ANNALS OF PHYSICS **108**, 69–78 (1977)

The Bound States of Weakly Coupled Long-Range One-Dimensional Quantum Hamiltonians*

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Received January 25, 1977

We study the small λ behavior of the ground state energy, $E(\lambda)$, of the Hamiltonian $-(d^2/dx^2) + \lambda V(x)$. In particular, if $V(x) \sim -ax^{-2}$ at infinity and if $\int V(x)dx < 0$, we prove that $(-E(\lambda))^{1/2} = -[\frac{1}{2}\lambda + a\lambda^2 \ln \lambda] \int dx V(x) + O(\lambda^2)$.

1. INTRODUCTION

It is well known that a sufficiently shallow square well in three dimensions will not bind. By contrast, in one or two dimensions, there is a special situation, due essentially to an infrared divergence, in which an attractive short-range potential always produces a bound state no matter how small the coupling. For the case of the one-dimensional Hamiltonian

$$H = -(d^2/dx^2) + \lambda V(x)$$

Abarbanel *et al.* [1] derived a formal series for the ground state, $E(\lambda)$, for an attractive V of short range of the form

$$(-E(\lambda))^{1/2} = -\tfrac{1}{2}\lambda \int dx\, V(x) - \tfrac{1}{4}\lambda^2 \int dx\, dy\, V(x) \mid x - y \mid V(y) + O(\lambda^2). \qquad (1.1)$$

This situation was further studied by Simon [2] who proved that so long as $\int dx\, V(x) \leqslant 0$, and $\int dx(1 + x^2) \mid V(x)\mid < \infty$, there is a unique bound state for small λ and its energy is given by (1.1). It was also shown that if $\int dx\, e^{y|x|} V(x) < \infty$, then $(-E(\lambda)))^{1/2}$ is analytic at $\lambda = 0$.

In this note we wish to consider the case where $V(x)$ is of sufficiently long range that

$$\int dx(1 + x^2) \mid V(x)\mid = \infty.$$

* Research supported by the U. S. National Science Foundation under grants MPS-75-11864 and MPS-75-22514 and in part by the Energy Research and Development Administration.

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ISSN 0003-4916

There are three cases to consider with

$$V(x) \simeq -ax^{-\beta} \tag{1.2}$$

as $x \to \infty$.

(A) If $2 < \beta < 3$, then a simple modification of the argument in [2] allows one to prove that (1.1) is still valid.

(B) If $\beta = 2$, there is still a unique bound state for small λ so long as $\int dx\, V(x) \leqslant 0$. However, if this integral is nonzero, then (1.1) is not valid because the λ^2 term is infinite; there is, in fact, a $\lambda^2 \ln \lambda$ term which we explicitly isolate. The situation here is reminiscent of some recent work of Greenlee [3, 4] and Harrel [5] who study perturbations of the operator $(-d^2/dx^2)$ by potentials with $x^{-\gamma}$ singularities at the origin on the interval $[0, \alpha]$ or $(-(d^2/dx^2) + x^2)$ on the interval $[0, \infty]$ with $\psi(0) = 0$ boundary conditions. If $\gamma \geqslant 3$, then first-order perturbation theory is infinite but there is an explicit λ^{-g}, $g = (\gamma - 2)^{-1}$ leading term if $\gamma > 3$ and a $\lambda \ln \lambda$ leading term if $\gamma = 3$. Harrel also considers situations in which the first-order term is finite but the second order is infinite; for example, if $\gamma = 2.5$, there is a $a\lambda + b\lambda^2 \ln \lambda + O(\lambda^2)$ behavior analogous to what we find in the $\beta = 2$ case.

(C) If $1 < \beta < 2$, then there are infinitely many bound states for any $\lambda > 0$ in one and three dimensions. One may ask in this case if there is any difference between the one and three-dimensional case as there is for a short-ranged potential $V(x)$. In fact, there is a difference; all the bound state energies in three dimensions and all but the ground state in one dimension are of order λ^h, $h = 2(2 - \beta)^{-1}$, as λ approaches zero from above. The ground state energy in one dimension is special in that it is of order λ^2. One still finds that

$$(-E(\lambda))^{1/2} = -\tfrac{1}{2}\lambda \int dx\, V(x) + O(\lambda). \tag{1.3}$$

We shall not consider the case $\beta = 1$ or $0 < \beta < 1$ although on the basis of the work by Greenlee and Harrel and our case (C), there is a natural *conjecture*: At $\beta = 1$, all states but the ground state are of order λ^2, while the ground state is of order $\lambda^2(\ln \lambda)^2$; for $0 < \beta < 1$, all states are of order λ^h, $h = 2(2 - \beta)^{-1}$ and the ground state is of order λ^g, $g = 2(3 - 2\beta)^{-1}$.

The outline of this note is as follows. In Section 2, we consider, for motivation, the special case $V(x) = -\frac{1}{4}(|x| + d)^{-2}$, which is solvable in terms of Bessel functions. In addition to verifying our general results for $\beta = 2$ (see also formula (4.2)) and small λ, we check explicitly a curious behavior at $\lambda = 1$; for $\lambda < 1$, there is only one bound state, while for $\lambda > 1$, there ate as infinite number. Such a behavior was proven in general for $\beta = 2$ potentials by Simon [6]. In Section 3, we consider the cases (A), (C) and certain general features of (B) as defined above, and in Section 4 the $\lambda^2 \ln \lambda$ term in case (B) is explicitly isolated.

2. An Example

In this section we shall discuss the potential

$$\lambda V(x) = -\tfrac{1}{4}\lambda(|x| + d)^{-2}, \tag{2.1}$$

and the solution to the Schrödinger equation

$$\psi''(x) + [k^2 - \lambda V(x)]\,\psi(x) = 0.$$

The outgoing wave solution for positive x can be written in terms of Hankel functions [7]

$$\psi_+(x) = T(k(x+d))^{1/2}\,H_\nu^{(1)}[k(x+d)], \tag{2.2}$$

where T is constant and $\nu = \frac{1}{2}(1-\lambda)^{1/2}$. The solution for negative x includes an incident plane wave and a reflected wave at infinity and is written

$$\psi_-(x) = (k(d-x))^{1/2}\{H_\nu^{(2)}[k(d-x)] + RH_\nu^{(1)}[k(d-x)]\}. \tag{2.3}$$

Matching boundary conditions at the origin yields a reflection coefficient, r, and a transmission coefficient, t, of the form

$$r = Re^{2iz}e^{-i\pi(\nu+1/2)},$$
$$t = Te^{2iz}e^{i\pi(\nu+1/2)},$$

where $z = kd$ and

$$T = 4i/\pi D(z)$$

$$R = T - H_\nu^{(2)}(z)/H_\nu^{(1)}(z)$$

and

$$D(z) = (d/dz)[z(H_\nu^{(1)}(z))]^2. \tag{2.4}$$

The eigenvalue condition is equivalent to the vanishing of $D(z)$ for z pure imaginary and in the upper half plane. Defining $z = iy$, it becomes

$$y(d/dy)\,K_\nu(y) = -\tfrac{1}{2}K_\nu(y). \tag{2.5}$$

where $y = d(-E(\lambda))^{1/2}$ and recall that $\nu = \frac{1}{2}(1-\lambda)^{1/2}$ where $K_\nu(y)$ is a modified Bessel function of the second kind related to $H_\nu^{(1)}(z)$ by $K_\nu(y) = (\pi i/2)\exp(i\pi\nu/2) \times H^{(1)}(iy)$.

For $0 < \lambda < 1$, there is always one solution to (2.5), as can easily be seen by considering the limiting behavior of both sides of this equation. For small λ it is convenient to expand $K_\nu(y)$ as

$$K_\nu(y) = \frac{1}{2}\,\Gamma(\nu)\left(\frac{y}{\nu}\right)^{-\nu}\left[1 + \frac{\Gamma(-\nu)}{\Gamma(\nu)}\left(\frac{y}{2}\right)^{2\nu}\right] + O(y^{2-\nu})$$

and the eigenvalue condition can be expanded as

$$2\nu = \tanh[\nu \ln(2/y)] + O(y) \tag{2.6}$$

or

$$y = \tfrac{1}{2} - \nu[1 - (\tfrac{1}{2} - \nu) \ln 4/y^2] + O[(\tfrac{1}{2} - \nu)^2].$$

This can be rewritten in the form

$$(-E(\lambda))^{1/2} = -\tfrac{1}{2}[\lambda + 2a\lambda^2 \ln \lambda] \int dx\, V(x) + O(\lambda^2), \tag{2.7}$$

where $a = \frac{1}{4}$. We shall return in Section 4 and derive this expansion for a more general class of potentials that behave as $-ax^{-2}$ for large x.

If λ is larger than one, then one sees that the index of the Bessel function becomes complex. This introduces an oscillatory behavior and profoundly affects the spectrum. If λ is only slightly larger than unity, then a simple expansion is possible for small y. Defining $\nu = i\delta$, then the eigenvalue condition (2.6) becomes

$$2\delta = \tan[\delta \ln(2/y)] + O(y). \tag{2.8}$$

There are therefore an infinite number of bound states for $\lambda > 1$, and their energies are geometrically related for weak binding.

3. General Considerations

Throughout this section we shall consider the family of operators H defined earlier with

$$\int dx \mid V(x)| < \infty, \tag{3.1}$$

and will often add the condition

$$\int dx \mid x |^\gamma \mid V(x)| < \infty \tag{3.2}$$

for some $\gamma > 0$.

The discussion in Ref. [2] assumed Eq. (3.2) with $\gamma = 2$, and it is our purpose to extend the class of admissible potentials. We note that if $V(x) \sim -ax^{-\beta}$ for large x, and (3.1) holds, then (3.2) will also hold for $\gamma = \beta - 1 - \epsilon$ for any $\epsilon > 0$. Under condition (3.1) two results carry over from Ref. [2] (Propositions 2.1 and 2.2); namely, $E = -\alpha^2$ with $\alpha > 0$ is an eigenvalue of H if and only if

$$\det[1 + \lambda K_\alpha] = 0 \tag{3.3}$$

where K_α is the integral operator

$$K_\alpha(x, y) = \frac{1}{2\alpha} \mid V(x)|^{1/2} \exp(-\alpha \mid x - y \mid) V^{1/2}(y), \tag{3.4}$$

and $V^{1/2}(y) = | V(y)|^{1/2} \operatorname{sign}(V(y))$. Moreover,

$$H = -\frac{d^2}{dx^2} + \lambda V(x) \geqslant c\lambda$$

for some c and all small λ. Thus one needs only look for solutions to (3.3) with $0 < \alpha(\lambda) < (c\lambda)^{1/2}$ for amsll λ.

It is convenient to decompose K_α into two sets of integral operators

$$K_\alpha = Q_\alpha + P_\alpha = L_\alpha + M_\alpha .$$

First,

$$\begin{aligned} Q_\alpha(x, y) &= \frac{1}{2\alpha} e^{-\alpha|x|} \mid V(x)|^{1/2} e^{-\alpha|y|} V^{1/2}(y), \\ P_\alpha(x, y) &= \frac{1}{\alpha} \mid V(x)|^{1/2} [e^{-\alpha|x|_>} \sinh \alpha \mid x \mid_<] V^{1/2}(y), \end{aligned} \tag{3.5}$$

where $| x |_< = 0$ if $xy < 0$ and $| x |_< = \min | x |, | y |$ otherwise; $| x |_> = \max | x |, | y |$. Second,

$$\begin{aligned} L_\alpha(x, y) &= \frac{1}{2\alpha} \mid V(x)|^{1/2} V^{1/2}(y), \\ M_\alpha(x, y) &= \frac{1}{2\alpha} \mid V(x)|^{1/2} [e^{-\alpha|x-y|} - 1] V^{1/2}(y). \end{aligned} \tag{3.6}$$

In Ref. [2], the latter decomposition was used since it results in a simpler implicit equation for α as deduced from Eq. (3.3) than does the former decomposition (compare our Eq. (3.8) with (9) of Ref. [2]).

The advantage of the Q_α, P_α pair is that it is more convergent. As $\alpha \to 0$, the factor in brackets of Eq. (3.5) approaches $| x |_<$ rather than a factor of $\frac{1}{2} | x - y |$ as in Eq. (3.6). For fixed y, the latter approaches $\frac{1}{2} | x |$ for large x whereas for the former $| x |_< \to 0$ or $| y |$ as $| x | \to \infty$. Therefore P_0 is less singular than M_0.

We emphasize that P_α is very natural, it arises from replacing the Green's function in K_α by the Green's function in which a zero boundary condition is imposed at the origin. The fact that when Eq. (3.2) holds with $\gamma = 1$, then $\det(1 + \lambda P_\alpha) = 0$ has no solutions for λ small is intimately connected with Schwinger's proof [8] of Bargmann's bound [9].

In order to bound P_α independently of α, note that the elementary inequality for $x > 0$

$$x^{-1} \sinh x \leqslant \cosh x \leqslant e^x$$

leads to

$$| P_\alpha(x, y)| \leqslant | V(x)|^{1/2} | x |_< | V(y)|^{1/2} \leqslant | xV(x)|^{1/2} | yV(y)|^{1/2} \tag{3.7}$$

so that by letting

$$P_0(x, y) \equiv | V(x)|^{1/2} | x |_< V^{1/2}(y),$$

we have by the dominated convergence theorem

$$\int dx\, dy\, | P_\alpha - P_0 |^2 \to 0$$

as $\alpha \to 0$ so long as (3.2) holds with $\gamma = 1$. With this result, one can now mimic the proofs of Theorems 2.4 and 2.5 of Ref. [2] and obtain

THEOREM 3.1. *Suppose that* (3.1) *holds and that* (3.2) *holds with* $\gamma = 1$. *Then* H *has at most one negative eigenvalue for* λ *small and this occurs if and only if* $\int dx\, V(x) \leqslant 0$. *If this condition holds, then* $\alpha = (-E(\lambda))^{1/2}$ *is given by the implicit condition (expand the determinant using the fact that* Q_α *is a separable integral operator)*

$$\alpha = -\tfrac{1}{2}\lambda(e^{-\alpha|x|} V^{1/2}, (1 + \lambda P_\alpha)^{-1} e^{-\alpha|y|} | V(y)|^{1/2}) \tag{3.8}$$

and, in particular, Eq. (1.1) *holds. This immediately extends the results of Ref.* [2] *from* $x^{-3-\epsilon}$ *potentials to* $x^{-2-\epsilon}$ *potentials.*

To understand and to anticipate our next result, suppose that $V(x) = V(-x)$ and $V \sim -ax^{-\beta}$ at infinity. If ψ_0 and ψ_1 are two bound states with energies $E_0 < E_1$, then it is possible to find a linear combination $\phi(x)$ on $(0, \infty)$ that vanishes at the origin. Therefore,

$$E_1 \int_0^\infty dx\, \phi^2(x) \geqslant \int_0^\infty dx\, [(\phi')^2 + \lambda V(x)\, \phi^2(x)].$$

As is well known, if $\phi(0) = 0$, then [10]

$$\int_0^\infty dx\, (\phi')^2 \geqslant \tfrac{1}{4} \int_0^\infty dx\, \phi^2(x)\, x^{-2},$$

so that

$$E_1(\lambda) \geqslant \min(\tfrac{1}{4}x^{-2} + \lambda V(x)) \sim -\lambda^g$$

where $g = 2(2 - \beta)^{-1}$ if $\beta < 2$. Thus one expects that all bound states *except* for the ground state will have energies that behave as λ^g whereas the ground state energy will be $O(\lambda^2)$.

THEOREM 3.2. *Let* V *obey* (3.1) *and* (3.2) *for some* γ, *where* $0 < \gamma < 1$; *then there is a constant* C *so that at most one bound state occurs with an energy smaller than* $-C\lambda^h$, $h = 2(1 - \gamma)^{-1}$, *for amsll* λ. *Such a bound state will exist if* $\int dx\, V(x) < 0$, *and in that case its energy, $E(\lambda)$, is given by Eq. (3.8) with $\alpha = (-E(\lambda))^{1/2}$. In particular,*

$$(-E(\lambda))^{1/2} = -\tfrac{1}{2}\lambda \int dx\, V(x) + O(\lambda^{1+\gamma}). \tag{3.9}$$

Proof. Since

$$e^{-\alpha|x|_>} \sinh \alpha\, |x|_< \leqslant \tfrac{1}{2} e^{-\alpha(|x|_> - |x|_<)} \leqslant \tfrac{1}{2}\,,$$

then

$$|P_\alpha(x, y)| \leqslant \frac{1}{2\alpha} |V(x)|^{1/2} |V(y)|^{1/2}.$$

Recalling the bound on $|P_\alpha|$ given by Eq. (3.7), one has for $0 < \theta < 1$

$$|P_\alpha(x, y)| \leqslant \left(\frac{1}{2\alpha}\right)^{1-\theta} (xy)^{\theta/2} |V(x)|^{1/2} |V(y)|^{1/2}.$$

Therefore, the Hilbert–Schmidt norm for P_α, choosing $\theta = \gamma$, is bounded by

$$\| P_\alpha(x, y)\|_{\mathrm{HS}} \leqslant (2\alpha)^{\gamma-1} \int dx\, |x|^{\gamma} |V(x)|.$$

It now follows that if $\lambda \| P_\alpha \|_{\mathrm{HS}} < 1$, or equivalently

$$E(\lambda) < -\frac{1}{4}\left[\lambda \int dx\, |x|\, |V(x)|\right]^{h} \tag{3.10}$$

where $h = 2(1-\gamma)^{-1}$, then $(1 + \lambda P_\alpha)$ is invertible and thus for such α and λ, Eq. (3.3) has a solution if and only if (3.8) has a solution with $\alpha > 0$. Then $E = -\alpha^2$ is the unique eigenvalue and satisfies the inequality (3.10). The result now follows by mimicking the arguments given in Ref. [2].

The $O(\lambda^{1+\gamma})$ error comes from

$$\begin{aligned}\text{Error} = &-\tfrac{1}{2}\lambda\left(e^{-\alpha|x|} V^{1/2}(x), [(1 + \lambda P_\alpha)^{-1} - 1]\, |V|^{1/2} e^{-\alpha|y|}\right)\\ &-\tfrac{1}{2}\lambda \int dx\, V(x)(e^{-\alpha|x|} - 1).\end{aligned}$$

The first term in the error is of order $\lambda^2 \| P_\alpha \|_{\mathrm{HS}} = O(\lambda^2\alpha^{\gamma-1}) = O(\lambda^{1+\gamma})$ since $\alpha = O(\lambda)$. By using $(e^{-\alpha|x|} - 1) \leqslant (\alpha |x|)^\gamma$, the second term is also seen to be of order $O(\lambda\alpha^\gamma) = O(\lambda^{1+\gamma})$.

4. The Second-Order Term for $\beta = 2$ Potentials

In this final section, we will consider potentials that behave as $V \sim -ax^{-2}$ at infinity. For later convenience we will decompose V as

$$V(x) = V_1(x) + V_2(x) \tag{4.1}$$

where

$$V_1(x) = -a(1 + x^2)^{-1}$$

and demand that

$$\int dx \mid x \mid^{1+\delta} \mid V_2(x) \mid < \infty \tag{4.2}$$

for some $\delta > 0$. It will be proved that if $\int dx\, V(x) < 0$, the ground state energy obeys

$$(-E(\lambda))^{1/2} = -[\tfrac{1}{2}\lambda + a\lambda^2 \ln \lambda] \int dx\, V(x) + O(\lambda^2). \tag{4.3}$$

To motivate this result, consider the direct expansion of the determinant, Eq. (3.3); after some slight manipulations one finds to second order in λ,

$$\alpha = -\frac{1}{2}\lambda \int dx\, V(x) + \frac{\lambda^2}{8\alpha} \int_0^\infty dz\, (1 - e^{-2\alpha z}) \int_{-\infty}^{\infty} dx\, V(x)\, V(x + z).$$

The small α limit of the second term depends upon the large z behavior of the convolution integral between two V's. One estimates that

$$\int dx\, V(x)\, V(x + z) \simeq [V(z) + V(-z)] \int dx\, V(x)$$

and for even potentials one finds

$$\int_0^\infty dz\, (1 - e^{-2\alpha z})\, V(z) \simeq -2a\alpha \ln \alpha + O(\alpha^2).$$

Now by noting that $\alpha = O(\lambda)$, the expansion (4.3) immediately follows.

In order to prove this result, let us return to the eigenvalue condition (3.8) and using Theorem 3.2, where $\gamma = 1 - \epsilon$ for $\beta = 2$, one has

$$\begin{aligned} \alpha &= -\tfrac{1}{2}\lambda \int dx\, V(x)\, e^{-2\alpha|x|} + \tfrac{1}{2}\lambda^2(e^{-\alpha|x|} V^{1/2}, P_\alpha \mid V \mid^{1/2} e^{-\alpha|y|}) + O(\lambda^{3-\epsilon}) \\ &= -\tfrac{1}{2}\lambda \int dx\, V(x)\, e^{-2\alpha|x|} + \tfrac{1}{2}\lambda^2(V^{1/2}, P_\alpha \mid V \mid^{1/2}) + O(\lambda^{3-\epsilon}). \end{aligned} \tag{4.4}$$

Now the second term is most easily estimated by using the relation $P_\alpha = M_\alpha + (L_\alpha - Q_\alpha)$, and one has

$$\begin{aligned} &\tfrac{1}{2}\lambda^2(V^{1/2}, (L_\alpha - Q_\alpha) \mid V \mid^{1/2}) \\ &\quad = -\tfrac{1}{2}\lambda^2 \int dx\, dy\, V(x)\, V(y)(e^{-\alpha|x|} - 1)/\alpha + O(\lambda^{3-\epsilon}) \\ &\quad = \lambda \int dx\, V(x)(e^{-\alpha|x|} - 1)[(1/\alpha)(\alpha + O(\lambda^{2-\epsilon})] + O(\lambda^{3-\epsilon}) \\ &\quad = \lambda \int dx\, V(x)(e^{-\alpha|x|} - 1) + O(\lambda^{3-\epsilon}). \end{aligned}$$

Thus Eqs. (4.4) achieve the form

$$\alpha = -\tfrac{1}{2}\lambda \int dx\, V(x) + \tfrac{1}{2}\lambda^2(V^{1/2}, M_\alpha \mid V\mid^{1/2})$$
$$+ \tfrac{1}{2}\lambda \int dx\, V(x)[2e^{-\alpha|x|} - e^{-2\alpha|x|} - 1] + O(\lambda^{3-\epsilon}). \tag{4.5}$$

This result shows the advantage of using the P_α, Q_α decomposition, becuase if one had instead used M_α and L_α directly, the third term in (4.5) mught have been missed by assuming that $\lambda^3(V^{1/2}, M_\alpha^2, \mid V\mid^{1/2})$ is of order $\lambda^{3-\epsilon}$. However, we shall see that this term contributes to order λ^2 only and does not contribute to the $\lambda^2 \ln \lambda$ term that we wish to isolate.

Introducing the Fourier transform by

$$\hat{g}(k) = (2\pi)^{-1/2} \int dx\, g(x)\, e^{-ikx},$$

then we find that

$$\hat{V}_1(k) = -\tfrac{1}{2}a(2\pi)^{1/2}\, e^{-|k|}$$

and V_2 is continuously differentiable with

$$\mid \hat{V}_2(k) - \hat{V}_2(k')\mid \leqslant c \mid k - k'\mid^{\delta},$$

and hence by Taylor's theorem with remainder

$$\hat{V}(k) = \hat{V}(0) + \tfrac{1}{2}a(2\pi)^{1/2} \mid k\mid + ck + O(k^{1+\delta}). \tag{4.6}$$

Now using the fact that the Fourier transform of $\exp(-b|x|)$ is $(2\pi)^{-1/2}\, 2b(b^2+k^2)^{-1}$, the third term in (4.5) becomes

$$\tfrac{1}{2}\lambda \int dk\, [\hat{V}(k) - \hat{V}(0)](2\pi)^{-1/2}\, 4\alpha[(k^2 + \alpha^2)^{-1} - (k^2 + 4\alpha^2)^{-1}]. \tag{4.7}$$

The contribution to the integral in (4.7) outside the region $(-1 < k < 1)$ is easily seen to be of order $\lambda\alpha^3 = O(\lambda^4)$. From this region itself, one sees that the ck term in (4.6) contributes zero and the $O(k^{1+\delta})$ term contributes of order $\lambda\alpha^{1+1/2\delta} = O(\lambda^{2+1/2\delta})$. Finally the $\mid k\mid$ term yields (neglecting $O(\lambda^4)$ terms) $\lambda a\alpha \ln 4 = O(\lambda^2)$. As claimed, this contributes a term of order λ^2 to α.

The second term in (4.5) is

$$\frac{\lambda^2}{4\alpha} \int dx\, dy\, V(x)\, V(y)(e^{-\alpha|x-y|} - 1),$$

which can be written as

$$\tfrac{1}{2}\lambda^2 \int dk\, (k^2 + \alpha^2)^{-1}\, [\mid \hat{V}(k)\mid^2 - \mid \hat{V}(0)\mid^2]. \tag{4.8}$$

Using the expansion (4.6), we have

$$|\hat{V}(k)|^2 - |\hat{V}(0)|^2 = 2\,\mathrm{Re}[\hat{V}(0)(\hat{V}(k) - \hat{V}(0))] + |\hat{V}(k) - \hat{V}(0)|^2$$
$$= a(2\pi)^{1/2}\hat{V}(0)\,|k| + c'k + O(k^{1+\delta}).$$

Thus Eq. (4.8) is estimated to be

$$= \lambda^2 a(2\pi)^{1/2}\,\hat{V}(0)\int_0^1 dk\; k(k^2 + \alpha^2)^{-1} + O(\lambda^{2+\delta})$$

$$= -a\lambda^2 \ln \lambda \int dx\; V(x) + O(\lambda^2).$$

This then proves Eq. (4.3).

Acknowledgments

It is a pleasure to thank S. Chandrasekhar for arousing our interest in potentials with x^{-2} behavior at infinity. One of us (RB) wishes to thank the Princeton University Physics Department for their kind hospitality.

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