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A Canonical Decomposition for Quadratic Forms with Applications to Monotone Convergence Theorems

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We prove monotone convergence theorems for quadratic forms on a Hilbert space which improve existing results. The main tool is a canonical decomposition for any positive quadratic form $h = h_r + h_s$ where h_r is characterized as the largest closable form smaller than h. There is also a systematic discussion of nondensely defined forms.

1. INTRODUCTION

In this note, we wish to prove precise theorems for monotone convergence of quadratic forms on a complex Hilbert space, $\mathscr{H}$. For convenience we only consider positive forms although semibounded forms can be treated. In order to describe the existing theorems and to establish some notation, we first review some of the main ideas in the theory [2, 4]: a positive quadratic form is a sesquilinear form $t(\cdot, \cdot)$ on a domain $D(t) \times D(t)$ with $t(\varphi, \varphi) \geqslant 0$ for all $\varphi \in D(t)$. Until Section 4, we follow the standard theory and consider only the case of densely defined forms where $\overline{D(t)} = \mathscr{H}$. t is called *closed* if and only if $D(t)$ with the norm

$$\| \varphi \|_t = [t(\varphi, \varphi) + \| \varphi \|_{\mathscr{H}}^2]^{1/2}$$

is a Hilbert space. t is called *closable* if it has a closed extension.

There is a standard construction associated with forms which will be basic to our approach and which simply characterizes closable forms. $D(t)$ always has a completion $\mathscr{H}_t$ as a Hilbert space. At the risk of being pedantic, we will view the natural inclusion $D(t)$ in $\mathscr{H}_t$ as a formal mapping I_t: $D(t) \to \mathscr{H}_t$. Since

$$\| I_t \varphi \|_t \geqslant \| \varphi \|_{\mathscr{H}}$$

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the identity mapping of $D(t)$ into $\mathscr{H}$ extends to a contraction $i_t: \mathscr{H}_t \to \mathscr{H}$. If $\operatorname{Ker} i_t = \{0\}$, then $i_t[\mathscr{H}_t]$ with form $\bar{t}(\varphi, \Psi) = (I_t\varphi, I_t\Psi)_t - (\varphi, \Psi)$ is easily seen to be a closed extension of t. It is the smallest closed extension of t, called the *closure* of t. If $\operatorname{Ker} i_t \neq \{0\}$, then by choosing $\varphi_n \in D(t)$ so that $I_t(\varphi_n) \to \Psi \in \operatorname{Ker} i_t\backslash\{0\}$, we find $\| \varphi_n \|_{\mathscr{H}} \to 0$, $t(\varphi_n - \varphi_m, \varphi_n - \varphi_m) \to 0$, $t(\varphi_n, \varphi_n) \not\to 0$, showing that t is not closable. Of course $i_t I_t = 1$ on $D(t)$.

There is a one–one correspondence (see, e.g., [2, p. 331; or 4, p. 276]) between closed (positive) densely defined quadratic forms, t, and positive self-adjoint operators, T given by $D(t) = D(T^{1/2})$, $t(\varphi, \Psi) = (T^{1/2}\varphi, T^{1/2}\Psi)$. Given forms t_n, t_∞ we say that $t_n \to t_\infty$ in *strong resolvent sense* (s.r.s.) if $(T_n + 1)^{-1} \to (T + 1)^{-1}$ strongly.

Given two densely defined quadratic forms, t_1 and t_2, we write $t_1 \leqslant t_2$ if and only if $D(t_1) \supset D(t_2)$ and $t_1(\varphi, \varphi) \leqslant t_2(\varphi, \varphi)$ for all $\varphi \in D(t_2)$. One has the following basic result [2, Theorem VI.2.21]:

PROPOSITION 1.1. *Let t_1, t_2 be forms corresponding to self-adjoint operators T_1 and T_2. Then $t_1 \leqslant t_2$ if and only if $(T_2 + 1)^{-1} \leqslant (T_1 + 1)^{-1}$.*

Given a sequence of quadratic forms t_n, we define the "limit form" $\operatorname{Lim}(t_n)$ as the form given with domain, D, of those vectors $\varphi \in \bigcap_{n \geqslant N} D(t_n)$ for some N for which $\lim_n t_n(\varphi, \varphi)$ exists. We define

$$[\operatorname{Lim}(t_n)](\varphi, \varphi) = \lim t_n(\varphi, \varphi).$$

If D is a dense vector space, then we can define a quadratic form by polarization so that $[\operatorname{Lim}(t_n)](\varphi, \Psi) = \lim t_n(\varphi, \Psi)$. In this case we write t_∞ for the limit and $D(t_\infty)$ for D.

In this paper we want to consider two situations:

(A) $t_1 \geqslant t_2 \geqslant \cdots$; all t_i densely defined and *closed*;

(B) $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_0$; all t_i densely defined and *closed*.

In either case t_∞ is defined as above with $D(t_\infty) = \bigcup_n D(t_n)$ in case A and $D(t_\infty) = \{\varphi \in \bigcap_n D(t_n) \mid \sup_n t_n(\varphi, \varphi) < 0\} \supset D(t_0)$ in case B. Convergence theorems for these cases are due to Kato [2] (see also Faris [1]), who bases his proofs on the remark that by Proposition 1.1, $(T_n + 1)^{-1}$ is monotone so that it has a limit $(T + 1)^{-1}$ for some self-adjoint T: in case A, one uses $(T_n + 1)^{-1} \leqslant 1$ to be sure the limit exists and $(T_1 + 1)^{-1} \leqslant (T + 1)^{-1}$ to be sure that $(T + 1)^{-1}$ is invertible and in case B, $(T_0 + 1)^{-1} \leqslant (T + 1)^{-1}$ to be sure that $(T + 1)^{-1}$ is invertible. Kato also considers the connection between t_∞ and T. In case A, he proves that if t_∞ is closable then $t = \bar{t}_\infty$ [2, Theorem VIII.3.11] and gives an example [2, Example VIII.3.10, Remark VIII.3.12] to show that t_∞ may not be closable. In case B, he proves that if t_∞ is closable, then t_∞ is closed and $t_\infty = t$ (this is a remark in the second edition of his book) and in any case t_∞ is an extension of t.

Our goal in this paper is to prove two refinements of these results:

Case B. t_∞ is always closed.

Case A. $t = \overline{(t_\infty)_r}$ where $(\cdot)_r$ is the regular part of t_∞ defined in Section 2. Along the way we will also give direct proofs of some of Kato's results.

Our basic tool is a decomposition theorem for any quadratic form $t = t_r + t_s$ where t_r is the largest closable quadratic form less than t; the existence of such a largest object is the heart of the proof.

We also want to extend many of these ideas to nondensely defined forms. While this extension is simple, it is quite useful in applications, as we shall explain.

It is a pleasure to thank D. Mattis for raising a problem which led to my reconsideration of monotone convergence theorems for forms.

2. The Decomposition Theorem

Let t be a densely defined (positive) quadratic form. Decompose $\mathscr{H}_t = (\operatorname{Ker} i_t) \oplus (\operatorname{Ker} i_t)^\perp$ ($^\perp$ in $(\ ,\ )_t$), and let Q_t and P_t be the orthogonal projections onto the subspaces $\operatorname{Ker} i_t$ and $(\operatorname{Ker} i_t)^\perp$. Define two new quadratic forms on $D(t)$ by:

$$t_s(\varphi, \Psi) = (Q_t I_t(\varphi), I_t(\Psi))_t,$$
$$t_r(\varphi, \Psi) = (P_t I_t(\varphi), I_t(\Psi))_t - (\varphi, \Psi).$$

Clearly $t_s + t_r = t$ since

$$(t_s + t_r)(\varphi, \Psi) = (I_t(\varphi), I_t(\Psi))_t - (\varphi, \Psi) = t(\varphi, \Psi).$$

More importantly:

THEOREM 2.1. *t_r is a positive closable quadratic form.*

Proof. Let $\varphi \in D(t)$ and let $\eta = I_t(\varphi)$. Then $\varphi = i_t\eta = i_t P_t \eta$ since $i_t Q_t \eta = 0$. Thus:

$$\| \varphi \|^2 = \| i_t P_t \eta \|^2 \leqslant \| P_t \eta \|_t^2 = t_r(\varphi, \varphi) + (\varphi, \varphi)$$

so t_r is positive. Moreover, by this calculation, if we let $\tilde{I}\colon D(t) \to \operatorname{Ran} P_t \equiv \tilde{\mathscr{H}}$ by $\tilde{I}\varphi = P_t(I_t\varphi)$ and $\tilde{\imath}\colon \tilde{\mathscr{H}} \to \mathscr{H}$ by $\tilde{\imath} = i \restriction \tilde{\mathscr{H}}$, then

(i) $\operatorname{Ran} \tilde{I}$ is dense in $\tilde{\mathscr{H}}$ since P_t is continuous and $\operatorname{Ran} I_t$ is dense in $\mathscr{H}_t$,

(ii) $(\tilde{I}(\varphi), \tilde{I}(\Psi)) = t_r(\varphi, \Psi) + (\varphi, \Psi)$,

(iii) for $\varphi \in D(t)$, $\tilde{\imath}\tilde{I}(\varphi) = \varphi$ (since $i_t Q_t(I_t\varphi) = 0$),

(iv) $\tilde{\imath}$ is injective.

By (i) and (ii) we can identify $\tilde{\mathscr{H}}$ and $\mathscr{H}_{t_r}$ and by (iii) we can identify $\tilde{i}$ and i_{t_r}. Thus (iv) asserts that t_r is closable. ▮

The critical theorem in the theory of this decomposition is:

THEOREM 2.2. *Let t and s be two (positive) quadratic forms with $s \leqslant t$. If s is closable, then $s \leqslant t_r$.*

Proof. Since $D(s) \supset D(t) = D(t_r)$ we need only prove that for $\varphi \in D(t)$, $t_r(\varphi, \varphi) \geqslant s(\varphi, \varphi)$ or equivalently that:

$$\| P_t I_t \varphi \|_t \geqslant \| I_s \varphi \|_s . \tag{1}$$

Now, on $D(t)$, $\| I_s \varphi \|_s \leqslant \| I_t \varphi \|_t$ since $s \leqslant t$ by hypothesis. Thus, by extending the identity map we can define a map $j_{st}: \mathscr{H}_t \to \mathscr{H}_s$ so that $j_{st} I_t = I_s$ and $\| j_{st} \eta \|_s \leqslant \| \eta \|_t$. We claim that

$$i_s j_{st} = i_t \tag{2}$$

for (2) need only be checked on $\operatorname{Ran} I_t$ and clearly $i_s j_{st} I_t = i_s I_s = 1 = i_t I_t$.

Since s is closed, i_s is an injection so that (2) implies that $\operatorname{Ker} i_t = \operatorname{Ker} j_{st}$. Thus for $\varphi \in D(t)$, $j_{st} Q_t I_t \varphi = 0$. It follows that

$$\begin{aligned} \| I_s \varphi \|_s &= \| j_{st} I_t \varphi \|_s && \text{(by (2))} \\ &= \| j_{st} P_t I_t \varphi \|_s && (\text{since } j_{st} Q_t = 0) \\ &\leqslant \| P_t I_t \varphi \|_t && (\text{since } j_{st} \text{ is a contraction}) \end{aligned}$$

verifying (1). ▮

Theorems 2.1 and 2.2 mean that one can give a characterization of t_r and t_s which is independent of our explicit construction: namely, t_r is the largest closable quadratic form less than t and $t_s = t - t_r$. This is important because there is at least one a priori arbitrariness in the construction of t_r; namely, one could construct $\mathscr{H}_t$ with the equivalent inner product $(\ , \ )' = t(\ , \ ) + \alpha(\ , \ )$ for any fixed positive α. $\operatorname{Ker} i_t$ is not effected by this change but P_t and Q_t are. What we learn by the above characterization is that t_r and t_s are not. More generally:

COROLLARY 2.3. *Let t be a positive quadratic form and let b be the (everywhere defined) quadratic form of a bounded positive operator. Define $t + b$ on $D(t)$ in the obvious way. Then*

$$(t + b)_r = t_r + b; \qquad (t + b)_s = t_s .$$

Remarks 1. If we extend the notion here to semibounded forms the restriction that b be positive can be removed.

(2) This result does *not* extend to general closable b with $D(t) \subset D(b)$. For example, if $\mathscr{H} = L^2(R, dx)$

$$D(t) = D(b) = C_0^\infty; \qquad t(\varphi, \varphi) = |\varphi(0)|^2; \qquad b(\varphi, \varphi) = \int |\varphi'(x)|^2 \, dx \quad (3)$$

then t is easily seen to be purely singular and $b + t$ is well known to be closable [2]. Example (3) will be used several times again.

COROLLARY 2.4. *If $s \leqslant t$, then $s_r \leqslant t_r$.*

Proof. $s_r \leqslant s \leqslant t$. ∎

Remark. It is *not* true that if $s \leqslant t$, then $s_s \leqslant t_s$, for in example (3), $t \leqslant b + 1 = h$. $h_s = 0$ (h is closable) but $t_s \neq 0$. This asymmetry between singular parts and regular parts is responsible for the asymmetry between the two monotone convergence theorems of Section 3 (t_∞ automatically closed in one case but not in the other).

There is an elementary example of the decomposition that illustrates quite clearly some aspects of the theory by suggesting an analogy. Let X be a compact Hausdorff space and let μ be a (positive) Baire measure with $\int f \, d\mu > 0$ for every positive, nonzero continuous function. Let ν be another (positive) Baire measure and let $\nu = \nu_a + \nu_s$ be the Lebesgue decomposition of ν into the sum of a piece, ν_a, absolutely continuous with respect to μ and a piece, ν_s, mutually singular to μ. Let $\mathscr{H} = L^2(X, d\mu)$. Let $D(t) = C(X)$ and $t(\varphi, \Psi) = \int \bar{\varphi}(x) \Psi(x) \, d\nu$. Then $t_r(\varphi, \Psi) = \int \bar{\varphi}\Psi \, d\nu_a$ and $t_s(\varphi, \Psi) = \int \bar{\varphi}\Psi \, d\nu_s$. Thus our decomposition can be viewed as a kind of generalized Lebesgue decomposition theory. This is not surprising since our construction of the decomposition is reminiscent of von Neumann's proof [3] of the Lebesgue decomposition theory. From this point of view the fact that closed forms are associated to self-adjoint operators can be thought of as a kind of Radon–Nikodym theorem! The big difference between forms and measures is the existence of closable forms dominating singular ones. The above analogy suggests that one think of the decomposition of two positive sesquilinear forms on a vector space. In fact, by the methods above one finds without trouble that:

THEOREM 2.5. *Let $w(\cdot, \cdot)$ and $t(\cdot, \cdot)$ be two positive semidefinite forms on a fixed complex vector space V. Then there is a canonical decomposition $t = t_r + t_s$ where t_r is the largest form obeying* (i) $t_r \leqslant t$, (ii) $\mathrm{Ker}(t_r) \supseteq \mathrm{Ker}(w)$, (iii) t_r *"lifted" to the Hilbert space $\overline{V/\mathrm{Ker}(w)}^w$ is a closable form.*

This result may be of interest if applied to states on C^*-algebras. One interesting open question is whether there is a canonical description of t_s independent of either $t_s = t - t_r$ or the construction and analogous to mutual singularity of measures.

3. Monotone Convergence Theorems

We emphasize that parts of the two theorems below are due to Kato; see Section 1 and [2, Chap. VIII].

Theorem 3.1. *Let $h_0, h_1, \ldots,$ be closed densely defined (positive) forms on a Hilbert Space. Suppose that*

$$h_1 \leqslant h_2 \leqslant \cdots \leqslant h_0 .$$

Let $D(h_\infty) = \{\varphi \in \bigcap_n D(h_n) \mid \sup_n h_\infty(\varphi, \varphi) < \infty\}$ *with*

$$h_\infty(\varphi, \Psi) = \lim_{n\to\infty} h_n(\varphi, \Psi)$$

(which exists by polarization). Then h_∞ is a densely defined closed form and $h_n \to h_\infty$ in s.r.s.

Proof. Since $D(h_\infty) \supset D(h_0)$, h_∞ is densely defined. Moreover, since h_n is closable, $h_n \leqslant (h_\infty)_r$. It follows that $h_\infty \leqslant (h_\infty)_r$ so that $h_\infty = (h_\infty)_r$ is closable.

Let $\varphi_m \in D(h_\infty)$ with $h_\infty(\varphi_n - \varphi_m, \varphi_n - \varphi_m) \to 0$; $\| \varphi_n - \varphi_m \|_{\mathscr{H}} \to 0$. First note that $h_k(\varphi_n - \varphi_m, \varphi_n - \varphi_m) \to 0$ so $\varphi \in D(h_k)$ since h_k is closed. Then, since

$$|(h_\infty(\varphi_n, \varphi_n))^{1/2} - (h_\infty(\varphi_m, \varphi_m))^{1/2}| \leqslant (h_\infty(\varphi_n - \varphi_m, \varphi_n - \varphi_m))^{1/2}$$

$\sup_m h_\infty(\varphi_m, \varphi_m) < \infty$. Thus

$$\begin{aligned}
\sup_n h_n(\varphi, \varphi) &= \sup_n \lim_m h_n(\varphi_m, \varphi_m) \\
&\leqslant \sup_n \sup_m h_n(\varphi_m, \varphi_m) \\
&= \sup_m \sup_n h_n(\varphi_m, \varphi_m) \\
&= \sup_m h_\infty(\varphi_m, \varphi_m) < \infty.
\end{aligned}$$

Thus $\varphi \in D(h_\infty)$. Let $\Psi_n = \varphi_n - \varphi$. Then Ψ_n is Cauchy in $h_\infty^{1/2}$ and $\| \Psi_n \| \to 0$ so $h_\infty(\Psi_n, \Psi_n) \to 0$ since h_∞ is closable. It follows that h_∞ is closed.

By Proposition 1.1, $(H_n + 1)^{-1}$ is monotone decreasing to a limit $(H + 1)^{-1}$ which is invertible since $(H + 1)^{-1} \geqslant (H_0 + 1)^{-1}$. Let h be the quadratic form of h. Then $(H_n + 1)^{-1} \geqslant (H + 1)^{-1}$ implies that $h_n \leqslant h$ so $h_\infty \leqslant h$. It follows that $(H_n + 1)^{-1} \geqslant (H_\infty + 1)^{-1} \geqslant (H + 1)^{-1}$ from which it follows that $H = H_\infty$. ∎

Theorem 3.2. *Let $h_0, h_1, \ldots$ be closed densely defined (positive) forms on a Hilbert space $\mathscr{H}$. Suppose that*

$$h_1 \geqslant h_2 \geqslant \cdots.$$

Let $D(h_\infty) = \bigcup_n D(h_n)$ *and* $h_\infty(\varphi, \varphi) = \lim_n h_n(\varphi, \varphi)$. *Then* h_n *converges in* s.r.s. *to* $\overline{(h_\infty)_r}$.

Proof. By Proposition 1.1, $(H_n + 1)^{-1}$ is increasing to a limit $(H + 1)^{-1} \leqslant 1$. Let h be the form of H. Then $h_n \geqslant h$ so $h_\infty \geqslant h$. It follows that $(h_\infty)_r \geqslant h$ (by Theorem 2.2) and thus $\overline{(h_\infty)_r} \geqslant h$ (for, under general circumstances, if s and t are closable and $0 \leqslant s \leqslant t$, then $\bar{s} \leqslant \bar{t}$). It follows, as at the end of the proof of Theorem 3.1 that $\overline{h(_\infty)_r} = h$.

Remark. It can happen in Theorem 3.2 that h_∞ is not closable. See Remark VIII.3.12 of [2]. See Section 2 for a discussion of the asymmetry between Theorems 3.1 and 3.2.

4. Nondensely Defined Forms

The extension of the usual theory of densely defined forms to the general case is quite elementary. Our primary purpose here is not so much to present this extension as to present propaganda for the general case. Consider the following two examples:

Example 1. Let $\Omega \subset R^\nu$ be open. The form $t_\Omega{}^D(\varphi, \Psi) = \int \overline{(\nabla\varphi)}(\nabla\Psi)\, d^\nu x$ on $C_0{}^\infty(\Omega)$ is a densely defined closable form on $L^2(\Omega)$. The closure defines the Dirichlet Laplacian $-\Delta_\Omega{}^D$ on Ω. Now let $\Omega_1 \subset \Omega_2 \subset \cdots$ and let $\Omega = \bigcup_{n=1}^\infty \Omega_n$. In a natural sense $t_{\Omega_n}^D \downarrow t_\Omega{}^D$ as forms on $L^2(\Omega)$ but the $t_{\Omega_n}^D$ are not densely defined.

Example 2. Let $\Omega \subset R^\nu$ be an open set. Let h_C be the quadratic form of $-\Delta + C\chi$ where χ is the characteristic function of $R^\nu \backslash \Omega$. On the basis of Wiener path integrals, one can see that as $C \to \infty$

$$\exp[-t(-\Delta + C\chi)](x, y) \to \exp(-t\Delta_\Omega{}^D)(x, y)$$

if $x, y \in \Omega$ and zero for other x, y. In some sense, $h_C \uparrow t_\Omega{}^D$, but $-\Delta_\Omega{}^D$ is not densely defined.

Let t be a quadratic form on $D(t) \subset \mathscr{H}$, with $D(t)$ not necessarily dense. The notions of closed, closability, and closure are unchanged. If t is closed, we define its *resolvent* as the operator which is $(T - z)^{-1}$ on $\overline{D(t)}$ and zero on $D(t)^\perp$. More generally, if F is a bounded continuous function on R, then $F(t)$ is the operator which is zero on $D(t)^\perp$ and $F(t)$ on $D(t)$. We have

Theorem 4.1. *The convergence theorems of Section* 3 *extend to arbitrary closed forms with the addendum that in Theorem* 3.1, *one only needs* $h_1 \leqslant h_2 \leqslant \cdots$ *without the necessity of an upper bound* h_0.

The proofs of Section 3 require no changes. The notion of s.r.s. convergence is defined as before. Define $F(t)$ as above.

THEOREM 4.2. *Let $t_n \to t$ in* s.r.s. *If F is a continuous function vanishing at infinity, then $F(t_n) \to F(t)$ strongly. If F is any bounded continuous function, then $F(t_n)\varphi \to F(t)\varphi$ in (strong) $\mathscr{H}$-topology for any $\varphi \in \overline{D(t)}$.*

This can be proven by mimicking the proof of Theorem VIII 20 in [4]. With these two results, we can return to Examples 1 and 2. Clearly, in Example 1, $t_{\Omega_n}^D \downarrow t_{\Omega}{}^D$ in the sense of the extended Theorem 3.2. This gives a continuity of Dirichlet Green's functions from within. Such a result allows one to extend the well-known connection between Dirichlet Green's functions and Wiener path integrals from regions with smooth boundary to arbitrary open regions (this can also be proven directly [5]). Example 2 is discussed in the next section.

5. SOME DIFFERENTIAL OPERATORS

Consider first the one-dimensional case:

THEOREM 5.1. *Let $H_0 = \sum_{m=0}^{2n} a_m(-i\, d/dx)^m$ with $a_{2n} > 0$, a_m real on $D(H_0) = \{\Psi \mid \hat{\Psi}$ the Fourier transform obeys $\int | k^{2n}\hat{\Psi} |^2\, dk < \infty\}$. Let $H_C = H_0 + C\chi_{(-\infty,0)}$. Let H_∞ be the operator on $L^2(0, \infty)$, given by the differential operator $\sum a_m(-i(d/dx))^m$ with boundary condition $\Psi(0) = \Psi'(0) = \cdots = \Psi^{(n-1)}(0) = 0$. Then $H_C \to H_\infty$ in* s.r.s. *as $C \to \infty$.*

Proof. Let h_C, h_∞ be the corresponding quadratic forms and let h_0 be the form of H_0. Then $D(h_C) = D(h_0) = D((d/dx)^n) = \{\Psi \mid \Psi$ is C^{n-1} and its $(n-1)$ derivative is absolutely continuous with L^2 derivative$\}$. Now, if $\Psi \in D(h_C)$ and $\sup_C h_C(\Psi, \Psi) < \infty$, then $\Psi(x) = 0$ for a.e. $x < 0$ and thus for all $x \leqslant 0$. Since Ψ is $C^{(n-1)}$ it follows that

$$\{\Psi \mid \sup_C h_C(\Psi, \Psi) < \infty\}$$
$$= \{\Psi \in D(h_0) \mid \Psi(x) = 0,\ x < 0;\ \Psi(0) = \cdots = \Psi^{(n-1)}(0) = 0\}$$

which proves the theorem by Theorem 3.1 extended. ▌

For the multidimensional case, we conjecture that for Ω any open set in R^ν whose compliment is perfect $H_0 + c\chi_{R^\nu \backslash \Omega} \to H_{0,\Omega}^D$, the closure of the quadratic form on $C_0^\infty(\Omega)$, in s.r.s. as $c \to \infty$.

THEOREM 5.2. *Let Ω be an open set in R^ν with the property that each connected component of $R^\nu \backslash \Omega$ has the property of either having a smooth boundary or being* *star shaped. Let* $H_0 = [\sum_{|\alpha|=0}^{2n-1} a_{m,\alpha}(iD)^\alpha + (-\Delta)^n]$. *Then* $H_0 + c\chi_{R^\nu \setminus \Omega} \to H_{0,\Omega}^D$ *as* $c \to \infty$.

Proof. Clearly $H_0 + c\chi \uparrow$ so that in s.r.s. there is convergence to an operator H_∞ in s.r.s. Now clearly $D(h_\infty) = \{\Psi \in D(h_0) \mid \Psi = 0$ a.e. on $R^\nu \setminus \Omega\}$. It suffices to approximate such Ψ's in h_0-norm by functions in $C_0^\infty(\Omega)$. By a partition of unity argument we need only prove this for Ψ living in a neighborhood of the boundary of a component. If it is star shaped we dialate about the star shaped point and then mollify. The smooth case is similar.

Added Note. We have learned that T. Kato has found a very different proof that t_∞ is closed in Case B. Kato's work is independent and approximately simultaneous to ours.

Note added in proof. That t_∞ is closed in case B has been proven implicitly by D. W. Robinson, *The Thermodynamic Pressure in Quantum Statistical Mechanics*, Springer, 1971 and E. B. Davies, *Helv. Phys. Acta.* **48** (1975), 365–382. Both authors use methods similar to those that Kato has used and very different from ours.

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