Comments and Addenda

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Some Comments on the Jin-Martin Lower Bound*

BARRY SIMON
Department of Mathematics, Princeton University, Princeton, New Jersey 08540
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We present an extremely simple derivation of the Jin-Martin bound in a slightly weakened form. Explicitly, let $f^{AB\to AB}(s)$ be the spin-averaged forward-scattering amplitude for the process $AB\to AB$ [normalized by $\sigma_{\rm tot}=4\pi\,{\rm Im}f(s)/k\sqrt{s}$]. Then we prove that there is a constant C and a sequence $s_n\to\infty$ such that either $f^{AB\to AB}(s_n)>Cs_n^{-2}$ (all n) or $f^{A\bar{B}\to A\bar{B}}(s_n)>Cs_n^{-2}$ (all n). We also clarify the connection between allowable asymptotic behavior and the sign of the scattering length found by Jin and Martin.

SEVERAL years ago, Jin and Martin,[1] using fairly involved Herglotz-function arguments, proved that the forward-scattering amplitude[2] $f(s)$ obeys an inequality of the form

$$\int_0^s s'|f(s')|ds'>(\ln s)^{1/2}. \tag{1}$$

Their derivation used only "axiomatic" assumptions, specifically forward dispersion relations, crossing and the positivity implied by the optical theorem (i.e., ${\rm Im}f\geq 0$ on the right-hand cut, ${\rm Im}f\leq 0$ on the left-hand cut).

Equation (1) implies the limit condition

$$\lim_{s\to\infty}\sup s^2(\ln s)^{1/2}|F(s,0)|=\infty\,. \tag{2}$$

Martin,[3] using the Herglotz functions, and Sugawara,[4] using his phase representation, showed that (2) could be improved to the form

$$\lim_{s\to\infty}\sup s^2|F(s,0)|>0\,, \tag{3a}$$

or, equivalently, for some sequence $s_n\to\infty$ and some $C>0$,

$$|F(s_n,0)|>Cs_n^{-2}. \tag{3b}$$

A result of the form (3b) [i.e., $|F(s_n,0)|>Cs_n^{-M}$ for some M] is often crucial in various arguments. For example, (3b) implies[1] that $\sigma_T(s_n)>C'(\ln s_n)^{-2}s_n^{-6}$ and this in turn implies[5] that $\sigma_e(s_n)\geq D[\sigma_T(s_n)]^2(\ln s_n)^{-2}$, where σ_e is the elastic cross section and σ_T is the total cross section.

It is our purpose here to provide a proof of the slightly weakened version of (3):

There exists a sequence $s_n\to\infty$ and a constant C such that either

$$|f^{AB\to AB}(s_n)|>C|s_n|^{-2}$$

or

$$|f^{A\bar{B}\to A\bar{B}}(s_n)|>C|s_n|^{-2}. \tag{4}$$

This proof will be so elementary that one is tempted to classify the Jin-Martin bound as an "immediate" consequence of unitary and analyticity. We note that in the crossing-symmetric case ($B=\bar{B}$ or $A=\bar{A}$), (4) is equivalent to (3), and that in general (4) holding but (3) failing requires oscillations or $\lim_{s\to\infty}s^2f^{A\bar{B}\to A\bar{B}}=\infty$, since $\lim_{s\to\infty}s^2f^{AB\to AB}(s)$ and $\lim_{s\to\infty}s^2f^{A\bar{B}\to A\bar{B}}(s)$ must be complex conjugates (by Phragman-Lindelöf) if both exist and are finite.

The crucial idea behind our proof of (4) is that it is so weak its negation must be very strong. In fact *if (4) is false*, then (using crossing)

$$\lim_{s\to\pm\infty,\text{along real axis}} s^2|f(s)|=0\,, \tag{5a}$$

but then, by the Phragman-Lindelöf principle (and the polynomial boundedness condition for f)

$$\lim_{s\to\infty}s^2|f(s)|=0\text{ (uniformly in arg}s). \tag{5b}$$

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[1] Y. S. Jin and A. Martin, Phys. Rev. **135**, B1369 (1964).
[2] Our normalization is that of Jin and Martin; explicitly, $\sigma_{\rm tot}=4\pi\,{\rm Im}f(s)/k\sqrt{s}$, where $\sigma_{\rm tot}$ and f are spin-averaged.
[3] See Ref. 11 of T. Kinoshita, Phys. Rev. **154**, 1438 (1967).
[4] M. Sugawara, Phys. Rev. Letters **14**, 336 (1965).
[5] A. Martin, Nuovo Cimento **29**, 993 (1963).

Thus one has the superconvergence relation[6]

$$\int_{-\infty}^{\infty} (s'-s_0)\,\mathrm{Im}f(s')ds'=0\,, \qquad (6)$$

where s_0 is chosen with $(M_A-M_B)^2<s_0<(M_A+M_B)^2$. Since $\mathrm{Im}f(x')\geq 0$ for $s'\geq(M_A+M_B)^2$, $\mathrm{Im}f(s')\leq 0$ for $s'\leq(M_A-M_B)^2$, and $\mathrm{Im}f(s)=0$ in the gap, we immediately have a contradiction with $f\neq 0$. Thus (5) must be false and so (4) must be true.

We can also use this approach to see the connection between asymptotic behavior and low-energy scattering found by Jin-Martin and Sugawara. For suppose that

$$\lim_{s\to\infty}|f(s)|=0 \text{ (uniformly in } \arg s). \qquad (7)$$

Then f obeys an unsubtracted dispersion relation:

$$f(s)=\frac{1}{\pi}\left(\int_{-\infty}^{(M_A-M_B)^2}+\int_{(M_{-}+M_B)^2}^{\infty}\right)\frac{\mathrm{Im}f(s')}{s-s'}ds'. \qquad (8)$$

If $(M_A-M_B)^2\leq s\leq(M_A+M_B)^2$, the integrand in (8) is positive so that f is positive. Thus by an argument identical to the one above, we conclude as follows:

If $f(s)\leq 0$ in the gap, in particular, if the scattering length is negative, then either

$$\lim_{s\to\pm\infty}\sup f^{AB\to AB}(s)>0$$

or

$$\lim_{s\to+\infty}\sup f^{A\bar{B}\to A\bar{B}}(s)>0. \qquad (9)$$

Finally, we remark that it is easy to construct examples $f(s)$ obeying the positivity and analyticity requirements and with s^{-2} falloff.[7] Thus an improvement of the bound would require using additional input.

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[6] Instead of viewing (6) as a superconvergence relation, we can write an unsubtracted dispersion so that (s real)

$$\mathrm{Re}f(is+s_0)=\int_{-\infty}^{\infty}\frac{(s'-s_0)\,\mathrm{Im}f(s')}{(s')^2+s^2}ds' \underset{s\to\infty}{\simeq} \frac{1}{s^2}\int_{-\infty}^{\infty}(s'-s_0)\,\mathrm{Im}f(s')ds'.$$

Thus $\int_{-\infty}^{\infty}(s'-s_0)\,\mathrm{Im}f(s')=0$.

[7] In the crossing-symmetric case, $f(s)=G(z)$ with $z=(s-M_A^2-M_B^2)^2$. $f(s)\sim s^{-2}$ is equivalent to $G(z)\sim z^{-1}$. To construct such a G, let $\tilde{G}$ have the proper cut structure with $|\tilde{G}|<|z|^N$ [e.g., $\tilde{G}(z)=(-z+z_0)^{1/2}$] and let

$$G(z)=\frac{1}{z^{N+2}}\left[\tilde{G}(z)-\sum_{n=1}^{N+1}\tilde{G}^{(n)}(0)\frac{z^n}{n!}\right].$$

Then $G(z)\sim z^{-1}$.