=-\Delta$ on $L^2(\mathbb{R}^n)$ and suppose that any one of the following conditions holds:

(a) $V\in L^2_\alpha=\{f\,|\,(1+x^2)^{\alpha/2}f\in L^2\}$ with $\alpha>n/2$. V is H_0-bounded with relative bound smaller than one.

or

(b) $V^{1/2}\in L^2_\alpha$ with $\alpha>n/2$. V is H_0-form bounded with relative bound smaller than one.

or

(c) $V\in L^1$ and for some $\alpha>n/2$, $(1+x^2)^\alpha V$ is H_0-form bounded with relative bound smaller than one.

Then, the wave operators $\Omega^\pm(H_0+V,H_0)$ exist and are complete. For, in case (a) (see Appendix 2),

$$V(H_0+1)^{-\alpha}\in\mathscr{I}_1,$$

in case (b),

$$[(H_0+1)^{-1/2}V^{1/2}][V^{1/2}(H_0+1)^{-\alpha}]\in\mathscr{I}_1,$$

and in case (c),

$$[(H_0+1)^{-1/2}V^{1/2}(1+x^2)^{\alpha/2}][(1+x^2)^{-\alpha/2}V^{1/2}(H_0+1)^{-\alpha}]\in\mathscr{I}_1.$$

Thus, $(H_0+1)^{-k}-(H+1)^{-k}\in\mathscr{I}_1$ by the result above, so by a well-known theorem of Birman [3], de Branges [7], and Kato [14], the assertion on the wave operators follows.

We note that in low dimensions the classical result of Kuroda [17] ($V\in L^1\cap L^2$ in $\mathbb{R}^3$) improves the above. (c) is a slight strengthening of a result of Nenciu [20] who uses a very different method.

For applications to optical and acoustical scattering, the hypotheses of Theorem A.1 may not hold. The problem is that V may not have relative bound less than one since it will contain terms of the top order. Fortunately, ellipticity solves this problem. We state a result slightly weaker than the best possible to make it precisely what we need in the text.

Theorem A.2. *Let H and H_0 be positive self-adjoint operators so that $Q(H)=Q(H_0)$ and so that $V=H-H_0$ (defined apriori as a difference of forms) obeys: $V(H_0+1)^{-k-1}$*

is trace class. Then

$$(H_0+E)^{-k}-(H+E)^{-k}$$

is trace class for all $-E\in\rho(H)\cap\rho(H_0)$.

Proof. By the closed graph theorem, H and H_0 define equivalent norms on $Q(H_0)$, so for some $\alpha>0$:

$$\pm(\phi, V\phi)\leqq\alpha(\phi,(H_0+1)\,\phi),$$
$$\pm(\phi, V\phi)\leqq\alpha(\phi,(H+1)\,\phi).$$

Let $H(\theta)=\theta H_0+(1-\theta)H$ so that for $0\leqq\theta\leqq1$,

$$\pm(\phi, V\phi)\leqq\alpha(\phi,(H(\theta)+1)\,\phi). \tag{33}$$

Choose an integer N with $\alpha N^{-1}<1$, let $A_i=H(i/N)$, and let $N^{-1}V=W$. By (33), $\|(A_i+1)^{-1/2}W(A_i+1)^{-1/2}\|<1$. Moreover, by the hypothesis and the proposition above,

$$(A_0+1)^{-1/2}W(A_0+1)^{-k-1/2}$$

is trace class. By Theorem A.1 and its proof

$$(A_i+1)^{-1/2}W(A_i+1)^{-k-1/2} \quad\text{and}\quad (A_i+1)^{-k}-(A_{i-1}+1)^{-k} \tag{34}$$

are trace class for $i=1$. (34) is now established inductively for higher i so that $(A_N+1)^{-k}-(A_0+1)^{-k}$ is trace class. □

Appendix 2. A Trace Class Criterion

In this appendix, we prove an extension of a trace class criterion of Stinespring [29] which we feel places it in its natural setting:

Theorem A.3. *Let* $L^2_\alpha(\mathbb{R}^m)=\{f\,|\,(1+x^2)^{\alpha/2}f\in L^2\}$ *with the natural norm. Suppose that* F *and* G *are in* $L^2_\alpha(\mathbb{R}^m)$ *with* $\alpha>m/2$. *Then* $F(-i\nabla)\,G(x)$ *is trace class with*

$$\|F(-i\nabla)\,G(x)\|_1\leqq c_\alpha\|F\|_\alpha\|G\|_\alpha.$$

Remarks. 1. This theorem should be compared with the result that if $F, G\in L^p(\mathbb{R}^m)$ with $p\geqq2$, then $F(-i\nabla)\,G(x)$ is in the trace ideal, $\mathscr{I}_p$ (see Seiler-Simon [27]).

2. Our original proof was natural but longer than the one we give below. This proof was shown to us by M. Aiezeman and E. Lieb (private communication). T. Kato has informed us that he too has found and proven Theorem A.3.

Proof. We first note that if $H, M\in L^2(\mathbb{R}^m)$, then $H(-i\nabla)\,M(x)$ has an integral kernel $(2\pi)^{-n/2}\check{H}(x-y)\,M(y)$, where $\check{}$ is the inverse Fourier transform, so $H(-i\nabla)\,M(x)$ is clearly Hilbert-Schmidt. Writing

$$F(-i\nabla)\,G(x)=[F(-i\nabla)(1-\Delta)^{\alpha/2}(1+x^2)^{-\alpha/2}]\cdot[(1+x^2)^{\alpha/2}(1-\Delta)^{-\alpha/2}G(x)]$$

we see that the first factor is Hilbert-Schmidt since $\alpha > m/2$. Now, since $\alpha > m/2$, $(1-\Delta)^{-\alpha/2}$ is convolution with a function $J(x)$ with $J \in L^2$. Moreover, since $(1+k^2)^{-\alpha/2}$ is analytic in a strip, $e^{+|x|/2} J(x) \in L^2$ also. The second factor has an integral kernel $(1+x^2)^{\alpha/2} J(x-y) G(y)$ so it clearly suffices to prove that

$$\int (1+x^2)^\alpha |J(x-y)|^2 \, dx \leqq C(1+y^2)^\alpha.$$

But, since J and $e^{|x|/2} J$ are in L^2,

$$\begin{aligned} &\int (1+x^2)^\alpha |J(x-y)|^2 \, dx \\ &\quad \leqq \int (1+(x+y)^2)^\alpha |J(x)|^2 \, dx \\ &\quad \leqq c\left[\int (1+x^2)^\alpha |J(x)|^2 \, dx + (y^2)^\alpha \int |J(x)|^2 \, dx\right] \\ &\quad \leqq \mathrm{const}(1+y^2)^\alpha \end{aligned}$$

completing the proof. □

Our original proof came from writing $F(-i\nabla) G(x) = ABC$ where

$$\begin{aligned} A &= F(-i\nabla)(1-\Delta)^{\alpha/2}(1+x^2)^{-\alpha/2}, \\ C &= (1-\Delta)^{-\alpha/2}(1+x^2)^{\alpha/2} G(x), \\ B &= (1+x^2)^{\alpha/2}(1-\Delta)^{-\alpha/2}(1+x^2)^{-\alpha/2}(1-\Delta)^{\alpha/2}. \end{aligned}$$

A and C are Hilbert-Schmidt as in the proof. B is bounded. While this fact is both intuitive and well-known (see e.g. Agmon [1] or Prosser [21]), a first principles proof is not very short.

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