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A Time-Dependent Approach to the Completeness of Multiparticle Quantum Systems*

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1. Introduction

The basic questions of single channel scattering systems depend on the existence of the *generalized wave operators*

$$\Omega^{\pm}(A, B) = \operatorname*{s-lim}_{t \to \mp\infty} e^{iAt} e^{-iBt} P_{ac}(B),$$

where $P_{ac}(\cdot)$ is the projection onto the absolutely continuous space for a selfadjoint operator, (see [19], Chap. VII for the necessary spectral theory background) and their *completeness*

$$\operatorname{Ran} \Omega^{\pm}(A, B) = P_{ac}(A)\mathcal{H}.$$

In this context, the following elementary proposition is well-known and fundamental:

PROPOSITION. *Suppose that $\Omega^{\pm}(A, B)$ exist. Then $\Omega^{\pm}(A, B)$ are complete if and only if $\Omega^{\pm}(B, A)$ exist.*

The importance of this proposition is that it reduces the completeness question to the proof of the existence of a limit. This *time-dependent* approach to scattering has been raised to a high art by Kato, Kuroda, and Birman (see Kato [14] or Reed–Simon [20] for textbook presentations, or Pearson [18] for a recent and significant simplification). Our goal in this note is to prove an analogue of this basic proposition for multiparticle quantum

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systems. As a "kinematical" result, it is certainly not deep, but it could be very useful. It remains to be seen whether one can prove directly the existence of the limit which we show would imply completeness. But it is our hope that we shall focus interest on an approach which we find quite appealing and natural. We remark that our work here is motivated in part by ideas of Pearson [16], [17], [18], Combescure-Ginibre [3], and our own work in [4].

The time-dependent formalism is to be distinguished from the *time-independent* approach pioneered by Povzner and Ikebe for two-body systems and by Fadeev for three-body systems (see Reed–Simon [20] for a textbook presentation and Agmon [1] or Kuroda [15] for recent elegant presentations in the two-body case and Ginibre–Moulin [7] for the three-body case). Even in the two-body case where both time-dependent and time-independent methods are available, it is fashionable to denigrate the time-dependent approach since it generally requires estimates on V to be $O(|x|^{-n-\varepsilon})$ in order for $\Omega^{\pm}(-\Delta+V,-\Delta)$ to exist on $L^2(R^n)$ while the time-independent theory only requires V to be $O(|x|^{-1-\varepsilon})$. And thus far, only time-independent methods have succeeded in the proof of completeness at all energies in systems with more than one channel and more than two particles. In our opinion, this attitude does not do justice to the time-dependent approach. We shall emphasize some of the disadvantages of the time-independent approach below. Moreover, there is a significant, although apparently little known paper of Kuroda [15], which applies the trace class theory to potentials V which are only $O(|x|^{-1-\varepsilon})$ at infinity so long as they are centrally symmetric. Finally, with the exception of a paper of Combes [2], no systematic attempt at using time-dependent methods has been made in the N-body case. It is precisely such systematic attempts that we hope to encourage. Using, in part, ideas from the present paper, Simon [21] has simplified the proof of Combes [2].

Despite several recent papers on technical improvements for the time-independent approach to multiparticle scattering, [10], [7], [24], [11], [22], [9], the method seems to have a variety of undesirable features which seem to be intrinsic difficulties of the time-independent approach but *not* intrinsic to the basic phenomena of scattering; among these are:

(i) The method cannot accommodate the situation where there is a resonance at threshold so that restrictions to generic coupling constants seem to be necessary. While there are significant phenomena when such resonances occur (cf. [6], [25], [23]), it would seem unlikely that they destroy completeness.

(ii) Thus far, the time-dependent method has not been shown to work when there are infinitely many bound states. It is clear that there would be tremendous technical complications to solve before such a possibility could be accommodated.

(iii) In this method, scattering and spectral theory are intertwined. While it seems to us that spectral properties of sub-systems must enter in the completeness question (see below), it is undesirable to have the problems intertwined.

(iv) The disentangling of channels is a rather subtle process depending on fairly complicated resolvent equations whose complexities increase enormously with N.

A comparison with the method we propose is obviously difficult before our program has been completed, but in what we discuss here neither problem (i) nor (ii) enters. Also, as we shall see, the spectral theory is separated out. Finally, the separation of channels is effected by an appealingly geometric procedure.

To describe our main result for the three-body problem in ν-dimensions let us introduce some notation. Let μ_1, μ_2, μ_3 be the masses of the three particles. Let $\alpha = 1$ stand for the pair (23), etc. In general, β, γ denote the other two indices. Thus V_α is a function of $r_\beta - r_\gamma$. We use the coordinates

$$x_\alpha = r_\beta - r_\gamma ,$$

$$y_\alpha = r_\alpha - (\mu_\beta + \mu_\gamma)^{-1}[\mu_\beta r_\beta + \mu_\gamma r_\gamma]$$

and denote reduced mass by

$$m_\alpha = (\mu_\beta^{-1} + \mu_\gamma^{-1})^{-1} ,$$

$$M_\alpha = (\mu_\alpha^{-1} + (\mu_\beta + \mu_\gamma)^{-1})^{-1} .$$

Moreover,

$$V = \sum_{\alpha=1}^{3} V_\alpha(x_\alpha) .$$

Then, we denote

$$H = H_0 + V , \qquad H_0 = -(2m_\alpha)^{-1}\Delta_{x_\alpha} - (2M_\alpha)^{-1}\Delta_{y_\alpha} \quad \text{on} \quad L^2(R^{2\nu})$$

(H_0 is independent of α),

$$H_\alpha = H_0 + V_\alpha \qquad \text{on} \quad L^2(R^{2\nu}) ,$$

$$h_\alpha = -(2m_\alpha)^{-1}\Delta_{x_\alpha} + V_\alpha(x_\alpha) \qquad \text{on} \quad L^2(R^{\nu}) ,$$

$$h_{0,\alpha} = h_\alpha - V_\alpha , \qquad k_\alpha = -(2M_\alpha)^{-1}\Delta_{y_\alpha} \quad \text{on} \quad L^2(R^{\nu}) .$$

Let p_α be the projection onto the span of the eigenvectors for h_α and P_α the projection onto the span of functions of the form $\eta(y_\alpha)\phi(x_\alpha)$, $\eta \in L^2(R^\nu)$, $\phi \in \operatorname{Ran} p_\alpha$. A result of Hack [8] (see also [20]) shows that, under suitable hypotheses on the V_α, the limits (for $\alpha = 0, 1, 2, 3$)

$$\Omega_\alpha^\pm = \operatorname*{s\text{-}lim}_{t \to \mp\infty} e^{itH} e^{-itH_\alpha}$$

exist. The operators

$$\hat{\Omega}_\alpha^\pm = \Omega_\alpha^\pm P_\alpha$$

(for $\alpha = 1, 2, 3$, P_α is as above; for $\alpha = 0$, set $\hat{\Omega}_\alpha^\pm \equiv \Omega_0^\pm$) are maps onto those states which as $t \to \mp\infty$ look like a bound cluster of particles $(\beta\gamma)$ and particle α moving freely. A result of Jauch [13] (see also [20]) assures one that $\operatorname{Ran} \hat{\Omega}_\alpha^\pm \perp \operatorname{Ran} \hat{\Omega}_\beta^\pm$ for $\alpha \neq \beta$. *Completeness of the three-body system* is the pair of statements

$$\bigoplus_{\alpha=0}^{3} \operatorname{Ran} \Omega_\alpha^\pm = \operatorname{Ran} P_{\mathrm{ac}}(H), \tag{1}$$

separately for + and −.

We can now describe our results for the three-body system. Let (for $\alpha = 1$, 2, 3)

$$Q_\alpha(m, R) = \{(x_\alpha, y_\alpha) \mid |x_\alpha| \leqq m\, |y_\alpha|^{1/3},\ |y_\alpha| \geqq m^{-1} R\}.$$

Pick R so large that $Q_\alpha(2, R) \cap Q_\beta(2, R) = \varnothing$. Let χ_α be the characteristic function of $Q_\alpha(1, R)$ and let J_α be a function picked once and for all so that

$$0 \leqq J_\alpha(x) \leqq 1 \text{ (all } x), \qquad \operatorname{supp} J_\alpha \subset Q_\alpha(2, R), \qquad J_\alpha \equiv 1 \quad \text{on} \quad Q_\alpha(1, R).$$

Let $J_0 = 1 - \sum_\alpha J_\alpha$. [Remark: In what occurs below, one could take $J_\alpha = \chi_\alpha$; no smoothness of J_α is used. However, in applying our method it may be useful to allow the possibility of smooth J's so we take the choice given above. No particular significance should be attached to $|y_\alpha|^{1/3}$ in the definition of D_α. Any $f(y_\alpha)$ with $|y_\alpha|^{-1} f(y_\alpha) \to 0$, $f(y_\alpha) \to \infty$ as $|y_\alpha| \to \infty$ could be used; $|y_\alpha|^\gamma$ with $\gamma < \frac{1}{2}$ may be convenient since diffusion ideas might be important. Also, the disjointness of the regions is not really necessary; the proofs are slightly less wordy this way.] We shall be concerned here with the existence of the following limits for $\alpha = 0, 1, 2, 3$:

$$W_\alpha^\pm = \operatorname*{s\text{-}lim}_{t \to \mp\infty} e^{+itH_\alpha} J_\alpha e^{-itH} P_{\mathrm{ac}}(H). \tag{2}$$

Consider the following statements (under the supposition that the limits $\Omega^{\pm}(h_\alpha, h_{0,\alpha})$, $\Omega_\alpha^{\pm}$ exist).

(a) The three-body system is complete, i.e., (1) holds.

(b) The limits (2) exist.

(c) Each h_α has no singular continuous system.

(d) Each $\Omega^{\pm}(h_\alpha, h_{0,\alpha})$ is complete.

In Section 2, we shall prove that (a) $\Rightarrow$ (b) and (b), (c), (d) $\Rightarrow$ (a) and, in Section 3, we consider the generalizations to $N \geqq 3$. We believe that it is also true that (a) $\Rightarrow$ (c), (d) but we have not tried very hard to find a proof since our main interest is to give a method for proving (a). (As a side light we want to justify the "naturalness" of (b), so we prove (a) $\Rightarrow$ (b).) We shall, however, make some remarks about our conjecture that (a) $\Rightarrow$ (c), (d):

(i) Heuristics for (a) $\Rightarrow$ (c) can be found also in Combes [2].

(ii) If $P_\alpha^{(s)}$ is the projection onto functions of the form $\eta(y_\alpha)\phi(x_\alpha)$ with ϕ in the singular continuous subspace for h_α, then one should be able to show that $\Omega_\alpha^{\pm} P_\alpha^{(s)}$ is orthogonal to each $\operatorname{Ran} \hat{\Omega}_\beta^{\pm}$ and this would imply that (a) $\Rightarrow$ (c). To prove this orthogonality, it suffices to prove a general result that if A is purely absolutely continuous and B is purely singular continuous, then w-lim $e^{iAt}e^{-iBt} = 0$ since Jauch's proof of orthogonality of channels would then extend. Actually, it suffices that the limit be zero in a Cesaro or Abelian sense.

(iii) Under suitable hypotheses on the V_α, (a) certainly implies that $\operatorname{Ran} \Omega^{+}(h_\alpha, h_{0,\alpha}) = \operatorname{Ran} \Omega^{-}(h_\alpha, h_{0,\alpha})$ for (a) implies that the three-body S-matrix is unitary whence the cluster properties for S (Hunziker [12]; see also [20]) imply the unitarity of the two-body S-matrices.

We also note that by using Dollard's modified dynamics [5], our results (which are essentially kinematical) can be extended to Coulombic systems.

Finally, we remark that Uchiyama [26] uses geometric methods similar to ours to show that certain three-body systems have a finite number of eigenvalues below the continuum.

2. The Three-Body Case

THEOREM 1. *If the three-body system is complete, then the limits* $W_\alpha^{\pm}$ *exist,* $W_\alpha^{\pm} \hat{\Omega}_\beta^{\pm} = P_\alpha \, \delta_{\alpha\beta}$ *and* $W_\alpha^{\pm} = (\hat{\Omega}_\alpha^{\pm})^*$.

LEMMA 1. *For each* $\alpha = 1, 2, 3$,

$$\|(1-\chi_\alpha) e^{-itH_\alpha} P_\alpha \psi\| \to 0$$

as $t \to \pm\infty$. *In particular,*

$$\|J_\beta e^{-itH_\alpha} P_\alpha \psi\| \to 0 \quad \text{and} \quad \|(1-J_\alpha) e^{-itH_\alpha} P_\alpha \psi\| \to 0$$

as $t \to \pm\infty$ *for* $\beta \neq \alpha$ *with* $\beta = 0$ *allowed.*

Proof: It suffices to prove that, for any $\eta \in \mathcal{S}(R^\nu)$ and ϕ an eigenfunction of h_α,

$$\int_{\substack{|x_\alpha| \geqq |y_\alpha|^{1/2} \\ \text{or } |y_\alpha| \leqq R}} |(e^{-itk_\alpha}\eta)(y_\alpha)|^2 \, |\phi(x_\alpha)|^2 \, dx_\alpha \, dy_\alpha \to 0 . \tag{3}$$

Since $\phi \in L^2(R^\nu)$, given ε, pick $R_\varepsilon \geqq 0$ so that

$$\int_{|x_\alpha| \geqq R_\varepsilon^{1/3}} |\phi(x_\alpha)|^2 \leqq \tfrac{1}{2}\varepsilon \, \|\eta\|^{-2} .$$

Write the integral in (3) as a sum over the region with $|y_\alpha| \leqq R_\varepsilon$ and over the region with $|y_\alpha| \geqq R_\varepsilon$. By the choice of R_ε, the second integral is less than $\frac{1}{2}\varepsilon$ for all t. The first integral is dominated by

$$\|\phi\|^2 \int_{|y_\alpha| \leqq R_\varepsilon} |(e^{-itk_\alpha}\eta)(y_\alpha)|^2 \, dy_\alpha$$

which goes to zero as $t \to \infty$ by explicit calculation.

LEMMA 2. *Let f be a function in $L^\infty(R^{2\nu})$ so that*

$$Q(R) \equiv \operatorname*{ess\,sup}_{r \geqq R} \omega\,(x \in S^{2\nu-1} \mid r\hat{x} \in \operatorname{supp} f) \to 0$$

as $R \to \infty$, where ω is the usual normalized measure on the sphere, $S^{2\nu-1}$. Then

$$\operatorname*{s-lim}_{t \to \pm\infty} f e^{+it\Delta} = 0 .$$

In particular,

$$\operatorname*{s-lim}_{t \to \pm\infty} J_\alpha e^{-itH_0} = 0$$

for $\alpha = 1, 2, 3$ and

$$\operatorname*{s-lim}_{t \to \pm\infty} (1 - J_0) e^{-itH_0} = 0 .$$

Proof: It suffices to prove that $\|fe^{it\Delta}\phi\| \to 0$ for a total set of ϕ's. Take $\phi(x) = g(|x|)\,Y_{lm}(x/|x|)$, where Y_{lm} is a spherical harmonic. Then, since $e^{it\Delta}\phi =$

$(e^{-ith_l}g)Y_{lm}$ for suitable h_l, we see that

$$\int_{|x|\geqq R} |(fe^{-it\Delta}\phi)(x)|^2\,dx \leqq Q(R)\|f\|_\infty^2\|\phi\|^2\|Y_{lm}\|_\infty^2\,;$$

so by choosing R large, we can be sure it is less than $\frac{1}{2}\varepsilon$ for all t. As above, $\int_{|x|\leqq R}$ goes to zero as $t\to\infty$.

Proof of Theorem 1: By the hypothesis of completeness, any $\phi\in P_{ac}(H)$ can be written uniquely as $\phi=\sum_{\alpha=0}^{3}\psi_\alpha$ with $\psi_\alpha=\hat{\Omega}_\alpha^+\phi_\alpha$. Consequently, $e^{-itH}\psi_\alpha-e^{-itH_\alpha}P_\alpha\phi_\alpha\to 0$ as $t\to-\infty$. It follows that

$$\begin{aligned} J_\alpha e^{-itH}\phi &= \sum_\beta J_\alpha e^{-itH_\beta}P_\beta\phi_\beta + o(t)\\ &= J_\alpha e^{-itH_\alpha}P_\alpha\phi_\alpha + o(t) = e^{-itH_\alpha}P_\alpha\phi_\alpha + o(t)\,, \end{aligned}$$

by Lemmas 1 and 2. Thus

$$\lim_{t\to-\infty} e^{itH_\alpha}J_\alpha e^{-itH}\psi = P_\alpha\phi_\alpha\,.$$

This establishes the existence of the limit defining W_α and the formula $W_\alpha^+\hat{\Omega}_\beta^+=\delta_{\alpha\beta}P_\alpha$. From this formula, the orthogonality of the $\operatorname{Ran}\hat{\Omega}_\alpha^+$ and partial isometric nature of the $\hat{\Omega}_\alpha^+$ we conclude that W_α^+ is a partial isometry which "undoes" $\hat{\Omega}_\alpha^+$ and thus $W_\alpha^+=(\Omega_\alpha^+)^*$.

THEOREM 2. *Suppose that the limits defining $\Omega_\alpha^\pm$ and $W_\alpha^\pm$ exist, that each h_α has no singular continuous spectrum, and that $\Omega^\pm(h_\alpha, h_{0,\alpha})$ exist and are complete for $\alpha=1, 2, 3$. Then the three-body system is complete. Moreover,*

$$\sum_{\alpha=0}^{3}\hat{\Omega}_\alpha^\pm W_\alpha^\pm = P_{ac}(H)\,. \tag{4}$$

Proof: Let $\phi\in\operatorname{Ran}P_{ac}(H)$. Let $\phi_\alpha=W_\alpha^+\phi$. Then

$$\|e^{-itH_\alpha}\phi_\alpha - J_\alpha e^{-itH}\phi\|\to 0 \quad\text{as}\quad t\to-\infty$$

so that

$$e^{-itH}\phi=\sum_{\alpha=0}^{3}e^{-itH_\alpha}\phi_\alpha + O(t)\,,$$

since $\sum_{\alpha=0}^{3} J_\alpha = 1$. Therefore,

$$\phi = \sum_{\alpha=0}^{3} \Omega_\alpha^+ \phi_\alpha .$$

Completeness follows if we prove that for $\alpha = 1, 2, 3$

$$\operatorname{Ran} \Omega_\alpha^+ \subset \operatorname{Ran} \hat{\Omega}_\alpha^+ \oplus \operatorname{Ran} \hat{\Omega}_0^+ , \tag{5}$$

and once we have completeness, Theorem 1 is applicable so that $(\hat{\Omega}_\alpha^+)^* = W_\alpha^+$ and (4) follows. Thus, we need only prove (5).

By the assumptions on h_α,

$$\operatorname{Ran} \Omega^\pm(h_\alpha, h_{0,\alpha}) = 1 - p_\alpha . \tag{6}$$

Let $L^2(R^{2\nu}) = L^2(R^\nu) \otimes L^2(R^\nu)$ according to the coordinates (y_α, x_α). Then $H_0 = k_\alpha \otimes 1 + 1 \otimes h_{0,\alpha}$, $H_\alpha = k_\alpha \otimes 1 + 1 \otimes h_\alpha$, $P_\alpha = 1 \otimes p_\alpha$ so that (6) is equivalent to

$$\operatorname{Ran} \Omega^\pm(H_\alpha, H_0) = 1 - P_\alpha$$

or

$$\Omega^\pm(H_\alpha, H_0)\Omega^\pm(H_\alpha, H_0)^* = 1 - P_\alpha .$$

It follows that

$$\begin{aligned} \Omega_\alpha^+ &= \hat{\Omega}_\alpha^+ + \Omega_\alpha^+(1 - P_\alpha) \\ &= \hat{\Omega}_\alpha^+ + \Omega^+(H, H_\alpha)\Omega^+(H_\alpha, H_0)\Omega^+(H_\alpha, H_0)^* \\ &= \hat{\Omega}_\alpha^+ + \hat{\Omega}_0^+ \Omega^+(H_\alpha, H_0)^* \end{aligned}$$

which implies (5).

3. The N-Body Problem

The main difficulty in extending the results of Section 2 to $N > 3$ bodies is notational. We shall therefore introduce the notation and state the analogues of Theorems 1 and 2. The proofs of these analogues are substantially identical to the proofs in Section 2. Where possible, we follow the notation of [20]. A (non-trivial) cluster decomposition, D, is a partition of $\{1, \cdots, n\}$ into two or more disjoint subsets, $C_1, \cdots, C_l$. We write iDj if i and j are in the same cluster C_k, and $\sim iDj$ if they are in different clusters. H_0 is the kinetic energy of the N-particles with center of mass removed and

$$H = H_0 + \sum_{i<j} V_{ij}(r_i - r_j)\,;$$

$h(C_k)$ is the Hamiltonian of the cluster C_k with center of mass removed written in terms of coordinates $x^{(k)}$ and $H(C_k)$ is the "same" operator acting on the N-particle system. We use the symbol x_D for the totality of coordinates $x^{(1)}, \cdots, x^{(l)}$ and y_D for the coordinates of the relative center of mass motion. T_D is the relative kinetic energy of the clusters, i.e., the kinetic energy of all the centers of mass of $C_1, \cdots, C_l$ minus the kinetic energy of the total center of mass. We obtain

$$H_D = H - \sum_{\sim iDj} V_{ij} = \bigoplus_{k=1}^{l} H(C_k) + T_D\,,$$

and P_D is the projection onto the span of all vectors of the form $\eta(y_D)\phi_1(x^{(1)}) \cdots \phi_l(x^{(l)})$, where ϕ_k is an eigenfunction of $h(C_k)$. If the operators

$$\Omega_D^{\pm} = \operatorname*{s-lim}_{t \to \mp\infty} e^{itH} e^{-itH_D}$$

exist (and this is true (cf. [20]) under fairly general hypotheses on V_{ij}) we set

$$\hat{\Omega}_D^{\pm} = \Omega_D^{\pm} P_D$$

and the $\hat{\Omega}_D^{\pm}$ have orthogonal ranges. Completeness says that

$$\bigoplus_D \operatorname{Ran} \hat{\Omega}_D^{\pm} = \operatorname{Ran} P_{ac}(H)\,. \tag{7}$$

We define

$$Q_D(m, R) = \{(x_D, y_D) \,|\, |x_D| \leqq |y_D|^{1/2};\ y_D| \geqq m^{-1}R\}$$

and given m_D, R_D we pick $\tilde{J}_D$ so that $\tilde{J}_D$ is 1 on $Q_D(m_D, R_D)$ and zero off $Q_D(2m_D, R_D)$.

If D is a refinement of D', i.e., if D' is obtained by "lumping together" clusters in D, we write $D \rhd D'$. If neither $D \rhd D'$ nor $D' \rhd D$, we say that D and D' are incompatible. We now restrict the m_D and R_D so that $\tilde{J}_D$ is 1 on the support of $\tilde{J}_{D'}$ if $D \rhd D'$ and so that $\operatorname{supp} \tilde{J}_D \cap \operatorname{supp} \tilde{J}_{D'} = \phi$ if D and D' are incompatible. We define J_D inductively in the number of clusters in D by

$$J_D = \tilde{J}_D \quad \text{if } D \text{ has two clusters},$$

$$J_D = \tilde{J}_D - \sum_{D' \triangleright D} J_{D'}.$$

Then we have

THEOREM 1′. *If* (7) *holds, then the limits*

$$W_D^{\pm} = \operatorname*{s-lim}_{t \to \mp\infty} e^{+itH_D} J_D e^{-itH} P_{ac}(H) \tag{8}$$

exist and obey $W_D^{\pm}\hat{\Omega}_{D'}^{\pm} = P_D \delta_{DD'}$ *and* $W_D^{\pm} = (\hat{\Omega}_D^{\pm})^*$.

THEOREM 2′. *Suppose that the strong limits* $W_D^{\pm}$ *in* (8) *exist, and that the strong limits* $\Omega_D^{\pm}$ *exist. Moreover, suppose that, for every proper subset* $C \subset \{1, \cdots, N\}$ *with* $2 \leqq \#(C) \leqq N-1$, $h(C)$ *has no singular continuous spectrum, that* $\Omega_D^{\pm}$ *exists for every cluster decomposition* D *of* C, *and* $\oplus_D \operatorname{Ran} \hat{\Omega}_D^{\pm} = \operatorname{Ran} P_{ac}(h(C))$. *Then* (7) *holds.*

Theorem 2′ inductively reduces completeness of an N-body system to the existence of a large number of strong limits together with purely spectral information on the $h(C)$'s.

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