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Lower semicontinuity of positive quadratic forms

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SYNOPSIS

We develop various facets of the theory of quadratic forms on a Hilbert space suggested by a criterion of Kato which characterizes closed forms in terms of lower semicontinuity.

1. INTRODUCTION

In this note, we consider a variety of aspects of the theory of positive quadratic forms, t, on a complex Hilbert space, $\mathscr{H}$, that is sesquilinear forms, $t(.,.)$ on $D(t)\times D(t)$ with $D(t)\subset\mathscr{H}$, a subspace with $t(\varphi,\varphi)\geqq 0$ for all $\varphi\in Q(t)$. We recently proved the following result [**8**]: see added note therein for earlier proof of D. Robinson.

THEOREM 1. *If* $t_1\leqq t_2\leqq\ldots$ *is a sequence of increasing* ***closed*** *quadratic forms, then the form* t *with* $D(t)=\{\varphi\in\bigcap D(t_n)\mid \sup_n t_n(\varphi,\varphi)<\infty\}$ *and* $t(\varphi,\psi)=\lim_n t_n(\varphi,\psi)$ *is closed and the corresponding self-adjoint operators* T_n *converge to* T *in strong resolvent sense.*

To understand this result, we recall [**3, 6**] that t is called closed if $Q(t)$ is complete in the norm $\|\varphi\|_t=(t(\varphi,\varphi)+\|^2)^{\frac{1}{2}}$, that $t_1\leqq t_2$ means $D(t_1)\supset D(t_2)$ and $t_1(\varphi,\varphi)\leqq t_2(\varphi,\varphi)$(all $\varphi\in D(t_2)$) and that there is a one to one correspondence between closed quadratic forms and operators T which are self-adjoint on $\overline{D(T)}$. Theorem 1 is a strengthening of a result in [**3**].

This note had its genesis in an attempt to understand the relationship of our proof of Theorem 1 and a very different (unpublished) proof of Kato of the same result. (Our work was done without knowledge of each others interest and the two proofs were obtained within a single week!) His proof depends on an important preliminary result: Define $\tilde{t}$, a function from all of $\mathscr{H}$ to $[0,\infty]$ by:

$$\tilde{t}(\varphi)=t(\varphi,\varphi)\quad\text{if}\quad\varphi\in D(t)$$

$$=\infty\qquad\text{if}\quad\varphi\notin D(t).$$

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Then Kato notes:

THEOREM 2. *t is a closed quadratic form if and only if $\tilde{t}$ is lower semicontinuous (l.s.c.) i.e. if $\varphi_n \to \varphi$ in $\mathcal{H}$, then*

$$\tilde{t}(\varphi) \leqq \liminf \tilde{t}(\varphi_n).$$

Theorem 1 follows from Theorem 2 and the fact that a supremum of l.s.c. functions is l.s.c. We provide a proof of Theorem 2 in §2.

Our proof of [**8**, Theorem 1] relies on a decomposition $t = t_r + t_s$ of any quadratic form into two positive forms where t_r is defined as the largest closable form less than t. The connection between the proofs is given by the following result proven in §3:

THEOREM 3. $\underline{(\bar{t}_r)^{\sim}(\varphi)} = \liminf_{\psi \to \varphi} \underline{\tilde{t}(\psi)}$

In theorem 3, the —— indicates closure and the symbol $\liminf_{\psi \to \varphi}$ indicates that we take the inf over all $\liminf_{n \to \infty}$ for all ψ_n's with $\psi_n \to \varphi$.

This circle of ideas also allows one to solve the following problem: Suppose that T_n converges to T in strong resolvent sense, with T_n, T positive and self-adjoint. Let t_n, t be the corresponding quadratic forms. How is t related to t_n? The many pathologies that can occur with unbounded forms suggest that there might not be any kind of explicit formula for t in terms of t_n. There is such a formula but it is not simple. In §4, we will prove:

THEOREM 3. *Under the above hypothesis*:

$$\underline{\tilde{t}(\varphi)} = \liminf_{\psi_n \to \varphi} \underline{\tilde{t}_n(\psi_n)}.$$

To see that this determines t, we note the polarization formula:

$$t(\varphi, \psi) = \tfrac{1}{4}[\tilde{t}(\varphi+\psi) - \tilde{t}(\varphi-\psi) - i\tilde{t}(\varphi+i\psi) + \tilde{t}(\varphi-i\psi)] \tag{1}$$

We also note that $\tilde{t}$ defines a sesquilinear form via (1), if and only if $\tilde{t}$ obeys the parallelgram law

$$\tilde{t}(\varphi+\psi) + \tilde{t}(\varphi-\psi) = 2\tilde{t}(\varphi) + \tilde{t}(\psi) \tag{2}$$

In §4, we will see also how theorem 1 follows from theorem 4 and provide a new proof of the following result from [**8**]:

THEOREM 3. *If $t_1 \geqq \ldots \geqq t_n \geqq \ldots$ and t is defined on $\cup D(t_n)$ by $t(\varphi, \psi) = \lim t_n(\varphi, \psi)$, then $T_n \to T_\infty$ in strong resolvent sense where t_∞, the form of T_∞, obeys*

$$t_\infty = \bar{t}_r$$

Finally in §5, we use ideas from [**7**] to prove a result on the following problem: Let $t_n \uparrow t$ and $s_n \downarrow s$. When does $T_n + S_n$ converge to $T + S$ in strong resolvent senses. Our result is considerably stronger than that of Schechter.

It is a pleasure to thank M. Schechter for showing me [**7**] before publication

and T. Kato for valuable correspondence and for permission to publish a proof of his unpublished result, Theorem 2.

2. Kato's Convergence Criterion

We want to give here a proof of Kato's result, Theorem 2, which is motivated in part by the role that l.s.c. conditions play in the theory of maximal monotone operators [**1**]. We will actually prove a slightly stronger result. We will show

(a) If t is closed and $\varphi_n \to \varphi$ *weakly* then $\liminf \tilde{t}(\varphi_n) \geqq \tilde{t}(\varphi)$

(b) If t is closable but not closed, there exist $\varphi_n \to \varphi$ in norm with $\tilde{t}(\varphi_n)$ converging to a finite limit and $\tilde{t}(\varphi) = \infty$ (but if $\tilde{t}(\varphi) < \infty$, then $\liminf \tilde{t}(\varphi_n) \geqq \tilde{t}(\varphi)$).

(c) If t is not closable, then these exist $\varphi_n \to \varphi$ in norm with $\tilde{t}(\varphi) < \infty$, and $\lim \tilde{t}(\varphi_n) < \tilde{t}(\varphi)$.

The weak l.s.c. condition (a), for certain explicit t's is used heavily in [**4, 5**] on the existence of solutions of the non-linear equations of atomic physics.

Proof of (*a*). Let T be the operator corresponding to t. Then, we claim that

$$\tilde{t}(\varphi) = \sup\{|(\varphi, T\psi)|^2 \mid \psi \in D(T), (\psi, T\psi) = 1\} \tag{3}$$

(3) follows by noting that $|(\varphi, T\psi)|^2 \leqq \tilde{t}(\varphi)\tilde{t}(\psi)$ by the Schwarz inequality and choosing $\psi_n = ((\varphi, TP_{[0,n]}(T)\varphi))^{-\frac{1}{2}} P_{[0,n]}(T)\varphi$ with $P_{[0,n]}(T)$ the spectral projection for T. Now, let $\varphi_n \to \varphi$ weakly. Then for any $\psi \in D(T)$ with $(\psi, T\psi) = 1$:

$$|(\varphi, T\psi)| = \lim |(\varphi_n, T\psi)| \leqq \liminf \tilde{t}(\varphi_n)$$

so that (3) implies the l.s.c. $\tilde{t}(\varphi) \leqq \liminf \tilde{t}(\varphi_n)$.

Proof of (*b*). Let $s = \bar{t}$. Since $\tilde{s}$ is l.s.c. and $\tilde{s} \leqq \tilde{t}$, $\tilde{t}(\varphi) \leqq \liminf \tilde{t}(\varphi_n)$ so long as $\tilde{t}(\varphi) = \tilde{s}(\varphi)$, and, in particular if $\tilde{t}(\varphi) < \infty$. Since t is not closed, pick $\varphi \notin D(t)$, $\varphi \in D(s)$, $\varphi_n \in D(t)$ st. $\varphi_n \to \varphi$ in $\mathcal{H}_s$, $= D(s)$ with the norm $\|\cdot\|_s$. Then $\tilde{t}(\varphi_n) \to \tilde{s}(\varphi) < \infty = \tilde{t}(\varphi)$ so $\tilde{t}$ is not l.s.c.

Proof of (*c*). Let $\mathcal{H}_t$ be the completion of $Q(t)$ with the norm $\|\cdot\|_t$. Let $i_t: \mathcal{H}_t \to \mathcal{H}$ by extending the identity map from $D(t)$ to $\mathcal{H}_t$. Since $\mathcal{H}_t$ is not closable, *Ker* $i_t \neq \{0\}$ Pick $\varphi \in \mathcal{H}_t$ with $\|\varphi\|_t = 1$, $i_t(\varphi) = 0$. Choose $\varphi_n \in D(t)$ st. $\varphi_n \to \varphi$ in $\mathcal{H}_t$. Since i_t is continuous $\varphi_n \to 0$ in $\mathcal{H}$. (notice that the φ_n's obey $\tilde{t}(0) = 0 < \lim \tilde{t}(\varphi_n) = \|\varphi\|_t^2 - \|\varphi\|^2 = 1$ so that on the surface, the existence of this sequence seems only to imply that $\tilde{t}$ is not u.s.c.). Now, clearly $(\varphi_n, \varphi_m)_t \to 1$ as $n, m \to \infty$, so pick N with $\mathrm{Re}\,(\varphi_n, \varphi_m)_t > \frac{1}{2}$ if $n, m \geqq N$. Let $\psi_n = \alpha\varphi_n - \varphi_N$ with $\alpha \geqq 0$, to be picked below. Then $\psi_n \to \varphi_N$ in $\mathcal{H}$. Moreover, for $n, m \geqq N$,

$$\begin{aligned} \tilde{t}(\psi_n) &= \tilde{t}(\varphi N) + |\alpha|^2 \tilde{t}(\varphi_n) - 2\,\mathrm{Re}\,[\alpha t(\varphi_N, \varphi_n)] \\ &\leqq \tilde{t}(\varphi_N) + |\alpha|^2 \tilde{t}(\varphi_n) - \alpha \end{aligned}$$

Since $\tilde{t}(\varphi_n) \to 1$, $\sup_n \tilde{t}(\varphi_n) < \infty$. Picking α small, we see that $\tilde{t}(\psi_n) \leqq \tilde{t}(\varphi_N) - \frac{1}{2}\alpha$ for all $n \geqq N$ so $\tilde{t}\,(\lim \psi_n) > \liminf \tilde{t}(\psi_n)$, i.e. $\tilde{t}$ is not l.s.c. ■

3. The Regular Part as a Lim Inf

In this section, we prove Theorem 3. Before giving the proof, we make a remark: Suppose that s is closable and $s \leqq t$. Then

$$q(\varphi) \equiv \liminf_{\psi \to \varphi} \bar{t}(\psi) \geqq (\bar{s})(\varphi).$$

It follows that if one knew that q were the diagonal part of some form, that form would be $\bar{t}_r$ and we would have a proof that $\bar{t}_r$ existed independent of the construction in **[8]**. The point is that it is not a priori obvious that q obeys the parallelogram law (2).

Proof of Theorem 3. Let $s = \bar{t}_r$. Then $s \leqq t_r \leqq t$ so by Theorem 2:

$$\bar{s}(\varphi) \leqq \liminf_{\psi \to \varphi} \bar{s}(\psi) \leqq \liminf_{\psi \to \varphi} \bar{t}(\psi)$$

To prove the converse, we need to recall the construction of t_r in **[8]**. As in 2, let $\mathcal{H}_t$ be the completion of $D(t)$ in $\|\cdot\|_t$ and let $i_t : \mathcal{H}_t \to \mathcal{H}$. To avoid confusion, let us denote the identification map of $D(t)$ into $\mathcal{H}_t$ by I_t. Let P_t be the $\mathcal{H}_t$-orthogonal projection onto the $\mathcal{H}_t$-orthogonal complement of Ker i_t. Then for $\varphi, \psi \in D(t)$:

$$t_r(\varphi, \psi) = (P_t\varphi, \psi)_t - (\varphi, \psi) \tag{4}$$

Now, by construction [8], $D(s)$ is the image of Ran P_t under i_t. Given $\varphi \in D(s)$, choose $\omega \in \text{Pan } P_t$ so that $\varphi = i_t\omega$. Choose $\psi_n \in D(t)$ so that $I_t(\psi_n) \to \omega$ in $\mathcal{H}_t$. Then $\psi_n = i_t I_t(\psi_n) \to i_t\omega = \varphi$. Moreover, by (4)

$$\|\varphi\|^2 + \bar{s}(\varphi) = \|\omega\|_t^2 = \lim_{n \to \infty} \|I(\psi_n)\|_t^2$$

$$= \lim_{n \to \infty} [\bar{t}(\psi_n) + \psi_n, \psi_n)]$$

It follows that $\bar{s}(\varphi) = \lim_{n \to \infty} \bar{t}(\psi_n)$ so

$$\bar{s}(\varphi) \geqq \liminf_{\psi \to \varphi} \bar{t}(\psi) \tag{5}$$

Since this inequality is obvious for $\varphi \notin D(s)$, (5) holds for all φ. (3 and 5) complete the proof. ■

4. A Convergence Criterion

Proof of Theorem 4. Suppose first that $\varphi \in \overline{D(T)}$. For each x, pick a continuous function f_x with $f_x(y) \to 0$ as $y \to \infty$, $0 \leqq f_x(y) \leqq y$ for all $y \geqq 0$ and $f_x(y) = y$ for $y \leqq x$. Then $f_x(T_n) \to f_x(T)$ strongly (for densely defined operators, this is proven in **[6]**; see **[8]** for the elementary extension) so that, for any φ and $\psi_n \to \varphi$:

$$(\varphi, f_x(T)\varphi) = \lim_{n \to \infty} (\psi_n, f_x(T_n)\psi_n)$$

$$\leqq \liminf_{n \to \infty} (\psi_n, T_n, \psi_n) = \liminf_n \bar{t}_n(\psi_n)$$

Since $\varphi \in \overline{D(T)}$, $\bar{t}(\varphi) = \lim_{x \to \infty} (\varphi, f_x(T)\varphi)$ so

$$\bar{t}(\varphi) \leqq \liminf_{\psi_n \to \varphi} \bar{t}_n(\psi_n)$$

Conversely let $\varphi \in Q(T) = D(t)$. Let $\psi_n = (T_n + 1)^{-\frac{1}{2}}(T+1)^{\frac{1}{2}}\varphi$ Then $\psi_n \to \varphi$ and $\bar{t}_n(\psi_n) + \|\psi_n\|^2 = \bar{t}(\varphi) + \|\varphi\|^2$ so

$$\bar{t}(\varphi) \geqq \liminf_{\psi_n \to \varphi} \bar{t}_n(\psi_n)$$

Since this is obvious for $\varphi \notin D(t)$, we have established the theorem for $\varphi \in \overline{D(t)}$.

Now suppose $\varphi \notin \overline{D(T)}$ and for each x pick a continuous function g_x going to zero as $y \to \infty$ with $g_x(y) = 1$ if $y \leqq x$ and $0 \leqq g_x \leqq 1$. Then, since $\varphi \notin \overline{D(T)}$, $\lim_{x \to \infty} (\varphi, g_x(T)\varphi) < \|\varphi\|^2$. Since $g_x(T_n) \to g_x(T)$ strongly:

$$\liminf_n \bar{t}_n(\psi_n) \geqq \liminf_n [x(\psi_n, [1 - g_x(T_n)]\psi_n]$$

$$\geqq x(\varphi, (1 - g_x(T))\varphi)$$

Since $\lim_{x \to \infty} (\varphi, (1 - g_x(T))\varphi) > 0$, $\liminf_{n \to \infty} \bar{t}_n(\psi_n)$ must be ∞. ■

One might hope that Theorem 4 would have some kind of converse. This seems unlikely because even for bounded operators, any kind of convergence for t_n to t is no stronger than weak convergence of T_n to T. This weak convergence does not even imply weak convergence of the resolvents [since weak convergence of resolvents implies strong convergence of resolvents **6**].

Example 1. Let us see how Theorems 2 and 4 lead to another proof of Theorem 1. Let t_n be increasing and closed. Let t_∞ be the form with $D(t_\infty) = \{\varphi \in \cap D(t_n) \mid \sup_n \bar{t}_n(\varphi) < \infty\}$ and $\bar{t}_\infty(\varphi) = \sup_n \bar{t}_n(\varphi)$. Since the t_n increase, $(T_n + 1)^{-1}$ are decreasing and so they converge strongly to some $(T+1)^{-1}$. We want to show that $\bar{t} = \bar{t}_\infty$. By theorem 4,

$$\bar{t}(\varphi) = \liminf_{\psi_n \to \varphi} \bar{t}_n(\psi_n).$$

Taking $\psi_n = \varphi$, we see that $\bar{t}(\varphi) \leqq \bar{t}_\infty(\varphi)$. But since $\bar{t}_n \geqq \bar{t}_m$ for $n \geqq m$, and each t_m is closed:

$$\liminf_{\psi_n \to \varphi} \bar{t}_n(\psi_n) \geqq \liminf_{\psi_n \to \varphi} \bar{t}_m(\psi_n) \geqq \bar{t}_m(\varphi)$$

by Theorem 2. Thus $\bar{t}(\varphi) \geqq \bar{t}_\infty(\varphi)$. We conclude that $\bar{t} = \bar{t}_\infty$.

Example 2. Let us see how Theorems 3 and 4 lead to a new proof of Theorem 5. Let t_n be decreasing and closed. Let t_∞ be the form with $D(t_\infty) = \cup D(t_n)$ and $\bar{t}_\infty(\varphi) = \inf_n \bar{t}_n(\varphi)$. Since $(T_n + 1)^{-1}$ increase, they have a strong limit $(T+1)^{-1}$. We want to show that $t=\overline{(t_\infty)}_r \equiv s$. Now by Theorem 4 and $t_n \geqq t_\infty$

$$\bar{t}(\varphi)=\liminf_{\psi_n\to\varphi} \bar{t}_n(\psi_n) \geqq \liminf_{\psi_n\to\varphi} \bar{t}_\infty(\psi_n)=\bar{s}(\varphi)$$

by Theorem 3.

Given N, on account of Theorem 2, we can find η_N so that $\|\eta_N-\varphi\| \leqq \frac{1}{N}$ and $\bar{t}_\infty(\eta_N) \leqq \bar{s}(\varphi)+\frac{1}{N}$. Now define $m(N)$ inductively so that $m(N) \geqq N$, $m(N) \geqq m(N-1)$ and

$$t_n(\eta_N) \leqq s(\varphi)+\frac{2}{N} \quad \text{if} \quad n \geqq m(N)$$

Now let $k(n)=\sup\{N \mid m(N) \leqq n\}$, which is finite since $m(n+1)>n$. Moreover, $k(n)$ is monotone increasing and $k(n) \nearrow \infty$ as $n\to\infty$ since $k(m(N)) \geqq N$. Let $\psi_n=\eta_{k(n)}$. Then $\|\psi_n-\varphi\| \leqq 1/k(n)$ and $t_n(\psi_n) \leqq s(\varphi)+2/k(n)$. It follows that $\liminf_{n\to\infty} t_n(\psi_n) \leqq s(\varphi)$ so $\bar{t}(\varphi) \leqq \bar{s}(\varphi)$ completing the proof of Theorem 5.

§5. Sums of Monotone Forms

In this section we want to consider positive self-adjoint operators A_n, B_n with forms a_n, b_n. Suppose $a_1 \leqq \ldots \leqq a_n \leqq \ldots$ and $b_1 \geqq \ldots \geqq b_n \geqq \ldots$. Let a_∞ and b_∞ be the limit forms described in examples 1 and 2 of §4. Let A be the operator associated to $a=a_\infty$ (which is closed) and B the operator associated to $b=\overline{(b_\infty)}_r$ so that $A_n \to A$ and $B_n \to B$ in strong resolvent sense. Here we want to know when a_n+b_n (the form sum is always defined on $D(a_n)\cap D(b_n)$) has a corresponding operator $A_n \dotplus B_n$ converging to $A \dotplus B$, the operator associated to $a+b$. Unfortunately this is not always true as the following show:

Example 3. Let $\mathcal{H}=L^2(-\infty,\infty)$ and let $a_n=-d^2/dx^2$, $b_n=n^{-1}a_n+\delta(x)$. Then $b_n \downarrow b_\infty=\delta(x)$ and $b=0$. $a_n \uparrow a_\infty=a=-d^2/dx^2$. $a_n+b_n \downarrow -d^2/dx^2+\delta(x)$ which is a closed form different from $a+b=a$.

Example 4. In example 3, b_∞ is not closable. But even if it is, there can be trouble. For example, let $a_n=a=\exp(x^4)$ independent of n. Let $b_n=\frac{1}{n}b$ where $b=\exp((-\Delta)^2)$. Then $b_\infty=0$ on $D(b)$ and $\bar{b}_\infty=0$ on all of $\mathcal{H}$. Clearly $a+b=a$. But by a general theorem [2], $D(a)\cap D(b)=\{0\}$, so $a_n+b_n=0$ on $\{0\}$.

Remarks 1. Example 4 is somewhat artificial in that $D(a_n)\cap D(b_n)=\{0\}$. It should be possible to modify it so that a_n is n-dependent with, say a_n bounded but $D(a_\infty)\cap D(b)=\{0\}$.

2. As examples 3 and 4 show, if $a_n \downarrow a_\infty$, $b_n \downarrow b_\infty$, it can happen that $A_n \dotplus B_n$ does not converge to $A \dotplus B$. We do not consider this phenomena in detail since it is clear what the limit is, namely $\lim (A_n \dotplus B_n)$ has a form $\overline{(a_\infty+b_\infty)}_r$ which may not equal $\overline{(a_\infty)}_r+\overline{(b_\infty)}_r$.

Theorem 6. *Under the above hypothesis and notation, suppose also that b_∞ is closed. Then $A_n \dotplus B$ converges in strong resolvent sense to $A \dotplus B$ if*

Remarks 1. More generally, $A_n \dotplus B_n$ converges to $A+B$ if

$$\sup_n [\overline{(a_n+b_\infty)}_r]=a_\infty+\overline{(b_\infty)}_r.$$

2. Our proof is closely patterned after an argument in [7] and the result is generalized in Theorem 6.

Proof. Let $c_{n,m}=a_n+b_m$. Let $c_{n,\infty}=a_n+b_\infty$ and $c_{\infty,m}=a_\infty+b_m$. Since, b_∞ is closed by hypothesis, all the forms in question are closed. Let A_n, B_m, $C_{n,m}$ etc. denote the corresponding operators. Then $c_{n,\infty} \leqq c_{n,n} \leqq c_{\infty,n}$, so that

$$(C_{n,\infty}+1)^{-1} \geqq (C_{n,n}+1)^{-1} \geqq (C_{\infty,n}+1)^{-1}$$

But $c_{n,\infty} \uparrow c_{\infty,\infty}$ and $c_{\infty,m} \downarrow c_{\infty,\infty}$ so $(C_{n,\infty}+1)^{-1}$ and $(C_{\infty,n}+1)^{-1}$ both converse strongly as $n\to\infty$ to $(C_{\infty,\infty}+1)^{-1}$. It follows that $(C_{n,n}+1)^{-1}$ converges strongly to $(C_{\infty,\infty}+1)^{-1}$.

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