JOURNAL OF FUNCTIONAL ANALYSIS 32, 97–101 (1979)

Kato's Inequality and the Comparison of Semigroups*

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Communicated by Tosio Kato

Received June 27, 1977

Let A be the generator of a positivity preserving semigroup and let B be another semibounded self-adjoint operator. We give necessary and sufficient conditions in terms of the generators for the inequality $| e^{-tB}u | \leqslant e^{-tA} | u |$ to hold pointwise.

Throughout this note we fix a separable Hilbert space, $\mathscr{H}$ which is of the form $L^2(M, d\mu)$. A self-adjoint semi-group, e^{-tA}, is called positivity preserving if and only if $e^{-tA}u \geqslant 0$ for $u \geqslant 0$ or equivalently if $| e^{-tA}u | \leqslant e^{-tA} | u |$ for any u. There are simple elegant criteria in terms of A for e^{-tA} to be positivity preserving —these go back to Beurling and Deny [2]; (see also Reed and Simon [7]). Recently, Simon [11] found that the positivity preserving property were equivalent to the pair of conditions:

(P_i) $u \in D(A)$ implies $| u | \in Q(A)$, and

(P_{ii}) For any $u \in D(A)$ and $\phi \geqslant 0$, $\phi \in Q(A)$

$$(\phi, A \,| u |) \leqslant \mathrm{Re}((\mathrm{sgn}\, u)^* \phi, Au) \tag{1}$$

where $\mathrm{sgn}\, u = u^* | u |^{-1}$ (at points with $u \neq 0$ and $\mathrm{sgn}\, u = 0$ if $u = 0$) and $Q(\cdot)$ denotes quadratic form domain. The special case of (1) in case $A = -\Delta$ was discovered and applied to self-adjointness problems by Kato [6]. Kato also found inequalities like (1) where the A on the right side is replaced by another operator B with $A = -\Delta$ and $B = (i\nabla + \mathbf{a})^2$. Our goal here is translate this form of Kato's inequality into a "positivity" condition on semigroups.

* Research partially supported by U.S.N.S.F. Grant MPS-75-11864.

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0022-1236/79/040097-05$02.00/0

We will consider the pair of conditions:

(K_i) $u \in D(B)$ implies $|u| \in Q(A)$, and

(K_{ii}) For $u \in D(B)$ and $\phi \geqslant 0$, $\phi \in Q(A)$

$$(\phi, A\,|\,u\,|) \leqslant \operatorname{Re}((\operatorname{sgn} u)^* \phi, Bu). \tag{2}$$

Our main result here is the following result which we conjectured in [11]:

THEOREM 1. *Let A and B be semibounded self-adjoint operators. Suppose that A is the generator of a positivity preserving semigroup. Then conditions (K_i), (K_{ii}) hold if and only if*

$$|\,e^{-tB}u\,| \leqslant e^{-tA}\,|\,u\,| \tag{3}$$

for all u.

Proof. (3) $\Rightarrow$ (K_i), (K_{ii}). Given (3), we find that

$$(u, e^{-tB}u) \leqslant (|\,u\,|, e^{-tA}\,|\,u\,|)$$

so

$$(u, t^{-1}(1 - e^{-tB})\,u) \geqslant (|\,u\,|, t^{-1}(1 - e^{-tA})\,|\,u\,|)$$

letting $t \downarrow 0$, we find

$$(|\,u\,|, A\,|\,u\,|) \leqslant (u, Bu) \tag{4}$$

where $(u, Cu) = \infty$ if $u \notin Q(C)$ and $=(|\,C\,|^{1/2}\,u, (\operatorname{sgn} C)\,|\,C\,|^{1/2}\,u)$ for $u \in Q(C)$. (4) implies that $|\,Q(B)| \subset Q(A)$ and a fortiori (K_i).

(3) also implies that for $\phi \geqslant 0$

$$\operatorname{Re}((\operatorname{sgn} u)^* \phi, e^{-tB}u) \leqslant (\phi, e^{-tA}\,|\,u\,|).$$

Since both sides of this last expression are equal at $t = 0$, there is an inequality on the derivatives at $t = 0$. The derivative of the left side is $-(\operatorname{sgn} u^* \phi, Bu)$ since $u \in D(B)$ and, of the right $-(\phi, A\,|\,u\,|)$ since ϕ, $|\,u\,| \in Q(A)$. This verifies (K_{ii}).

(K_i, K_{ii}) $\Rightarrow$ (3). Adding $(\phi, \lambda\,|\,u\,|)$ to both sides of (2), we find that

$$(\phi, (A + \lambda)\,|\,u\,|) \leqslant \operatorname{Re}(\phi, (\operatorname{sgn} u)(B + \lambda)\,u) \leqslant (\phi, |(B + \lambda)\,u\,|) \tag{5}$$

for any u in $D(B)$ and any $\phi \in Q(A)$, $\phi \geqslant 0$. Now, let $v \in \mathscr{H}$ be arbitrary and

$\psi \geqslant 0$. Set $\phi = (A + \lambda)^{-1} \psi$ which is positive since $(A + \lambda)^{-1} = \int e^{-\lambda t} e^{-At}\, dt$ is positivity preserving, and let $u = (B + \lambda)^{-1} v$. Then (5) becomes

$$(\psi, |(B + \lambda)^{-1} v|) \leqslant (\psi, (A + \lambda)^{-1} |v|)$$

or equivalently, that

$$|(B + \lambda)^{-1} v| \leqslant (A + \lambda)^{-1} |v| \tag{6}$$

From (6) and induction we find that

$$|(B + \lambda)^{-n} v| \leqslant (A + \lambda)^{-n} |v|$$

(for $|(B + \lambda)^{-n-1} v| \leqslant (A + \lambda)^{-1} |(B + \lambda)^{-n} v|$ (by (6)) $\leqslant (A + \lambda)^{-n-1} |v|$ (by induction and $(A + \lambda)^{-1}$ positivity preserving). Using

$$e^{-tA} = s - \lim_{n\to\infty} \left(\frac{n}{t}\right)^n \left(A + \frac{n}{t}\right)^{-n}$$

(3) results. ▌

Remarks. (1) By looking at the proof, one sees that it suffices that (3) hold for u in any core for B.

(2) By an argument of Davies [4], (3) implies the following: If V is a multiplication operator with $D(V) \supset D(A)$ and $\| Vu \| \leqslant \alpha \|(A + b) u\|$ for all $u \in D(A)$, then $D(V) \supset D(B)$ and $\| Vu \| \leqslant \alpha \|(B + b) u\|$.

One especially interesting case of (3) is to the original application of Kato: viz $A = -\Delta$, $B = (i\nabla + \mathbf{a})^2$. If $\mathbf{a} \in L^2_{\text{loc}}$, then B can be defined by the method of quadratic forms and we conjecture that (3) holds for any such $\mathbf{a}$ with $\nabla \cdot \mathbf{a} = 0$. At this point, (3) is only known for $\mathbf{a}$ so that B is essentially self-adjoint on $C_0^\infty(\mathbb{R}^\nu)$; this includes $\mathbf{a}$ in C^1 [5] and also $\mathbf{a} \in L^p_{\text{loc}}$ (for p suitable, e.g. $p = 4$ if $\nu = 3$) with $\nabla \cdot \mathbf{a} = 0$ [8, 9]. We remark that (3) has recently been applied to the study of Schrödinger operators in magnetic fields [1, 3].

One proof of (3) for the case just mentioned is that in [11] (see also [12]) which was suggested to the author by Nelson: one writes an explicit formula for e^{-tB} using Wiener path integrals and Ito stochastic integrals, whence (3) follows by inspection. The only bar to extending this proof to arbitrary $\mathbf{a} \in L^2_{\text{loc}}$ is verifying the Feynman–Kac–Ito formula for such $\mathbf{a}$ (this problem is discussed in [12]).

A second proof of (3) in this situation is available using the methods of this note. (K_i) holds for arbitrary $\mathbf{a} \in L^2_{\text{loc}}$.

PROPOSITION 2. *Let* $\mathbf{a} \in L^2_{\text{loc}}$ *and let* $B = (i\nabla + \mathbf{a})^2$ *as a sum of forms. Then for any* $u \in D(B)$, *we have that* $|u| \in Q(-\Delta)$ *and*

$$(u, Bu) \geqslant (|u|, A|u|).$$

Proof. For $u \in C_0^\infty$ and $\mathbf{a} \in C^1$, one has that

$$|(\nabla - i\mathbf{a})\, u| \geqslant |\nabla| u||$$

see e.g. [6, 10]. Thus

$$\| \nabla | u| \|_2 \leqslant \|(\nabla - i\mathbf{a})\, u\|_2 \tag{7}$$

(2) holds for $\mathbf{a}$ in C^1 and $u \in C_0^\infty$ and so by a limiting argument for arbitrary $\mathbf{a}$ in L^2_{loc} and $u \in C_0^\infty$. Since $Q(B)$ is the closure of C_0^∞ in the norm $\|(\nabla - ia)\, u\|_2 + \|u\|_2$ and $Q(A)$ is the closure of C_0^∞ in the norm $\|\nabla u\|_2 + \|u\|_2$, the proof is complete. ▮

THEOREM 3. *Let* $\mathbf{a} \in L^4_{\mathrm{loc}}$ *with* $\nabla \cdot \mathbf{a} = 0$ *so that* $B = (i\nabla + \mathbf{a})^2$ *is essentially self-adjoint on* C_0^∞. *Then* (K_{ii}) *holds with* $A = -\Delta$ *and in particular* (3) *holds.*

Proof. By the remark following theorem 1, we need only check(2) for $u \in C_0^\infty$. By an approximation argument, this holds if we know (2) when $\mathbf{a} \in C_0^\infty$. For such $\mathbf{a}$, (2) is a result of Kato [6]. ▮

Notes Added in Proof. 1. Theorem 1 has been proven independently and simultaneously by Hess *et al.*, *Duke Math. J.* 44 (1977), 893–904. 2. The problem of proving (3) for Schrödinger operators with arbitrary $a \in L^2_{\mathrm{loc}}$ is solved in B. Simon, *J. Optimization Theory Appl.* 1 (1979), to appear.

ACKNOWLEDGMENT

It is a pleasure to thank J. Avron and I. Herbst for valuable discussions and M. Guenin for the hospitality of the University of Geneva where this work was completed.

REFERENCES

1. J. AVRON, I. HERBST, AND B. SIMON, Schrödinger operators with magnetic fields. I. General theory, *Duke Math. J.* 45 (1978), 847–883.
2. A. BEURLING AND J. DENY, Espaces de Dirichlet, *Acta Math.* 99 (1958), 203–224.
3. J. COMBES, R. SCHRADER, AND R. SEILER, Classical bounds and limits for energy distributions of Hamiltonian operators in electromagnetic fields, preprint, *Ann. Phys. (New York)* 111 (1978), 1–18.
4. E. B. DAVIES, Properties of Green's function of some Schrödinger operators, *J. London Math. Soc.* 7 (1973), 473–491.
5. T. IKEBE AND T. KATO, Uniqueness of the self-adjoint extension of singular elliptic differential operators, *Arch. Rational Mech. Anal.* 9 (1962), 77–92.
6. T. KATO, Schrödinger operators with singular potential, *Israel J. Math.* 13 (1972), 135–148.

7. M. REED AND B. SIMON, "Methods of Modern Mathematical Physics. IV. Analysis of Operators," Academic Press, New York, 1978.
8. M. SCHECHTER, "Spectra of Partial Differential Operators," North-Holland, Amsterdam, 1971.
9. B. SIMON, Schrödinger operators with singular magnetic vector potential, *Math. Z.* **131** (1973), 361–370.
10. B. SIMON, Universal diamagnetism of spinless Bose systems, *Phys. Rev. Lett.* **36** (1976), 1083–1084.
11. B. SIMON, Abstract Kato's inequality for generators of positivity preserving semigroups, *Indiana Univ. Math. J.* **26** (1977), 1067–1073.
12. B. SIMON, "Functional Integration and Quantum Physics," Academic Press, New York, 1979.