Separatum
HELVETICA PHYSICA ACTA
Vol. 43, Fasc. 6, p. 607–630 (1970)

On the Infinitude or Finiteness of the Number of Bound States of an N-Body Quantum System, I

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(15. IV. 70)

Abstract. We present a general discussion of when an N-body quantum system with two-body forces will have infinitely many bound states. A physically-motivated criterion for infinitude is presented and proved sufficient. In a restricted class of cases, it is proven to be necessary. Two applications are made: first, we recover the Zhislin and Zhislin-Sigalov results on the number of bound states of atoms; secondly, we discuss the coupling constant dependence of the number of bound states and find it very different from the two-body situation. Finally, we present examples of N-body systems with infinitely many bound states even though the two-body forces are too weak to bind any states.

CONTENTS

1. Introduction

While the last twenty years have seen the development of a mathematically complete theory of most aspects of two-body non-relatavistic quantum mechanics [1], there are still glaring gaps in our understanding of n-body problems, even on a heuristic level[1]). It is our purpose here to present a first look at the question of when an n-body system with only two-body forces has an infinity of bound states. Because

[1]) The most glaring gap is that unitarity of the S-matrix has been proven only in a small class of cases [2].

of the presence of negative energy continuum states, the n-body case is a priori more complicated than the two-body case; what we will see is that the n-body question is essentially reducible to a two-body problem and so it is tractable.

In some sense, our results have their roots in the work of Zhislin [3], who proved atoms have infinitely many bound states; Zhislin and Sigalov [4] who showed this remains true even if Fermi-Dirac statistics are included; and Uchiyama [5], who included the possibility of certain kinds of magnetic fields in the atoms. Our only strong theorem, that proven in section 3 is very similar in spirit to the results they prove. We feel the technical details of our proof are more transparent and most importantly that we emphasize the physical ideas behind the result. However, we would emphasize to the reader that our strong results in section 3 are contained to some extent in reference [3–5].

In what follows (and in the above) we intend 'bound state' to indicate square integrable eigenfunction *below the continuum*. There are cases in which one can conceive of (or prove the existence of!) square integrable eigenfunctions above the continuum [2]) but these are somewhat pathological beasts and are expected to be unstable under perturbations – we choose to ignore them by defining bound state suitably!

Physically, continuum states should be those states which are capable of breakup into at least two spatially separated clusters. Thus, one would expect the bottom of the continuum to be determined as follows: first consider breakup of $\{1, \ldots, n\}$ into two disjoint sets D_1, D_2. Let H_{D_i} be the Hamiltonian for the cluster D_i (so that $H_{D_1} + H_{D_2}$ is H minus the interaction *between* D_1 and D_2). Next let $\inf$ [spectrum $(H_D)] \equiv \sigma_D$. One expects:

$$\sum \equiv \inf\,[\text{continuous spec}\ (H)] = \min_{\substack{D_1 \cap D_2 = \emptyset \\ D_1 \cup D_2 = \{1, \ldots, n\}}} [\sigma_{D_1} + \sigma_{D_2}]\,. \tag{1}$$

That (1) is true for locally square integrable two-body potentials vanishing at infinity is a theorem of Hunziker [7] [3]) [4]).

Suppose we are in the situation where $\sum = \sigma_{D_1} + \sigma_{D_2}$ for some decomposition $\{D_1, D_2\}$ and for no other. Suppose also that H_{D_1} and H_{D_2} have discrete ground states at the bottom of their spectra. We will call this case 'the two-cluster continuum limit'.

In this situation, let us consider *heuristically* when we might expect an infinitude of bound states. A set of states bounded in energy and in spatial spread is compact [5]). Thus, an infinite number of orthogonal states cannot be bounded in spatial spread if

[2]) Potentials with wiggles and long range can produce such functions: see [6]. Also, if one has identical particles without statistics, bound states of one permutation symmetry can be embedded in the continuum of another symmetry.

[3]) Equation (1) was first proven for L^2-potentials by Van Winter [8]. Hunziker's proof has been abstracted to allow certain types of n-body forces and non-local potentials in a paper of Combes [9]. Simon [10] has shown that worse local singularities than L^2, for example $L^{3/2}$ singularities, can occur without affecting the truth of (1). In the case where the minimum is realized with D_1 a one-element set, (1) was proven by a different method by Zhislin [3] whose proof was abstracted and expanded by Jörgens [11].

[4]) In appendices 1 and 2, we present certain kinematical considerations not explicitly treated by Hunziker.

[5]) For an exact statement, see Ruelle [12].

they are bounded in energy. If there are infinitely many bound states, they must spread out in space into two or more clusters. If the clustering is into $F_1, \ldots, F_m$, the high lying (weakly bound) states have energy very near or above $\sigma_{F_1} + \ldots + \sigma_{F_m}$. For these states to lie below $\sum$, we must have that the F_i breakup is just D_1, D_2 (for $\sum = \sigma_{D_1} + \sigma_{D_2}$ is a unique breakup) and moreover, the weakly bound states must look like a product of the ground state wave functions for D_1 and D_2 and a wave function for the relative position of the centers of mass for D_1 and D_2.

Thus, when there are infinitely many bound states and we have a two-cluster continuum limit, the loosely bound states look like 'two-body' states (where the bodies have internal structure). As we will recall in section two, the finiteness of infinitude of the number of bound states for a two-body system depends only on the large r behavior of $V(r)$. For large separations of D_1 and D_2, where the 'internal' structure doesn't matter, the potential between D_1 and D_2 looks like

$$\tilde{V}_{D_1 D_2}(r) = \sum_{i \in D_1;\, j \in D_2} V_{ij}(r)\,. \tag{2}$$

If there are infinitely many bound states for the N-body system, then the two-body system with potential $\tilde{V}$ should have infinitely many bound states. Conversely, if $\tilde{V}$ has infinitely many bound states, we expect the n-body system to have infinitely many bound states which look like the product of the ground states of D_1, D_2 and a bound state of $\tilde{V}$.

To summarize, *we expect that an n-body system with a two-cluster continuum limit D_1, D_2 will have infinitely many bound states when and only when the two-body system with mass m_D* [6]) *and potential $\tilde{V}_{D_1 D_2}$ has infinitely many bound states.* Obviously, in cases where the V_{ij} wiggle a lot or where large cancellations are involved in (2), we should be prepared for our expectation to be wrong.

This picture actually suggests that one can make more detailed statements about the high-lying spectrum; explicitly, it suggests the conjectures:

Conjecture 1. Suppose we are in the two-cluster continuum limit and that the n-body system has infinitely many bound states of energies $\mu_1, \ldots, \mu_n, \ldots$. Suppose $E_1, \ldots, E_n, \ldots$ are the bound state energies of $H_2 \equiv (2\,m_D)^{-1} p^2 + \tilde{V}_{D_1 D_2}$. Then $\lim_{n \to \infty} (\mu_n - \sum)/E_{n+m} = 1$ for some *fixed* m.

Conjecture 2. Suppose we have the notation of conjecture 1. Let ψ_k be the wave function of the bound state of the n-body system. Let ϕ_n be the n-th bound state of H_2. Let η_1, η_2 be the ground states of H_{D_1}, H_{D_2}. Then, for some m, fixed:

$$\lim_{k \to \infty} \| \psi_k - \eta_1\, \eta_2\, \phi_{k+m} \|_2 = 0\,.$$

After reviewing the two-body theory (section 2), we will prove the above criterion for infinitude is more or less sufficient (section 3). In section 5, we apply this theorem to atomic Hamiltonians without and with statistics. To handle the latter case, we first discuss Hunziker's theorem with statistics (section 4). As preparation for the discussion of some examples of coupling constant dependence (section 7), we prove

[6]) That $m_D^{-1} = (m_{D_1})^{-1} + (m_{D_2})^{-1}$ with $m_{D_i} = \sum_{j \in D_i} m_j$ is shown in appendix 1.

the above criterion is more or less necessary for infinitude in a very restricted class of cases (section 6). It is our feeling that this result can be improved with some simple (but clever!) proof. Finally, we give some discussion of what happens when one doesn't have a two-cluster continuum limit (section 8). We do not treat conjectures 1 and 2.

2. A Remembrance of Things Past

Let us recall the situation with regard to the infinitude or finiteness of the number of bound states in a two-body quantum system. We do this not only for comparison with the n-body case but also because, as we have discussed heuristically, we *expect* the n-body case to reduce to a two-body problem.

The results we discuss are contained, more or less, in the classic work of Courant and Hilbert [13], especially pp. 445–450. The crucial facts to remember are that the infinitude or finiteness is dependent only on the long-distance behavior of V and that the dividing line between the two types of behavior is $-r^{-2}$ falloff. The reason for the first fact is best seen in the proof of proposition 3 below and the reasons for the power (-2) is seen in the proof of:

Proposition 1. Let V be a Kato potential[7]) with $V(r) < -C\,r^{-\gamma}$ for $|r| \equiv r > R_0$ where $\gamma < 2$ and $C > 0$. Then the Hamiltonian $H = -\Delta + V(r)$ has infinitely many bound states.

Proof. We first recall[8]) three facts:

(i) If A is any operator which is self-adjoint and bounded below and if:

$$\mu_n(A) = \max_{\phi_1, \dots, \phi_{n-1}} \left[\min_{\psi \in D(A)} \{\langle\psi, A\,\psi\rangle \mid \langle\psi, \phi_i\rangle = 0;\ \ \|\psi\| = 1\}\right]$$

then either $\mu_n(A)$ is the n-th eigenvalue (counting multiplicity) from the bottom of the spectrum or it is the bottom of the continuous spectrum. In the latter case $\mu_n(A) = \mu_{n+1}(A) = \dots$ and there are no more than $n-1$ eigenstates below the continuum.

(ii) For a two-body system with a Kato potential, the continuous spectrum begins at $E = 0$.

(iii) if one can find an N-dimensional subspace, V, of $D(A)$ with $\langle\psi, A\,\psi\rangle \leqslant C\,\|\psi\|^2$ for all $\psi \in V$, then $\mu_n(A) \leqslant C$ for $n = 1, 2, \dots, N$.

By (ii), if we can prove $\mu_n(H) < 0$ for all n, the proposition will be proven, and by (iii) we can show this by finding, for arbitrary N, an N-dimensional space $V_N \subset D(H)$ with $\max_{\psi \in V, \|\psi\| = 1} \langle\psi, H\,\psi\rangle \equiv V_N < 0$[9]).

Pick any C^∞ function ϕ with $\operatorname{supp} \phi \subset \{r \mid 2\,R_0 > |r| > R_0\}$ and $\|\phi\|_2 = 1$. Let $A = \langle\phi, (-\Delta)\,\phi\rangle$ and $B = \langle\phi, (C\,r^{-\gamma})\,\phi\rangle > 0$. Let $\phi_R(r) = R^{-3/2}\,\phi(r\,R^{-1})$ $(R \geqslant 1)$. Then $\|\phi_R\|_2 = 1$ and:

$$\langle\phi_R, H\,\phi_R\rangle \leqslant \langle\phi_R, (-\Delta)\,\phi_R\rangle + \langle\phi_R, (-C\,r^{-\gamma})\,\phi_R\rangle = A\,R^{-2} - B\,R^{-\gamma}\,.$$

[7]) By Kato potential, we mean $V \in L^2 + (L^\infty)_\varepsilon$, i.e. a function V so that for any ε, there is $V_{1,\varepsilon} \in L^2$, $V_{2,\varepsilon} \in L^\infty$, $\|V_{2,\varepsilon}\|_\infty < \varepsilon$, $V = V_{1,\varepsilon} + V_{2,\varepsilon}$ (e.g. r^{-1}). Such potentials were first treated by T. Kato [14].

[8]) (i) is an exercise in spectral analysis and is essentially Weyl's min-max principle; (ii) is a special case of Hunziker's theorem (!) and is due to Weyl originally; (iii) is an elementary consequence of the definition of μ_n.

[9]) This method of studying the number of bound states, which we will use over and over goes back at least as far as Kato [15].

Because $\gamma < 2$, the negative potential wins out over the positive kinetic energy for R large enough so $\langle\phi_R, H\,\phi_R\rangle \equiv \nu_1 < 0$ for some R. If $\psi_n = \phi_{2^n R}$ then the ψ_n's have disjoint support and each $\langle\psi_n, H\,\psi_n\rangle \equiv U_n < 0$. Because H is local, $\max_{\|\psi\| = 1,\ \psi \in [\psi_1, \dots, \psi_N]} \langle\psi, H\,\psi\rangle \equiv \max_{i = 1, \dots, N} (\nu_i) < 0$. Thus taking $V_N = [\psi_1, \dots, \psi_N]$, the proposition is proven.

For the borderline behavior $V \underset{r \to \infty}{\simeq} -C\,r^{-2}$, the finer borderline is $C = 1/4$:

Proposition 2. Let V be a Kato potential with $V(r) < -C\,r^{-2}$ for $r > R_0$ where $C > 1/4$. Then $H = -\Delta + V$ has infinitely many bound states.

Proof. It is enough to find one ϕ with compact support in $\{r \mid r > R_0\}$ with $\langle\phi, (-\Delta - C\,r^{-2})\,\phi\rangle \equiv \alpha < 0$ for letting ϕ_R be defined as in the last proof:

$$\langle\phi_R, H\,\phi_R\rangle \leqslant \alpha\,R^{-2} < 0$$

and we can find ϕ_R's with disjoint supports and complete the argument as in the previous proof. It is a simple exercise to find such a ϕ[10]).

In the other direction, we have:

Proposition 3. Let V be a Kato potential with $V(r) \geqslant -C\,r^{-2}$ for some $C < 1/4$ (for example, suppose $V(r) \geqslant D\,r^{-\gamma}$; $r > R_0$ for some $\gamma > 2$). Then $H = -\Delta + V$ has only finitely many bound states.

Proof. It is proven in [13] (p. 446) that $\langle\psi, (-\Delta - 1/4\,r^{-2})\,\psi\rangle \geqslant 0$ for any $\psi \in C_0^\infty$[11]) and this can be extended to $D(-\Delta)$ by a closure argument. Thus, for $a = (1/4 - C)$:

$$H \geqslant -a\,\Delta + \tilde{V} \equiv \tilde{H}$$

where $\tilde{V}(r) = 0$ if $r > R_0$ and say $\tilde{V}(r) = V(r)$ if $r \leqslant R_0$. Since V is L^2 and L^1, one can show that [10]:

$$\int \frac{|\tilde{V}(x)|\ |\tilde{V}(y)|}{|x-y|^2}\, d^3x\, d^3y < \infty\,.$$

By a theorem of Schwinger [16], $\tilde{H}$ has only finitely many bound states[12]), so $\mu_n(\tilde{H}) = 0$ for some n. Thus $\mu_n(H) = 0$ for some n so H has only finitely many bound states.

Having reviewed the two-body case, let us, following a suggestion of Nelson[13]), compare this result with classical mechanics. As Nelson has remarked[13]), the classical analogue of an infinitude of bound states below the continuum is to have the phase space orbits which are bounded in x-space and which lie *below 'the continuum'* fill an infinite volume in phase space. 'The continuum' here is the set of energies for which there are orbits of unbounded extent. If $V \to 0$ at ∞, then the continuum is $(0, \infty)$ or $\{[0, \infty]\}$. There is a direct relation between the quantum mechanical notions in the two-body case provided one takes the 'right' classical limit[14]):

[10]) In appendix 5, we do a computation needed in section 8. The computations there can be used to construct such a ϕ. Alternately, see [5].

[11]) This result can be viewed as a quantitative form of the uncertainty principle.

[12]) In many cases, if we take $\hat{V} = \min(0, V + (r^{-2})$, $\hat{V}$ will be bounded below so that we can dispense with Schwinger's theorem and compare H with a square well Hamiltonian.

[13]) E. Nelson, private communication.

[14]) The proposition is false with $H_{Cl} = p^2 + V$ for take $V = -1/4\,r^{-2}$.

Proposition 4. Let V be a smooth, bounded potential[15]) with:

(a) $\lim_{r \to \infty} V(r) = 0$,

(b) $\lim_{r \to \infty} r^2 V(r)$ exists[16]) and $\neq -1/4$.

Then the quantum mechanical Hamiltonian $H = -\Delta + V$ has infinitely many bound states, if and only if the classical Hamiltonian

$$H_{cl} = p^2 + V + \frac{1}{4} r^{-2}$$

has a set of orbits, each of finite spatial extent and each with $E < 0$, which fills an infinite volume of phase space.

Note: The extra $1/4\, r^{-2}$ is, of course, familiar from W.K.B. theory. It also arose in Nelson's analysis of Feynman path integrals for r^{-2} potentials [17].

Proof. The orbits of finite extent and $E < 0$ are precisely those orbits with $E < 0$. Thus, the total phase space volume in question is just $\int_{H_{cl} < 0} d^3r\, d^3p$. From this and propositions 1–3, the theorem easily follows.

It is worth remarking that the analogue of proposition 4 doesn't hold in the n-body case, essentially because different mechanisms determine the continuum limit in the classical case $\left\{\text{where } \sum_{cl} = \min_{D_1 \cap D_2 = \phi,\, D_1 \cap D_2 = \{1, \dots, n\}} \left[\inf_{p, r} (H_{D_1}(P_1, r_1) + H_{D_2}(P_2, r_2))\right]\right\}$ and the quantum mechanical case {where (1) holds}.

3. Two Cluster Continuum Limit: Infinitude

Our goal in this section is to prove:

Theorem 1. Let H be an n-body Hamiltonian system with center of mass removed, and suppose H has a two cluster continuum limit $\{D_1, D_2\}$. For each $i \in D_1$, $j \in D_2$, suppose

$$V_{ij}(r) \leqslant C_{ij}\, r^{-\gamma}$$

for $r > R_0$, γ independent of (i, j). If *either:*

(a) $\gamma < 2, \sum C_{ij} < 0$

or

(b) $\gamma = 2, \sum C_{ij} < -\dfrac{1}{2} m_D$,

then H has infinitely many bound states below the continuum. ■

15) These conditions are supposed merely to avoid the need of discussing any singularities in the classical flow.

16) The limit is allowed to be $\pm \infty$. In fact, we only need $\overline{\lim}_{r \to \infty} r^2 V < -\frac{1}{4}$ or $\underline{\lim}_{r \to \infty} r^2 V > -\frac{1}{4}$.

This theorem depends on:

Proposition 5. Let ψ_i be a bound state of H_{D_i}, a k_i-body system with Kato potentials ($i = 1, 2$). Let V_{ij} be a Kato potential with

$$\overline{\lim_{r \to \infty}}\, r^\gamma\, V_{ij}(r) \leqslant C$$

for some $\gamma \leqslant 2$. Let

$$\tilde{V}_{ij}(R) = \int |\psi_1(r_1)|^2\, |\psi_2(r_2)|^2\, V_{ij}(r_{ij}(R, r_1, r_2))\, d^{3(k_1-1)} r_1\, d^{3(k_2-1)} r_2$$

where $r_{ij}(R, r_1, r_2)$ is the distance between particle $i \in D_1$ and $j \in D_2$ in terms of the internal coordinates r_1, r_2 or D_1, D_2 and the distance, R, between the centers of mass of D_1 and D_2[17]). Then $\overline{\lim}_{R \to \infty} R^\gamma\, \tilde{V}_{ij}(R) \leqslant C$.

Proof of Theorem 1. By a kinematical computation[17]):

$$H = H_{D_1} + H_{D_2} + (2\, m_D)^{-1}\, P_{12}^2 + \sum_{i \in D_1;\, j \in D_2} V_{ij}(r_{ij}) \tag{3}$$

where P_{12} is the momentum conjugate to R. Let us consider trial wave functions of the form:

$$\phi(r_1, r_2, R) = \psi_1(r_1)\, \psi_2(r_2)\, \eta(R)$$

where ψ_1, ψ_2 are the ground states of H_{D_1} and H_{D_2}. Then:

$$\langle \phi, H\, \phi \rangle = \sum + \langle \eta, \tilde{H}\, \eta \rangle$$

where $\sum$ is the continuum limit and

$$\tilde{H} = (2\, m_D)^{-1}\, P_{12}^2 + \sum_{i \in D_1;\, j \in D_2} \tilde{V}_{ij}(R)\,.$$

By Propositions 1, 2, 5 and the assumptions for the theorem, given any N, we can find an N-dimensional space, V_N, of $\eta \in D(-\Delta)$ so $\langle \eta, \tilde{H}\, \eta \rangle < 0$, all $\eta \in V_N$. Correspondingly, we can find an N-dimensional space $W_N \subset D(H)$, $W_N = \{\psi_1 \psi_2 \eta \mid \eta \in V_N\}$ with $\langle \phi, H\, \phi \rangle < \sum$ all $\phi \in W_N$. Thus for any N, H has at least N bound states. ■

Remark. This proof did not use the fact that D_1, D_2 uniquely determines $\sum$. If several competing decompositions determine $\sum$, there will be infinitely many bound states as long as one of the decompositions obeys the conditions of the theorem.

All that remains is to prove proposition 5. As preparation, we note[17]):

$$r_{ij}(R, r_1, r_2) = R - \sum b_k\, \eta_k \equiv R - r_0$$

where the η_k are internal coordinates. By changing internal coordinates, we find:

$$\tilde{V}_{ij}(R) = \int d^3r_0\, \varrho(r_0)\, V_{ij}\,(R - r_0) \tag{4}$$

where

$$\varrho(r_0) = \int d^{3(k_1 + k_2 - 3)}\, r\, |\psi_1(r_1)\, \psi_2(r_2)|^2 \tag{5}$$

with the integration over all internal coordinates but r_0. To prove proposition 5, we first show:

17) See appendix 1.

Lemma:

(a) $\varrho(r_0) \in L^1 \cap L^2(R^3)$,

(b) $r_0^2 \varrho(r_0) \in L^1 \cap L^2(R^3)$.

Proof. (a) Since $\psi_1 \psi_2 \in L^2(R^{3(k_1+k_2-2)})$, we see $\varrho \in L^1$. We have $\psi_1 \psi_2 \in D(H_{D_1} + H_{D_2})$. Thus, by Kato's theorem [14], for any $f \in L^2$, $\psi_1 \psi_2 \in D(f(r_0))$. In particular, $\langle \psi_1 \psi_2, |f| \psi_1 \psi_2 \rangle \equiv \int \varrho(r_0) f(r_0) d^3 r_0 < \infty$. Thus $\varrho \in L^2$.

(b) By a result of Hunziker [7, 18], $r_0 \psi_1 \psi_2 \in L^2$ since $\psi_1 \psi_2$ is a bound state of $H_{D_1} + H_{D_2}$. Thus $r_0^2 \varrho(r_0) \in L^1$. Since $[H_{D_1} + H_{D_2}, r_0] = C p_0$, we see that $(H_{D_1} + H_{D_2})(r_0 \psi_1 \psi_2) = C p_0 \psi_1 \psi_2 + E r_0 \psi_1 \psi_2 \in L^2$ so that $r_0 \psi_1 \psi_2 \in D(H_{D_1} + H_{D_2})$. As in (a), this implies $r_0^2 \varrho(r_0) \in L^2$. ■

Finally, we conclude this section with:

Proof of proposition 5.

$$\tilde{V}(R) = \int \varrho(r_0) V(R - r_0) d^3 r_0 = \int \varrho(R - r) V(r) d^3 r.$$

Without loss of generality, we can suppose $V(r) = C r^{-\gamma}$ for $r > R_0$ since the proposition follows easily from this special case. In this case

$$\int \varrho(R - r) r^\gamma V(r) d^3 r \underset{R\to\infty}{\to} C$$

so we need only show

$$\int |\varrho(R - r)| \, |R^\gamma - r^\gamma| \, |V(r)| \, d^3 r \underset{R\to\infty}{\to} 0. \tag{6}$$

But, for any ε:

$$|R^\gamma - r^\gamma| \leqslant [(1 + \varepsilon)^\gamma - 1] r^\gamma + (1 + \varepsilon^{-1})^\gamma |R - r|^\gamma$$

[for if $(1 + \varepsilon)^{-1} r \leqslant R \leqslant (1 + \varepsilon) r$, the first term majorizes the left and if $R \geqslant (1 + \varepsilon) r$ or $r \geqslant (1 + \varepsilon) R$, the second term majorizes the left.]

Thus:

$$\begin{aligned}\overline{\lim_{R\to\infty}} \left| \int \varrho(R - r) \right| R^\gamma - r^\gamma |V(r) \, dr| &\leqslant [(1 + \varepsilon)^\gamma - 1] \\ &\times \left[\overline{\lim_{R\to\infty}} \int |\varrho(R - r)| \, |r^\gamma V(r)| \, dr \right] + (1 + \varepsilon^{-1})^\gamma \\ &\times \left[\overline{\lim_{R\to\infty}} \int (|R - r|^\gamma \varrho(R - r)) \, |V(r)| \, dr \right].\end{aligned}$$

Since $V \in L^2 + (L^\infty)_\varepsilon$ and $|R - r|^\gamma \varrho(R - r) \leqslant (1 + |R - r|^2)^2 \varrho(R - r) \in L^1 \cap L^2$ the second term goes to 0 as $R \to \infty$. So for all ε

$$\overline{\lim_{R\to\infty}} \left| \int |\varrho(R - r)| \, |R^\gamma - r^\gamma| \, |V(r)| \, d^3 r \right| \leqslant |C| \, [(1 + \varepsilon)^\gamma - 1].$$

This implies the limit in (6) is true so the proof is completed.

4. Hunziker's Theorem with Statistics

Our goal in this section (and appendix (4)) is to prove Hunziker's theorem and an analogue of Theorem 1 for n-body systems with non-trivial statistics. Here, we consider Fermi-Dirac statistics and in appendix (4) we consider general statistics with the aid of a little symmetric group theory.

Let us suppose $\{1, \ldots, n\}$ is partitioned into sets $C_1, \ldots, C_m$. The particles in C_i are all identical and are assumed to be Fermions; thus the allowable wave functions must be anti-symmetric under exchange of coordinates[18]) within one cluster. We will call $\{C_i\}_{i=1}^m$ the statistical partition. Let D_1, D_2 be another partition of $\{1, \ldots, n\}$. We will say $\{C_i \cap D_1\}_{i=1}^m$ is the induced statistical partition for D_1 and similarly for D_2. Then:

Theorem 2. Let H be an n-body Hamiltonian with center of mass removed, symmetric in the coordinates of each cluster $C_1, \ldots, C_n$ and restricted to the space of allowable wave functions antisymmetric in each C_i. Given $D = \{D_1, D_2\}$, a partition of $\{1, \ldots, n\}$, let $H_D = H_{D_1} + H_{D_2}$ where H_{D_1} is the part of H depending on the particles in D_1 restricted to the space of functions antisymmetric in the induced statistical partition. Let $\Sigma = \inf(\sigma_{cont}(H))$. Then

$$\Sigma = \min_{D_1, D_2} \{\inf(\sigma(H_{D_1})) + \inf(\sigma(H_{D_2}))\}.$$

Proof. Let us first show that $[\Sigma, \infty)$ is in the spectrum of H[19]). Suppose $\lambda \in [\Sigma, \infty)$ and ε are given. We need only show some $u \in D(H)$ exists with $\|(H - \lambda) u\| < \varepsilon \|u\|$. Suppose $\Sigma = s_1 + s_2$ with $s_i = \inf \sigma(H_{D_i})$ and write $\lambda = s_1 + s_2 + \lambda_R$; $\lambda_R \geqslant 0$. Let us decompose H in the form (3). Pick a function, ψ_1, C^∞ of compact support in the coordinates of D_1, and with the restricted symmetry so $\|(H_{D_1} - s_1) \psi_1\| < 1/4 \, \varepsilon \|\psi_1\|$. Pick ψ_2 in a similar manner. Now pick a function f of compact support in R^3 with $\|(2 m_D^{-1} P^2 - \lambda_R) f\| < 1/4 \, \varepsilon \|f\|$. Let $f_a(r) = f(r - a)$ for any $a \in R^3$. As in [7], $\lim_{a\to\infty} \langle \psi_1 \psi_2 f_a, \left(\sum_{i \in D_1; j \in D_2} V_{ij}\right) \psi_1 \psi_2 f_a \rangle = 0$. Thus, pick a so that this last expectation value is less then $1/4 \, \varepsilon \|\psi_1 \psi_2 f\|$ and so that for any $i \in D_1$, $j \in D_2$:

$$\mathcal{D}_i [\operatorname{supp} \psi_1] \cap [\mathcal{D}_j (\operatorname{supp} \psi_2) + \operatorname{supp} f_a] = \phi \tag{7}$$

where $\mathcal{D}_i$ is the projection of $R^{3m} \to R^3$ given by projecting onto the coordinate r_i. Let $v = \psi_1 \psi_2 f_a$. By our choices, $\|(H - \lambda) v\| \leqslant \varepsilon \|v\|$. Let $\mathcal{A}$ be the operator that antisymmetrizes in the clusters C_i, is

$$\mathcal{A} = \frac{1}{(\# C_1)! \ldots (\# C_m)!} \sum_{\pi | \pi C_i \subset C_i} (\operatorname{sgn} \pi) \, \pi$$

where π is a generic permutation. Let $u = \mathcal{A} v$. Then u has the required symmetry. If π does not exchange particles between D_1 and D_2, $\pi v = v$. If it does, $\operatorname{supp} v \cap \operatorname{supp} \pi v = \phi$

[18]) The coordinates may be purely spatial or combined spin-spatial. If the Hamiltonian contains combined spin-spatial terms (e.g. spin-orbit terms) we must exchange the full spin-spatial coordinates; otherwise, 'exchange' can mean either spin or space, or both.

[19]) The proof is similar to the analogous proof in Hunziker's paper [7]. The big difference is that some care is needed to assure antisymmetrization doesn't kill the trial function.

by condition (7) so that there are no 'cross-terms' in $\|u\|$ or $\|(H-\lambda)u\|$. Thus $\|(H-\lambda)u\| \leqslant \varepsilon \|u\|$. This proves $[\Sigma, \infty) \subset \sigma(H)$.

Now let $D_1, \ldots, D_k$ be a partition of $\{1, \ldots, n\}$. Then

$$\Sigma \leqslant \inf \sigma(H_{D_1}) + \cdots + \inf \sigma(H_{D_k}) \tag{8}$$

for by what we have just proven,

$$\inf \sigma(H_{D_1}) + \inf \sigma(H_{D_2}) \geqslant \inf \sigma(H_{D_1 \cup D_2})$$

etc.

Finally, using (8), we show the spectrum of H in $(-\infty, \Sigma)$ is discrete. It is sufficient to show the Weinberg connected kernel [19], $I(z)$, is compact and analytic in the plane cut by $[\Sigma, \infty)$, for then we need only apply Hunziker's beautiful argument [7]. By using the ideas sketched in appendix (3), it is enough to prove analyticity in the cut plane (since we have compactness for $Re\, z$ very negative). $I(z)$ is a sum of terms essentially called $I_s(z)$ by Hunziker so we need only prove each $I_s(z)$ is analytic. S here stands for a collection of partitions $P_1, \ldots, P_N$ where P_i has i elements and P_{i+1} is a refinement of P_i[20]). Let $G_{P_i}(z)$ be the Green's function for $H_{P_i} = \sum_{D \in P_i} H_D$ (i.e. the Hamiltonian, H, with interactions between clusters in P_i removed). If P_{i+1} is obtained by partitioning $A \in P_i$ into $B_1 \cup B_2$, we write

$$P_{P_{i+1} P_i} = \sum_{i \in B_1, j \in B_2} V_{ij} .$$

Then

$$I_s(z) = G_{P_N}(z)\, V_{P_N P_{N-1}}\, G_{P_{N-1}}(z) \ldots G_{P_1}(z)\, V_{P_2 P_1} .$$

By (8), each $G_{P_i}(z)$ (when restricted to states with the symmetry induced on P_i) is analytic in the plane cut by (Σ, ∞). If ψ has the symmetry of the full statistical partition, $V_{P_2 P_1}\psi$ has the symmetry induced on P_2 so $G_{P_2}(z)\, V_{P_2 P_1}\psi$ has this same symmetry and is analytic in the cut plane. $V_{P_3 P_2}\, G_{P_2}(z)\, U_{P_2 P_1}\psi$ has the symmetry induced on P_3, etc. This completes the proof[21]). ∎

We can now prove the analogue of theorem 1 (slightly weakened):

Theorem 3. Let H be an n-body Hamiltonian with center of mass removed, symmetric in the coordinates of each cluster $C_1, C_2, \ldots, C_n$ and restricted to the space of allowable wave functions antisymmetric in each C_i. Let the continuum limit be determined (according to theorem 2) by a two-cluster breakup D_1, D_2 and suppose:

(a) H_{D_i} has an eigenstate ψ_i at the bottom of its spectrum.

(b) For each $i \in D_1$, $j \in D_2$:

$$V_{ij}(r) \leqslant C_{ij}\, r^{-\gamma} \qquad (r > R_0)$$

with

$$\gamma < 2; \qquad \sum C_{ij} < 0 .$$

Then H has infinitely many bound states below the continuum.

[20]) Thus, P_{i+1} comes from P_i by splitting one set of P_i into two parts.

[21]) We have proven $I_s(z)$ is analytic as a map of the fully symmetrized functions to functions with no particular symmetry. Since $I(z) = \sum_s I_s(z)$ leaves the former invariant, $I(z)$ is an analytic family of maps of this space to itself.

Proof. Let us only sketch the proof. The idea is to take trial functions of the form $\psi_1 \psi_2 \eta$ (as in theorem 1) and antisymmetrize them. Pick a small sphere, S, about $(0, 0, 1)$ so that $|r_1 - r_2| < Q$ if $r_1, r_2 \in S$ where Q is to be determined shortly. We pick an N-dimensional space, V, of η's with support in this sphere and are prepared to try trial wave functions of the form $\psi_1 \psi_2 \eta_R$ with $\eta_R(r) = R^{-3/2}\,\eta(r/R)$. Let us introduce the shorthand $\mathcal{A}\, u = [u]$. Then it is sufficient to show (given N, a priori) for some R and some $\varepsilon > 0$ and all $\eta \in V$:

$$\langle [\psi_1 \psi_2 \eta_R], H[\psi_1 \psi_2 \eta_R] \rangle \leqslant (\Sigma - \varepsilon)\, \|[\psi_1 \psi_2 \eta_R]\|^2 . \tag{9}$$

In antisymmetrizing, the permutations [really $(\mathrm{sgn}\,\pi)\,\pi$] fall into classes which produce identical results when applied to $\psi_1 \psi_2 \eta_R$. If it were not for the cross terms between classes, the inequality (9) would follow as it did in theorem 1. If we can show each cross term falls off as R^{-2}, (9) can still be proven since the 'direct' terms have a ratio below $\Sigma - C\, R^{-\gamma}$ $(C > 0)$.

Let $R\,S = \{R\,x \mid x \in S\}$ so supp $\eta_R \subset R\,S$. Consider a cross-term in $\|[\psi_1 \psi_2 \eta_R]\|^2$. It is of the form:

$$\mathcal{D}_R \equiv \int \overline{\psi_1(y_1)\, \psi_2(y_2)\, \eta_R(R_{12})}\, \psi_1(y_1')\, \psi_2(y_2')\, \eta_R(R_{12}')\, dy_1\, dy_2\, dR_{12}$$

where y_i are the internal coordinates for D_i and $'$ means the coordinates with $r_k \in D_1$ and $r_l \in D_2$ interchanged. Then by the Schwartz inequality:

$$|\mathcal{D}| \leqslant \int_{\substack{R_{12} \in RS \\ R_{12}' \in RS}} |\psi_1(y_1)|^2\, |\psi_2(y_2)|^2\, |\eta_R(R_{12})|^2\, dy_1\, dy_2\, dR_{12} .$$

By computations in appendix 1, $R_{12} = r_{kl} + x_1 + x_2$, $R_{12}' = (1 - M_1^{-1} m_k - M_2^{-1} m_l)\, r_{kl} + x_1 + x_2$ where x_1 and x_2 only involve internal coordinates from clusters 1 and 2 respectively. If $R_{12}', R_{12} \in R\,S$, then $|r_{kl}| = C_{kl}\, |R_{12} - R_{12}'| \leqslant C_{kl}\, Q\, R$. By choosing Q sufficiently small, we can assume $|r_{kl}| < R/3$; $|R_{12}|, |R_{12}'| > 2\,R/3$. Thus

$$|\mathcal{D}| \leqslant \int_{|x_1 + x_2| \geq R/3} |\psi_1(y_1)|^2\, |\psi_2(y_2)|^2\, |\eta_R(R_{12})|^2\, dR_{12}\, dy_1\, dy_2$$

$$\leqslant \int_{|x_1| \geq R/6} \varrho(x_1)\, dx_1 + \int_{|x_2| \geq R/6} \varrho_2(x_2)\, dx_2$$

where $\varrho_1(x_i)$ is the probability density for x_1 in the state ψ_1. As in the proof of theorem 1, $x_1^2\, \varrho(x_1) \in L^1$. Thus

$$\int_{|x| \geq R/6} \varrho(x)\, dx \leqslant \int_{|x| \geq R/6} \left(\frac{6\,|x|}{R}\right)^2 \varrho(x)\, dx \leqslant \frac{36}{R^2} \int_{|x| \geq R/6} x^2\, \varrho(x)\, dx$$

so $|\mathcal{D}| = 0(R^{-2})$. The other cross terms may be treated similarly. ∎

5. Bound States of Atoms and Molecules[22])

One of the more intriguing results concerning the infinitude of the number of bound states is the fact that the 'Helium atom Hamiltonian' (i.e. a three body system

[22]) Some of the material of this section is contained in a thesis [10] submitted to the Department of Physics of Princeton University in partial fulfillment of the requirements of Doctor of Philosophy.

with two attractive Coulomb forces of magnitude 2 and a repulsive Coulomb force of magnitude 1) has infinitely many bound states. That this can be shown for this problem which defies exact solution is a testament to the power of the qualitative methods functional analysis.

The earliest results on this problem were obtained by T. Kato [15] who showed that if the Hughes-Eckart terms were ignored (so-called infinite nuclear mass) then there were infinitely many bound states. When the Hughes-Eckart terms were accounted for and the physical masses used, he showed there were at least 25585 bound states[23]). The only reason Kato did not obtain infinitely many bound states was that he was unable to show that the continuum was where common sense (and Hunziker's theorem[24]) !) say it should be. *Given Hunziker's theorem, Kato's original 1951 argument implies that the 'Helium atom Hamiltonian' (with the physical masses) has an infinitude of bound states.*

By using methods similar to the ones we presented in section 3, Zhislin [3] was able to prove any atom or positive ion has infinitely many bound states[25]). Zhislin and Sigalov [4] extended this result to allow arbitrary statistics. Using theorems 1, 2, 3 and 2′, 3′ (see appendix 4) and the long range nature of the Coulomb force, we can recover their results easily:

Theorem 4. Let H be a Hamiltonian

$$H = \sum_{i,j=1} a_{ij} P_i P_j - \sum_{i=1}^{n} b_i |r_i|^{-1} + \sum_{i<j} c_{ij} |r_{ij}|^{-1}$$

where (a_{ij}) is a positive definite matrix and all $b, c \geqslant 0$. If $\sum_{j\neq i} c_{ij} < b_i$ for each fixed i, then H has an infinitude of bound states. If H has identical particles and is restricted to states of a certain symmetry[26]) the result is still true. ■

This theorem can be paraphrased as: 'Any atom or positive ion has infinitely many bound states'. If our criterion for necessity is correct, negative ions never have infinitely many bound states.

Supposing the heuristic necessary condition for the infinitude of the number of bound states is correct, we can also ask when certain 'molecules' will have infinitely many bound states. Consider first diatomic molecules, AB. Here the situation is simple. If A–B determines the continuum, there will only be finitely many bound states. If $A^+ - B^-$ or $A^- - B^+$ determines the continuum, we will have long range Coulomb forces and infinitely many bound states. To find an example of the latter situation, one must only find some A whose electron affinity is bigger than the ionization potential of some B. Alas, a look at the tables[27]) [20] shows the biggest

[23]) This corresponds to 42 principle shells. If rotationally degenerate states are not counted separately, i.e. if eigenvalues are counted, Kato showed there are at least 903.

[24]) The exact status of the Hughes-Eckart terms is not made explicit in Hunziker's paper. See appendix 2 for a discussion of this point.

[25]) Zhislin did not have Hunziker's theorem; he had to prove a special case of it.

[26]) See appendix 4 for the definition of 'states of a certain symmetry'.

[27]) These tables list experimental numbers for the *physical* atoms and molecules. Since the purely coulombic non-relativistic Hamiltonians are only approximately correct, our conclusions (about the pure Coulomb case) have a small possibility of error.

atomic affinity is smaller than the tiniest atomic ionization potential[28]). When one gets to polyatomic molecules, the situation gets more complicated and many more possibilities open. For infinitude, one needs the continuum to be determined by ions of opposite charge. I was unable to find any candidates for this situation. Since there are many competing two-cluster breakups, care is needed: for example the continuum limit for $Cs^+ + (Cl\,O_3)^-$ does lie below that for $Cs + Cl\,O_3$ [20] but for the total system, the lowest continuum state turns out to be $CsCl + O_2$.

6. Two Cluster Continuum Limit: Finiteness

Our goal in this section is to prove a *weak* theorem which implies that there are only finitely many bound states in a certain family of cases. Our main reason for including this rather trivial result is to provide rigorous proofs for the details of the next section.

Theorem 5. Let H be a three body Hamiltonian (with center of mass removed) where one of the bodies is infinitely heavy[29]) and where the force between the finite mass bodies is short range and repulsive, i.e.

$$H = (2\,m_1)^{-1} p_1^2 + (2\,m_2)^{-1} p_2^2 + V_1(r_1) + V_2(r_2) + V_{12}(r_{12}) ,$$

$$V_{12}(r_{12}) \geqslant 0 ;\ V_{12}(r_{12}) \leqslant C\, r_{12}^{-\gamma} ,\ \text{all } r > R_0 ;\ \gamma > 2 ;\ C > 0 .$$

Suppose H is in a two cluster continuum limit with a short range force between the clusters; explicitly, suppose

$$\inf\,[\sigma((2\,m_1)^{-1} p_1^2 + V_1)] < \inf\,[\sigma((2\,m_2)^{-1} p_2^2 + V_2)]$$

and

$$V_2(r) + V_{12}(r) \geqslant -D r^{\alpha} \qquad \alpha > 2 ;\ D > 0 .$$

Then H has only finitely many bound states[30]).

Proof. Let $\tilde{H} = H - V_{12}$. By Hunziker's theorem, H and $\tilde{H}$ have the same continuum limit. Since $V_{12} > 0$, $\tilde{H} < H$ so the Weyl min-max principle tells us the number of bound states for H cannot exceed the number of bound states for $\tilde{H}$. Thus, it is sufficient to show $\tilde{H}$ has only finitely many bound states. But $\tilde{H} = H_1 + H_2$ where $H_i = (2\,m_i)^{-1} p_i^2 + V_i$. H_1 and H_2 commute so $\sigma(\tilde{H}) = \sigma(H_1) + \sigma(H_2)$. Since $\{0, 1\}$, $\{2\}$ determines the continuum $\inf(\sigma(H_1)) < \inf(\sigma(H_2)) = \varepsilon_0 + \inf(\sigma(H_1))$. Thus bound states[30]) are of the form $\psi_1 \psi_2$ where $H_i \psi_i = E_i \psi_i$, and $E_1 + E_2 < \inf(\sigma(H_1))$. Thus $E_1 < -\varepsilon$ so only finitely many ψ_1's can enter. By the short range nature of V_2, H_2 has only finitely many bound states so that the number of $\psi_1 \psi_2$ which are acceptable is finite. Thus $\tilde{H}$, and with it H, has only finitely many bound states. ■

Notes. 1. $\tilde{H}$ will in general have eigenfunctions above the continuum. Presumably, the addition of V_{12} makes these states 'dissolve' into the continuum.

2. I have barely failed several times in proving a strong version of theorem 5 allowing Hughes-Eckart terms and without any assumptions on V_{12}. I have an uneasy feeling a simple proof of the stronger conjecture exists.

[28]) I.P. (Cs) = 3.89 eV; E.A. (Cl) = 3.76 ± 0.05 eV.

[29]) This means we surpress the Hughes-Eckart terms.

[30]) Remember, bound states are, *by definition*, *below* the continuum.

7. Coupling Constant Dependence

In the body case, the number of bound states, $H(\lambda) \equiv H_0 + \lambda V$ is a monotonically increasing function of the coupling constant, λ (we take $\lambda > 0$). The reason for this is quite simple: The functions $\mu_n(H(\lambda))$ defined in the proof of proposition 1 are monotonically decreasing functions of λ[31]). Moreover, the continuum limit is fixed at 0. Thus:

$N(\lambda) = \#$ of $\mu_n(\lambda)$ with $\mu_n(\lambda)$ strictly less than 0 is monotone in λ. In particular, if $N(\lambda_0) = \infty$, then $N(\lambda) = \infty$ for all $\lambda \geqslant \lambda_0$.

In the N-body case, this argument breaks down. The μ_n are still monotone decreasing but the continuum limit is also decreasing. There thus arises the possibility that the continuum might overtake some of the bound states, gobble them up and presumably digest them (i.e. the eigenvalues probably do not persist but 'dissolve' in the continuous spectra). If the criterion in the introduction is correct, this possibility is quite common – in fact, the possibility of an infinite number of states being absorbed is common in systems with some long range and some short range forces.

Let us consider some examples. In order to be able to make rigorously proven statements, we only consider Hamiltonians of the form considered in theorem 5: $H(\lambda) = p_1^2 + p_2^2 + \lambda V_1 + \lambda V_2 + \lambda V_{12}$.

As a first example, let V_{12} be a repulsive square well, V_1 an attractive square well and V_2 an attractive Coulomb so normalized that the ground state energy of $p_1^2 + V_1$ is less than that of $p_2^2 + V_2$. Consider first what clustering determines the continuum for $H(\lambda)$. When λ is small, $H_1(\lambda) = p_1^2 + \lambda V_1$ has no bound states, while $H_2(\lambda)$ has a ground state energy, $G_2(\lambda)$, equal to $C\lambda^2$ $(C < 0)$, so for λ small $\{2, 0\}$, $\{1\}$ determines the continuum. At some critical $\lambda_0 < 1$, the two ground state energies become equal since at $\lambda = 1$, the clustering $\{1, 0\}$, $\{2\}$ determines the continuum. For large λ, $G_2(\lambda) < G_1(\lambda)$ since $G_1(\lambda) > D\lambda$ $(D < 0)$. Thus, for some $\lambda_1 > 1$, the clustering $\{2, 0\}$, $\{1\}$ takes over again. When have we an infinitude of bound states? By theorems 1 and 5, when and only when 0 and 2 are in different clusters in a clustering determining the continuum. Thus $H(\lambda)$ has infinitely many bound states only when $\lambda \in [\lambda_0, \lambda_1]$.

We thus see explicitly the phenomena we noted as a possibility above. As λ passes λ_1, infinitely many bound states are 'gobbled up' by the continuum. What happens near $\lambda = \lambda_1$ is quite interesting from a perturbation theoretic point of view. At $\lambda = \lambda_1$, there are infinitely many discrete bound states. By the Kato-Rellich theory [21], each state is analytic (both the eigenvalue and eigenvector) in some small circles about $\lambda = \lambda_1$, but the size of the circle can depend on the level. In fact the size of the circles shrinks as the quantum numbers of the unperturbed level go to infinity. This shrinkage takes place in such a way that at any $\lambda < \lambda_1$ (i.e., a *finite* distance below) there are only finitely many levels left. We also note that if the two body case[32]) is a reliable guide, the *eigenvalues* may have an *analytic continuation* into larger circles

[31]) The reason for this is simple: $\mu_n \leq 0$ and $\mu_n < 0$ if there is an n-dimensional subspace on which $H_0 + \lambda V < 0$. Since H_0 is positive, $V < 0$ on this subspace so $\tilde{\lambda} > \lambda$ implies $H_0 + \lambda V < H_0 + \lambda V$ on this subspace also.

[32]) For a discussion of this case, see § II.3 of [22].

than the circles in which they represent actual levels – it is thus quite possible that by looking only at the perturbation series for eigenvalues, the catastrophic change at $\lambda = \lambda_1$ might not be noticeable.

By adjusting the relative strength of V_1 and V_2, we can obtain a system with the amusing property that $H(\lambda)$ has an infinitude of bound states for precisely one value of λ. We can also modify V_1 into a Yukawa and V_2 into a Coulomb cutoff at small distances and thereby obtain a system with an infinity of bound states when $\lambda \in [\lambda_0, \infty)$[33]).

More bizarre possibilities can occur. Suppose V_1 is short range and V_2 long range and that $G_1(\lambda)$ and $G_2(\lambda)$ cross infinitely often, i.e. there are $\lambda_0 = 0, \lambda_1, \lambda_2, \ldots$ so that $G_1(\lambda) < G_2(\lambda)$ if $\lambda \in (\lambda_{2i+1}, \lambda_{2i+2})$ and $G_1(\lambda) > G_2(\lambda)$ of $\lambda \in (\lambda_{2i}, \lambda_{2i+1})$. Then $H(\lambda)$ has infinitely many bound states if $\lambda \in [\lambda_1, \lambda_2] \cup [\lambda_3, \lambda_4] \cup \ldots$ so the continuum keeps absorbing and re-emitting an infinity of bound states. A priori, it is not clear that such a V_1 and V_2 exist. In reference [23], it was shown the large λ behavior of $G(\lambda)$ mirrors the small r behavior of $V(r)$, so it is to be expected the requirement that V_2 be long range and V_1 be short range should present no problem. In fact, using the techniques of the last section of [23], V_1 and V_2 with the properties above can be constructed. One takes $V_1 \sim r^{-\alpha}$ as $r \to 0$ and V_2 to grow alternately as $r^{-(\alpha+\epsilon)}$, $r^{-(\alpha-\epsilon)}$ as $r \to 0$, adjusting the size of the intervals of growth for V_2 suitably.

To summarize: *In any N-body system, the number of bound states is not necessarily a monotone function of coupling constant. In fact, as the coupling is increased, an infinity of bound states can disappear.* These facts of N-body life do not bode well for any attempt to carry over the beautiful theorems on the number of bound states in the two-body case (see, e.g. [24]).

8. N-Cluster Continuum Limit

We have thus far presented a fairly complete discussion of the question of infinitude or finiteness when the continuum is determined by a clustering D_1, D_2 where H_{D_1} and H_{D_2} have eigenvalues at the bottom of their spectrum. In this section, we wish to make a preliminary analysis of what happens when this is not the case.

Suppose H_{D_1} has continuum at the bottom of its spectrum. Then $D_1 = D_{11} \cup D_{12}$ (D_{11}; D_{12} disjoint) with $\sigma(H_{D_1}) = \inf \sigma(H_{D_{11}}) + \inf \sigma(H_{D_{12}})$. We can continue this process until we find a decomposition $C_1, \ldots, C_m$ with $(\sigma_{cont}(H)) = \sum_{i=1}^{m} \inf \sigma(H_{C_i})$ and so that each H_{C_i} has an eigenfunction at the bottom of its spectrum[34]). We will now slightly change terminology and only say $C_1, \ldots, C_m$ determines the continuum. We will say H has an m-cluster continuum limit. Continuum states near the continuum limit look like *bound* clusters $C_1, \ldots, C_m$ moving relative to one another in an unbound manner.

[33]) For comparison, we note in the two-body case, say $V = -r^{-2}$, $N(\lambda) = \infty$ in the *open* interval $(1/4, \infty)$.

[34]) If C_i has one particle, H_{C_i} (which has the center of mass removed!) is 0 on the one dimensional space $\mathbf{C}$. Thus, one particle clusters always have an eigenvalue at the bottom of their spectrum.

We remark that by the way we obtained the clustering, $\inf \sigma(H_{C_1 \cup C_2}) = \inf \sigma(H_{C_1}) + \inf \sigma(H_{C_2})$. In particular, there cannot be any really long range force (i.e. $-r^{-\gamma}$, $\gamma < 2$) between the clusters C_i. For if there were $H_{C_1 \cup C_2}$ would have an infinitude of bound states thereby violating the inf condition above.

If there are only really short range forces (i.e. $-r^{-\gamma}$, $\gamma > 2$) or repulsive forces, we *expect* that there cannot be an infinitude of bound states.

That leaves the borderline situation of forces $V \underset{r \to \infty}{\sim} -r^{-2}$. Consider, for example, an N-body system of identical particles of mass m with r^{-2} potentials and center of mass removed, i.e.

$$H_N(C) = \sum_{i=1}^{N-1} (m)^{-1} p_i^2 + \sum_{i<j,\,1}^{N-1} (2\,m)^{-1} p_i\, p_j - C \sum_{i=1}^{N-1} V(r_i) - C \sum_{i<j,\,1}^{N-1} V(r_{ij})$$

where

$$V(r) = \begin{cases} r^{-2} & r \geqslant 1 \\ 1 & r \leqslant 1\,. \end{cases} \tag{10}$$

We have cut V off to ensure V is a Kato potential. We have removed the center of mass by using coordinates relative to the Nth particle (see appendix 1).

Let us set $m = 1$, and let us temporarily take $C = 1/4$. We wish to show for the case at hand that one has the following situation: for some $N_0 \geqslant 3$, H_N has *no* bound states for $N < N_0$ and an infinitude for $N \geqslant N_0$. We suspect $N_0 = 3$ but are unable to prove it. In appendix 5, we prove $N_0 \leqslant 7$!

Lemma a. If H_{N-1} has spectrum $[0, \infty)$ and $\langle \varphi, H_N \varphi \rangle < 0$ for some φ, then H_N has an infinitude of bound states.

Proof. If such a φ exists, we can find ψ, normalized and C^∞ of compact support in $R^{3N-3} - \{0\}$ with $\langle \psi, H_N \psi \rangle < 0$. If $\psi_R = R^{-1/2(3N-3)}\, \psi(r\, R^{-1})$ a simple computation shows $\langle \psi_R, H_N \psi_R \rangle \leqslant R^{-2} \langle \psi, H_N \psi \rangle$ if $R > 1$. We can thus find an infinity of ψ_i's $\equiv \psi_{R_i}$'s with disjoint supports and $\langle \psi_i, H_N \psi_j \rangle = 0$ if $i \neq j$; < 0 if $i = j$; $\langle \psi_i, \psi_j \rangle = \delta_{ij}$. Thus $\mu_n(H_N) < 0$ all n. If H_j had any spectrum in $(-\infty, 0)$ for $j < N - 1$, then H_{N-1} would have continuous spectrum in $(-\infty, 0)$ contrary to assumption. Thus $\sigma(H_j) = [0, \infty)$ all $j < N$ which implies $\sigma_{cont}(H_N) = [0, \infty)$. This combined with $\mu_n(H_N) < 0$ all n implies H_N has an infinitude of bound states. ■

Lemma b. For some N and some φ, $\langle \varphi, H_N \varphi \rangle < 0$.

Proof. Let f be spherically symmetric and C^∞ of compact support in R^3 with $\|f\|_2 = 1$. Let $\varphi_N(r_1, \ldots, r_{N-1}) = f(r_1) \ldots f(r_{N-1})$. Then

$$\langle \varphi_N, H_N \varphi_N \rangle = (N-1) < f, \left(p^2 - \frac{1}{4} V\right) f > - \frac{1}{8} (N-1)(N-2) \langle f \otimes f,\ V(r_{12})\, f \otimes f \rangle$$

where $\langle f \otimes f, V(r_{12})\, f \otimes f \rangle = \int V(r_1 - r_2)\, |f(r_1)|^2\, |f(r_2)|^2\, dr_1\, dr_2$. It is clear that this is negative for some N.

Let N_0 be the smallest N for which H_N has negative expectation values. By Lemma a, H_{N_0} has an infinity of bound states. By definition, H_N has no bound states for $N < N_0$. The situation for $N > N_0$ is described by:

Lemma c. If $N > N_0$, H_N has an infinitude of bound states.

Proof. Let $C_1, \ldots, C_m$ be the clustering determining the continuum. Since $\inf \sigma(H_{N_0}) + \inf \sigma(H_{N-N_0}) < 0$, some C_i, say C_1, must have $N_1 = \# C_1 \geqslant N_0$. We shall show $H_{C_1 \cup C_2}$ has an infinity of bound states. This will imply $C_1 \cup C_2 = \{1, \ldots, n\}$ and thereby the theorem. The $C_1 - C_2$ intercluster mass is $M^{-1} = (N_1)^{-1} + (N_2)^{-1} < 2$ so $(2\,M)^{-1} < 1$. The long range component of the intercluster force is $(N_1\, N_2)\,(-1/4\, R^{-2})$. Thus, the intercluster Hamiltonian is of the form for r large: $a\, p^2 - b\, r^{-2}$ with $b > 1/4$, $a < 1$. Thus by theorem 1, $H_{C_1 \cup C_2}$ has an infinity of bound states. ■

We have thus proven:

Proposition 6. For some $3 \leqslant N_0 \leqslant 7$, $\sigma(H_N) = [0, \infty)$ for $N < N_0$ and H_N has an infinity of bound states for $N \geqslant N_0$. ■

As we have remarked before, we present in appendix 5, a proof that $\langle \psi, H_0 \psi \rangle < 0$ for some ψ.

The proof of Lemma b makes it clear that with any C ($< 1/4$) at all, Lemma b would hold as would Lemma a. Lemma c does not obviously carry through but it is probably also true. We can also see it is possible for N to find a $-c\, r^{-2}$ potential so that the N-body system with $-c\, r^{-2}$ potential (cutoff) has no bound states but so that for some $m > N$ (probably $m = N + 1$ will do for suitable c!), the m-body system has an infinitude of bound states.

To summarize, we *expect* the following: *An N-body continuum limit* ($N > 2$) *is only possible if there are no really long range* ($V \sim -r^{-\gamma}$, $\gamma < 2$) *forces between the clusters determining the continuum. If there are only really short range forces* ($V \sim -r^{-\gamma}$, $\gamma > 2$) *or repulsive potentials between the clusters determining the continuum, a system with N-cluster continuum limit has only finitely many bound states. In the border line* r^{-2} *case, there can be an N-body continuum limit with either a finite or infinite number of bound states depending on the size of the total coupling constant.*

Acknowledgements

It is a pleasure to thank Sam Treiman for shaming me into consideration of the material in the last section and for several valuable conversations.

Appendix 1

Some Kinematics

Let $T \equiv H_0 = \sum_{i=1}^{n} (2\, m_i)^{-1} p_i^2 = \sum_{i=1}^{n} 1/2\, m_i\, \dot{r}_i^2$. We want to find H_0 as a function of the momenta in different coordinate systems $Q_1, \ldots, Q_n$. We will write T as a function of the $\dot{Q}_i$'s and then use

$$P_i = \frac{\partial T}{\partial \dot{Q}_i}$$

to find P_i. Solving for $\dot{Q}_i$ as a function of P_i we can find H_0 as a function of P_i.

(a) *Coordinates relative to* $\boldsymbol{r}_n$

Consider the new system:

$$\boldsymbol{R} = M^{-1}\left(\sum m_i\, \boldsymbol{r}_i\right),$$

$$\boldsymbol{x}_i = \boldsymbol{r}_i - \boldsymbol{r}_n \quad (i = 1, \ldots, n-1)$$

where $\sum = \sum_{i=1}^{n}$ and $M = \sum m_i$. Solving for $\boldsymbol{r}_i$, we find:

$$\boldsymbol{r}_n = \boldsymbol{R} - \boldsymbol{\eta}\,,$$

$$\boldsymbol{r}_i = \boldsymbol{x}_i + \boldsymbol{R} - \boldsymbol{\eta}$$

where $M\,\eta = \sum' m_i\, x_i$ and $\sum' \equiv \sum_{i=1}^{n-1}$. Then:

$$T = \sum{}' \frac{1}{2}\, m_i\, \dot{x}_i^2 + \frac{1}{2}\, M\,(\dot{R} - \dot{\eta})^2 + \sum{}' m_i\, \dot{x}_i\,(\dot{R} - \dot{\eta}) =$$

$$= \sum{}' \frac{1}{2}\, m_i\, \dot{x}_i^2 + M\,\dot{R}^2 - \frac{1}{2}\, M\,\dot{\eta}^2\,.$$

Thus

$$P = M\,\dot{R}\,,$$

$$k_i = m\,\dot{x}_i - M\,\dot{\eta}\,\frac{\partial \dot{\eta}}{\partial \dot{x}_i} = m_i\,(\dot{x}_i - \dot{\eta})$$

so $\sum' k_i = M\,\dot{\eta} - (M - m_n)\,\dot{\eta} = m_n\,\dot{\eta}$. Therefore

$$H_0 = (2\,M)^{-1}\,P^2 + \sum{}' (2\,m_i)^{-1}\,k_i^2 + m_i\,\dot{\eta}^2 - \frac{1}{2}\,M\,\dot{\eta}^2$$

$$= (2\,M)^{-1}\,P^2 + \sum{}' (2\,m_i)^{-1}\,k_i^2 + \frac{1}{2}\,(M - m_n)\,\dot{\eta}^2 + m_n\,\dot{\eta}^2 - \frac{1}{2}\,M\,\dot{\eta}^2$$

$$= (2\,M)^{-1}\,P^2 + \sum{}' (2\,m_i)^{-1}\,k_i^2 + \frac{1}{2}\,m_n^{-1}\left(\sum{}' k_i\right)^2.$$

Thus, finally:

$$H_0 = (2\,M)^{-1}\,P^2 + \sum{}' (2\,\mu_i)^{-1}\,k_i^2 + \sum_{i<j}{}' m_n^{-1}\,k_i\,k_j$$

with $(\mu_i)^{-1} = (m_i)^{-1} + (m_n)^{-1}$. The $i - j$ cross terms were originally discovered by Hughes and Eckart [25] and are known variously as Hughes-Eckart, mass polarization or specific mass terms.

(b) *Two cluster breakup*

Let us consider the breakup into two clusters $\{1, \ldots, l\}$; $\{l+1, \ldots, n\}$. One 'natural' set of coordinates for the breakup is:

$$\boldsymbol{y}_i = \boldsymbol{r}_i - \boldsymbol{r}_l \quad i = 1, \ldots, l-1\,, \qquad \boldsymbol{y}_i = \boldsymbol{r}_i - \boldsymbol{r}_n \quad i = l+1, \ldots, n-1\,,$$

$$\boldsymbol{R}_{12} = \boldsymbol{\varrho}_1 - \boldsymbol{\varrho}_2\,, \qquad \boldsymbol{R} = M^{-1}\left(\sum_{i=1}^{n} m_i\, \boldsymbol{r}_i\right)$$

where

$$\boldsymbol{\varrho}_1 = M_1^{-1} \sum_{i=1}^{l} m_i\, \boldsymbol{r}_i \qquad \left(M_1 = \sum_{i=1}^{l} m_i\right)$$

$$\boldsymbol{\varrho}_2 = M_2^{-1} \sum_{i=l+1}^{n} m_i\, \boldsymbol{r}_i \qquad \left(M_2 = \sum_{i=l+1}^{n} m_i\right).$$

To find H_0, we first go to a coordinate system $\varrho_1, \varrho_2, y_i$ $(i = 1, \ldots, l, \ldots, n)$, where by (a) we have

$$H_0 = H_{0,\{1,\ldots,l\}} + H_{0,\{l+1,\ldots,n\}} + (2\,M_1)^{-1}\,P_1^2 + (2\,M_2)^{-1}\,P_2^2$$

where $H_{0,\{1,\ldots,l\}}$ is the free Hamiltonian for $\{1, \ldots, l\}$ with the center of mass removed and relative coordinates used. Now using

$$R_{21} = \varrho_1 - \varrho_2\,,$$

$$R = M^{-1}\,(M_1\,\varrho_1 + M_2\,\varrho_2)\,.$$

we see:

$$H_0 = H_{0,\{1,\ldots,l\}} + H_{0,\{l+1,\ldots,n\}} + (2\,M_{12})^{-1}\,P_{12}^2 + (2\,M)^{-1}\,P^2$$

with

$$M_{12}^{-1} = M_1^{-1} + M_2^{-1}\,.$$

This is the form used throughout this paper. In several places, and in section 3 in particular, we need the fact that:

If $k \in \{1, \ldots, l\}$ *and* $h \in \{l+1, \ldots, n\}$, then $\boldsymbol{r}_{kh} \equiv \boldsymbol{r}_k - \boldsymbol{r}_h = \boldsymbol{R}_{12} + \sum a_i^{(kh)}\,y_i$. Since $\boldsymbol{r}_{kh} = y_k - y_h + \boldsymbol{r}_{ln}$, we need only shows $\boldsymbol{r}_{ln} = R_{12} + \sum b_i\,y_i$. This in turn follows from the formulae in (a):

$$r_l = \varrho_1 - M_1^{-1} \sum_{i=1}^{l-1} m_i\,y_i\,,$$

$$r_n = \varrho_2 - M_2^{-1} \sum_{i=l+1}^{n-1} m_i\,y_i\,.$$

(c) *Exchange between clusters*

We have just seen if $k \in D_1$, $l \in D_2$ then $R_{12} = r_{kl} + \sum a_i\,y_i$ where the y's are relative coordinates within D_1 or D_2. Let D_1', D_2' be D_1 and D_2 after interchanging k and l and let R_{12}' be the relative center of mass for D_1', D_2'. Suppose $m_k = m_l$. Then, we wish to show:

$$R_{12}' = (1 - M_1^{-1}\,m_k - M_2^{-1}\,m_l)\,r_{kl} + \sum a_i\,y_i$$

a formula we need in section 4 [same a's; the y's are still relative coordinates within D_1, D_2 (*not* D_1', D_2')]. Since $m_k = m_l$, the only difference between (ϱ_1, ϱ_2) and (ϱ_1', ϱ_2') is the exchange of k and l. Thus

$$R_{12} - R_{12}' = (\varrho_1 - \varrho_1') + (\varrho_2' - \varrho_2) = m_k\,M_1^{-1}\,(r_k - r_l) + m_l\,M_2^{-1}\,(r_k - r_l)$$

which proves the required formula.

Appendix 2

Hughes-Eckart Terms and Hunziker's Theorem[22])

It is our goal in this appendix to make explicit the statement of Hunziker's paper [7] that kinematic center of mass corrections don't matter. Let us present the Helium atom as a paradigm. If the nucleus has mass M and the reduced mass of the electron is μ, the Helium atom Hamiltonian with center of mass removed and coordinates relative to the nucleus is:

$$H_{He} = (2\,\mu)^{-1}\,(p_1^2 + p_2^2) + M^{-1}\,p_1 \cdot p_2 - 2\,e^2\,(r_1^{-1} + r_2^{-1}) + e^2\,r_{12}^{-1}\,.$$

It is this Hamiltonian whose continuous spectrum should begin at the binding energy for He⁺, i.e. the ground state energy of:

$$H_{He^+} = (2\,\mu)^{-1}\,p^2 - 2\,e^2\,r^{-1}\,.$$

Hunziker's argument explicitly tells us we get the continuum limit by breaking $\{1, 2, 3\}$ into clusters and considering the ground state energy of the Hamiltonians with the intercluster *potentials* thrown away. Thus the continuum for the H_{He} begins at the bottom of the spectrum of

$$\tilde{H}_{He^+} = (2\,\mu)^{-1}\,(p_1^2 + p_2^2) + M^{-1}\,p_1 \cdot p_2 - 2\,e^2\,r_1^{-1}\,.$$

We are thus faced showing $\inf \sigma(\tilde{H}_{He^+}) = \inf \sigma(H_{He^+})$. On the surface, since we have gotten rid of the intercluster potential but ***not the intercluster Hughes-Eckart terms***, it is not even clear that this equality should hold (although physically we do expect $\inf \sigma_{cont}(H_{He}) = \inf \sigma(H_{He^+})$). Let P be the center of mass momentum, m the mass of the electron, K the momentum of the nucleus and k_i of the ith electron. Let X, x_1, x_2 be the positions of the particles. Then:

$$\tilde{H}_{He^+} + (2\,M + 4\,\mu)^{-1}\,P^2 = \hat{H}_{He^+}$$

where

$$\hat{H}_{He^+} = (2\,m)^{-1}\,(k_1^2 + k_2^2) + (2\,M)^{-1}\,k^2 - 2\,e^2\,|x_1 - X|^{-1}\,.$$

Since P and $\tilde{H}_{He^+}$ involve independent coordinates:

$$\inf \sigma(\hat{H}_{He^+}) = \inf \sigma(\tilde{H}_{He^+}) + \inf\,[(2\,M + 4\,\mu)^{-1}\,P^2] = \inf \sigma(\tilde{H}_{He^+})\,.$$

Since k_2^2 and $\hat{H}_{He^+} - (2\,m)^{-1}\,k_2^2 \equiv H'_{He^+}$ involve independent coordinates:

$$\inf \sigma(\hat{H}_{He^+}) = \inf \sigma(H'_{He^+}) + \inf \sigma[(2\,m_2)^{-1}\,k_2^2] = \inf \sigma(H'_{He^+})\,.$$

Finally, since $H'_{He^+} = H_{He^+} + (2\,m + 2\,M)^{-1}\,P_{10}^2$ with $P_{10} = k_1 + K$, we see

$$\inf \sigma(H'_{He^+}) = \inf \sigma(H_{He^+})\,.$$

Thus

$$\inf \sigma(H_{He^+}) = \inf \sigma(\tilde{H}_{He^+})$$

as we desired to prove.

Appendix 3

Compactness of $I(z)$[22])

We give here a simple proof of the compactness of the Weinberg connected interaction [19] when the potentials are locally L^2 falling to zero at infinity. This fact is used crucially in the proof of Hunziker's theorem. Hunziker's proof depends on an involved inductive argument [26]. The proof we will give depends on the following elementary lemma:

Lemma. Let $A(z)$ be an analytic operator valued function on a connected set D. Suppose $A(z_n)$ is compact for a sequence z_n with a limit point. Then $A(z)$ is compact for all $z \in D$.

Proof. The compact operators are closed. Thus given any $A \notin$ Compact Ops., we can find a continuous linear functional l on Hom(H) so that l[Compact Ops.] $= 0$ and $l(A) = 1$ (by the Hahn-Banach theorem). Since $l(A(z_n)) = 0$, and $l(A(z))$ is analytic, $l(A(z)) \equiv 0$ so $A(z)$ is never A. ∎

Note. Hunziker proves a similar theorem in an appendix of [9].

As a result of the lemma, we need only prove $I(z)$ is compact for $Re\,z$ very negative. When $Re\,z$ is very negative, $I(z)$ is the norm convergent sum of all connected Weinberg diagrams. Since the compacts are norm closed, one need only show every connected diagram is compact. For $V \in L^2$, a direct computation shows the diagrams are Hilbert-Schmidt. For $V \in (L^2)_{loc}$, $V \to 0$ at ∞, we need only use a limiting argument.

Appendix 4

Particle Statistics

We wish to develop here the machinery that will allow one to extend theorems 2 and 3 to systems with arbitrary statistics. Suppose H is an n-body Hamiltonian. Let $C_1, \ldots, C_m$ be a partition of $\{1, \ldots, n\}$ so that H is invariant under permutations which leave the sets C_i invariant. Let $\sum(C_1, \ldots, C_m)$ be the group of such permutations. We define the action of $\pi \in \sum(C_i)$ on $L^2(R^{3n})$ by $(\pi\,\psi)\,(x_1, \ldots, x_n) = \psi(x_{\pi^{-1}(1)}, \ldots\, x_{\pi^{-1}(n)})$. Then we are supposing $H\,\pi = \pi\,H$ for all $\pi \in \sum(C_i)$.

Let $A(C_i)$ be the group algebra generated by $\sum(C_i)$, i.e. all formal sums of elements in $\sum$ with complex coefficients and a multiplication generated by the group operation. 'Statistics' should be thought of as restrictions on the allowable wave functions involving relations among the various $\pi\,\psi$. For example, full Fermi-Dirac statistics is just $\pi\,\psi = (\mathrm{sgn}\,\pi)\,\psi$ for all π. Alternatively, certain elements of $A(C_i)$ when applied to allowable ψ must be 0. If $a \in A(C_i)$ and $a\,\psi = 0$, then $b\,a\,\psi = 0$ for all $b \in A(C_i)$. Thus the set of elements of A which annihilate any fixed ψ or any set of ψ's is a left ideal. We thus define a ***'statistics' for an n-body system with clusters of identical particles*** $C_1, \ldots, C_m$ ***is a left ideal***, I, *of* $A(C_1, \ldots, C_m)$. The space of allowable wave functions is then $\mathcal{H}_I = \{\psi \mid a\,\psi = 0$ for all $a \in I\}$.

Given a statistics for such a system and a decomposition $\{D_1, D_2\}$ of $\{1, \ldots, n\}$ we define the induced clustering of identical particles to be $\{D_1 \cap C_1,\ D_2 \cap C_1,\ D_1 \cap C_2, \ldots, D_2 \cap C_m\}$. Then $I_{D_1, D_2} \equiv I \cap A(D_1, D_2)$ is a left ideal in $A(C_i) \cap A(D_j) \equiv A(D_j \cap C_i)$. This ideal defines the ***induced statistics***.

One important object used in the proofs of theorems 2 and 3 was the antisymmetrizer, $\mathcal{A}\cdot\mathcal{A}$ had two crucial properties. It was a self-adjoint projection[35]) and $1-\mathcal{A}$ generated the statistical ideal $I_{\mathcal{A}}$, in the sense that $I_{\mathcal{A}} = \{b\,(1-\mathcal{A}) \mid b \in A(C_i)\}$ so that $\mathcal{H}_I = \operatorname{Ran} \mathcal{A}$. If I is any left ideal, an analogous self-adjoint idempotent, e, for which $(1-e)$ generates I will be called a (actually the) *natural projection* for I. The crucial group theoretical result we will need is:

Lemma. Every ideal $I \subset A(G)$ has a natural projection.

Proof. This is actually a simple consequence of the Wedderburn structure theorem (see [27], pp. 239–243 or [28], pp. 12–27) but let us sketch a proof for the reader's convenience. G has a natural representation on $A(G)$ (the left regular representation) given by $U_g a = g\,a$. In the inner product $\langle \sum a_i g_i, \sum b_i g_j \rangle = \sum \bar{a}_i b_i$, this representation is unitary. The left ideals $I \subset A(G)$ are precisely the invariant subspaces. Since every invariant subspace in a unitary representation has an orthogonal invariant subspace, every left ideal $I \subset A(G)$ has an associated complementary ideal $I^\perp$ with $I \oplus I^\perp = A(G)$. Corresponding to this decomposition, we can write $1 = e_1 + e_2$; $e_1 \in I$, $e_2 \in I^\perp$. It is not hard to prove that e_2 is a natural projection for I. ∎

Using this natural projection in place of $\mathcal{A}$, the analogues of theorems 2 and 3 can be proven[36]):

Theorem 2′. Let H be an n-body Hamiltonian with center of mass removed symmetric in the coordinates of each cluster C_i and restricted to the space, $\mathcal{H}_I$, generated by some ideal $I \subset A(C_i)$. Given $D = \{D_1, D_2\}$, a partition of $\{1, \ldots, n\}$, let $H_D = H_{D_1} + H_{D_2}$ where H_D is the part of H depending only on the coordinates in one cluster and restricted to the space $\mathcal{H}_{I_{D_1, D_2}}$ generated by the induced ideal $I_{D_1, D_2} \subset A(C_i \cap D_j)$. Let $\Sigma = \inf(\sigma_{cont}(H))$. Then:

$$\Sigma = \min_{D_1, D_2} \{\inf \sigma(H_{D_1}) + \inf \sigma(H_{D_2})\}\,.$$

Theorem 3′. Let H be an n-body Hamiltonian with center of mass removed symmetric in the coordinates of each C_i and restricted to the space, H_1, generated by some ideal $I \subset A(C_i)$. Let the continuum limit be determined (according to theorem 2′) by a two cluster breakup D_1, D_2 and suppose:

(a) Each H_{D_i} has an eigenstate ψ_i at the bottom of the spectrum.

(b) For each $i \in D_1$, $j \in D_2$

$$V_{ij}(r) \leqslant C_{ij}\, r^{-\gamma} \text{ if } r > R_0$$

with

$$\gamma < 2;\ \sum C_{ij} < 0\,.$$

Then H has an infinitude of bound states.

[35]) $A(G)$ has a natural $*$ operation given by $(\sum a_i g_i)^* = \sum \bar{a}_i g_i^{-1}$.

[36]) One should first write $I^\perp$ as a sum of minimal ideals. $\mathcal{H}_I$ correspondingly breaks up into a direct sum of spaces left invariant by H. Every minimal ideal in $A(C_1, \ldots, C_m)$ is a tensor of product of minimal ideals in $A(C_1), \ldots, A(C_m)$ (since every irreducible representation of $\sum(C_1, \ldots, C_m)$ is the direct product of irreducible representations of $\sum(C_i)$). It is easier to prove these theorems for these minimal ideals and then patch together.

Appendix 5

A Computation: $N_0 \leqslant 7$

We want to prove for some Ψ, $\langle \Psi, H_7 \Psi \rangle < 0$ where $H_N = \sum_{i=1}^{N-1} (p_i^2 - 1/4\, V(r_i)) + \sum_{i<j} (\boldsymbol{p}_i \cdot \boldsymbol{p}_j - 1/4\, V(r_{ij}))$ with V is given by equation (10) (in 8), i.e. $V = V_1$ with

$$V_\varepsilon(r) = \begin{cases} r^{-2} & r \geqslant \varepsilon \\ \varepsilon^{-2} & r \leqslant \varepsilon\,. \end{cases}$$

We first not that it is sufficient to prove the result with H_7 replaced by $\tilde{H}_7$ where $\tilde{H}_7$ is obtained by taking $\varepsilon = 0$. For, if $\langle \Psi, \tilde{H}_7 \Psi \rangle < 0$, then $\langle \Psi, H_{7,\varepsilon} \Psi \rangle < 0$ for some $\varepsilon > 0$ where $H_{7,\varepsilon}$ is obtained by replacing V in H_7 by V_ε. But letting $\Psi_\varepsilon(r) = \varepsilon^{3/2}\, \Psi(\varepsilon r)$, we see:

$$\langle \Psi_\varepsilon, H_7 \Psi_\varepsilon \rangle = \varepsilon^2 \langle \Psi, H_{7,\varepsilon} \Psi \rangle < 0\,.$$

Let us use the trial wave function $\Psi_\alpha(r_1, \ldots, r_6) = \Psi_\alpha(r_1), \ldots, \Psi_\alpha(r_6)$ with $\Psi_\alpha(r) = r^\alpha e^{-1/2 r}$ $(\alpha > -1/2)$. We will show $\lim_{\alpha \to -\frac{1}{2}} \langle \Psi_\alpha, \tilde{H}_7 \Psi_\alpha \rangle < 0$ which will be sufficient[37]).

Now we compute:

$$\langle \Psi_\alpha, \Psi_\alpha \rangle = 4\pi\, \Gamma(2\alpha + 3) \xrightarrow[\alpha \to -\frac{1}{2}]{} 4\pi\,,$$

$$\langle \Psi_\alpha, r^{-2} \Psi_\alpha \rangle = 4\pi\, \Gamma(2\alpha + 1)\,,$$

$$\langle \Psi_\alpha, -\Delta \Psi_\alpha \rangle = 4\pi \left[\alpha^2\, \Gamma(2\alpha + 1) - \alpha\, \Gamma(2\alpha + 2) + \frac{1}{4}\, \Gamma(2\alpha + 3)\right]^{38)}$$

so

$$\left\langle \Psi_\alpha, \left(-\Delta + \frac{1}{4} r^{-2}\right) \Psi_\alpha \right\rangle \xrightarrow[\alpha \to -\frac{1}{2}]{} \pi\,.$$

Finally, let us compute

$$A = \langle \Psi_{-1/2} \otimes \Psi_{-1/2}, r_{12}^{-2}\, \Psi_{-1/2} \otimes \Psi_{-1/2} \rangle$$
$$= \int (x^{-1} e^{-x})\, (y^{-1} e^{-y})\, |x - y|^{-2}\, d^3x\, d^3y\,.$$

Using [10]:

$$\int \overline{f(x)}\, g(y)\, |x - y|^{-2}\, d^3x\, d^3y = \frac{1}{4\pi} \int \overline{\hat{f}(p)}\, \hat{g}(p)\, p^{-1}\, d^3p$$

(with $\hat{f}(p) = \int e^{ip\cdot x} f(x)\, dx$) and the well-known Yukawa Fourier transform $(4\pi/p^2 + 1)$, we see:

$$A = (8\pi)^2\,.$$

[37]) For $\alpha > -1/2$, Ψ_α is in the domain of H_7 as a quadratic form. We can thus find $\Phi_\alpha \in D(H_7)$ so $\langle \Phi_\alpha, H_7 \Phi_\alpha \rangle$ is arbitrarily near $\langle \Psi_\alpha, H_7 \Psi_\alpha \rangle$.

[38]) We see explicitly $\Psi_{-1/2}\ \varepsilon$ torm domain of H.

Thus:

$$\lim_{\alpha\to-\frac12}\langle\Psi_\alpha, H_N\Psi_\alpha\rangle = (N-1)\lim_{\alpha\to-\frac12}\langle\Psi_\alpha, \left(-\Delta-\frac14 r^{-2}\right)\Psi_\alpha\rangle\langle\Psi_{-1/2},\Psi_{-1/2}\rangle^{N-2}$$

$$-\frac12 (N-1)(N-2)\langle\Psi_{-1/2}\otimes\Psi_{-1/2}, \frac14 r_{12}^{-2}\Psi_{-1/2}\otimes\Psi_{-1/2}\rangle\langle\Psi_{-1/2},\Psi_{-1/2}\rangle^{N-3}$$

$$=\langle\Psi_{-1/2},\Psi_{-1/2}\rangle^{N-3}(\pi^2)\,[4(N-1)-(N-1)(N-2)$$

$$=\langle\Psi_{-1/2},\Psi_{-1/2}\rangle^{N-3}(\pi^2)\,[(N-1)(6-N)]\,.$$

This becomes negative at $N = 7$.

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