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Separation of Center of Mass in Homogeneous Magnetic Fields*†

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We show that a system of particles in a homogeneous magnetic field, with translation invariant interaction, has a constant of motion analogous to the total momentum when $B = 0$. Next, we consider the separation of the center of mass. When the total charge of the system is zero, the situation is similar to (but more complicated than) the $B = 0$ case. When the total charge is nonzero, the analysis is quite different. The two-body problem is worked out in some detail and we also state and prove a version of the HVZ theorem in homogeneous magnetic field.

1. INTRODUCTION

The total momentum of N mutually interacting particles with translation invariant interaction is a constant of motion. Furthermore, the center of mass separates, i.e., the Hamiltonian can be factored into the Hamiltonian with the c.m. fixed and a part that describes the uniform c.m. motion.

The separation of c.m. is sufficiently useful to need no propaganda but we shall not resist the temptation of making some. First, the absorption and emission spectrum is, in fact, the spectrum of the Hamiltonian with the c.m. removed and so is more natural. Second, one gets rid of three degrees of freedom and third, for the Hamiltonian with the c.m. removed one has the HVZ theorem. The HVZ theorem for atoms identifies the bottom of the essential spectrum for the N electron atom with the lowest eigenvalue of the ion with one electron removed (the theorem is, in fact, much more general). Technically, Hamiltonians with the c.m. removed are easier to analyze because the potential interaction is, in many cases, a (relatively) compact perturbation. This is very convenient and is responsible for the HVZ theorem.

In the present work we describe the separation of c.m. in constant magnetic field B. Not surprisingly, $B = 0$ is a special case of the general theory. The theory has an interesting structure, apparent for $B \neq 0$, but which trivializes when $B = 0$.

With an eye toward concrete application we mention that the separation of c.m. facilitates a more accurate treatment of the Zeeman effect for hydrogen and posi-

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† This paper is the second in our three-part series: "Schrödinger Operators with Magnetic Fields" [3, 4].

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tronium, reducing them to a one-body problem. (The reduction is more complicated than merely the occurrence of the reduced mass.) This is discussed in Section 4.

We do not consider perturbation theories in B since an exact treatment is preferable. Because of the singularity of the perturbation, rigorous perturbation theory is limited; see, however, [3, Section 6].

In Section 2 we start by looking at conserved quantities of the full Hamiltonian analogous to the momentum. A "discrete" version of some of the ideas in this section has already appeared in connection with the magnetic groups in the one-electron theory of solids [5, 7, 11, 31, 32]. The conservation laws, and the theory that evolves from them, lead to phenomena which are very different, qualitatively, from the $B = 0$ case. In fact $B = 0$ intuition is quite often misleading. We say more on this in Sections 3–5. Let us make a small digression on the analogous situation for a constant electric field. The results are

(a) The Hamiltonian is translation invariant iff the total charge $Q = \sum_{i=1}^{n} e_i = 0$.

(b) The center of mass is always separable in the sense that it undergoes a uniform acceleration depending on Q.

(Separation of c.m. and momentum-like constants of motion are evidently distinct notions.)

We return now to discussion of the magnetic case. The vector potential is fixed only up to gauge transformations. This arbitrariness enables a realization of the translation invariance of the physics in constant magnetic field, as an invariance of the Hamiltonian under a "translation group." More precisely, consider N particles in constant magnetic field B with a translation invariant potential interaction. The kinetic energy is $\sum_{i=1}^{n} \frac{1}{2} m_i v_i^2$ with $m_i v_i = p_i - e_i A_i$. We shall henceforth stick to the gauge where $A_i = \frac{1}{2} B \times r_i$. Let $\alpha \in \mathbb{R}^3$. The classical Hamiltonian, being a function of v_i and coordinate difference $x_i - x_j$, is invariant under the phase space translation g_α [11]

$$g_\alpha[x_i, p_i] = (x_i + \alpha, p_i + (e_i/2) B \times \alpha).$$

g_α describes a combined translation and gauge transformation. When $B = 0$, this is the usual coordinate translation, a "horizontal" translation in phase space. In general, it is a "skewed" translation in phase space. The translation group in quantum mechanics is not necessarily Abelian because in that case, phase space translations do not commute. Clearly, the generators of the group are the constants of the motion for the Hamiltonian, analogous to the total momentum for $B = 0$.

In Section 3 we show that these generators are physical and not only formal pseudomomentum: Interaction with radiation conserves the sum of the photon momentum and the pseudomomentum.

Sections 4 and 5 describe the invariance group and reduction of the c.m., including a detailed analysis of the two-body problem. Section 4 is devoted to a neutral system, $Q = 0$, and 5 to a charged system $Q \neq 0$. For the neutral system the invariance group is Abelian (this is remarked already in [35]) and so there is a close analogy with the $B = 0$ case. There is one important difference, however. The reduced c.m. Hamiltonian depends, in general, in a nontrivial way on the total pseudomomentum. When $B = 0$, the "energy bands" are parabolas; for $B \neq 0$ they are asymptotically flat as the transverse quasi-momentum goes to infinity.

Section 4 contains a discussion of the two-body problem (hydrogen, positronium) which can be worked out rather explicitly. In Section 5, we analyze $Q \neq 0$ where the invariance group is non-Abelian. Here not all the components of the pseudomomentum commute and so they cannot be diagonalized simultaneously. Fortunately, a certain notion of c.m. separation holds also in this case. It corresponds to restricting the Hamiltonian to harmonic oscillator eigenstates.

Since separation of c.m. in magnetic fields is so different from the usual one, one naturally wonders why is it relatively unimportant for most applications in atomic physics. The reason is the extreme smallness of B. In atomic units, the natural unit for the magnetic field is 2×10^9 G. Most laboratory experiments are carried out with tiny B on this scale. Two situations where separation of c.m. and the associated conservation laws may have observational effects are: Neutron stars physics where magnetic fields of the order of 10^{12} G are believed to exist, and precision experiments such as the $2s$–$2p$ degeneracy lifting in hydrogen. This splitting (also in the relativistic case) is partly due to the finite mass of the proton and so on the c.m. separation and the associated conservation laws for radiation transitions (see Sections 3 and 4).

It may be remarked that corrections due to the finite proton mass in magnetic field were calculated by Lamb in his celebrated papers [18, 19]. The external magnetic field, in this case, was an rf field. The corrections are of the same order as relativistic effects so Lamb naturally considered the Dirac hydrogen atom. Lamb got around c.m. separation by a perturbation expansion in the electron–proton mass ratio.

A related notion of c.m. separation can be developed for excitons in solids [17]. This combines the ideas of Zak [32] and Brown [7] with the methods of the present work.

Section 6 is devoted to the HVZ theorem. In fact, Propositions 4.7 and 5.5 are special cases of Theorem 6.1.

This paper is the second in a sequence on the spectral theory of Schrödinger operators in magnetic fields [3, 4]. Although the paper is self-contained, there is some interdependence of these papers, so we only consider the Schrödinger case. An extension of our results to the Dirac case should be straightforward. Indeed, parts of our results have been obtained in the Dirac case already by Grotch and Hegstrom [35].

2. The Pseudomomentum

The Hamiltonian

$$H_0 = (1/2m)(p - eA)^2, \qquad A = \tfrac{1}{2}B \times x \tag{1}$$

has a constant of motion, the pseudomomentum, which was recognized at least as early as Johnson and Lippmann [13]. It is the natural analog of the momentum for the Laplacian. There are three ways of looking at this constant of motion: The center of the Landau orbit [13]; the generator of skewed (phase space) translations [11, 22]; and a pair of creation–destruction operators. These descriptions are given in the next three propositions.

PROPOSITION 2.1. *Let* $c = x + (m/e)[(v \times B)/B^2]$, $mv = p - eA$. *Then under the action of* H_0:

(a) *c is the center of the classical Landau orbit (also called the guiding center).*

(b) $dc/dt = (B \cdot p)B/mB^2$.

Thus, the perpendicular components of c are constants of the motion.

(c) $d^2c/dt^2 = 0$.

(d) *c has commutation relation summarized by* $c \times c = -iB/eB^2$.

Proof. The Larmour frequency vector ω is equal to $-eB/m$. Hence the Larmour radius (classical)

$$R = -\frac{\omega \times v}{\omega^2} = -\frac{m}{e}\frac{v \times B}{B^2}.$$

Parts (b) and (c) follow from the (Heisenberg) equation of motion $m\dot{v} = ev \times B$ (Lorentz force). Part (d) follows from CCR. ∎

Although c is very natural due to its geometrical meaning, a closer analog of the momentum is a suitable multiple of c:

PROPOSITION 2.2. *Let* $\alpha \in \mathbb{R}^3$ *and* $U(\alpha)$ *the strongly continuous unitaries* $U(\alpha) = e^{-i(e/2)x \cdot B \times \alpha} e^{ip \cdot \alpha}$. *Then*

(a) $[U(\alpha), v] = 0$,

(b) $k = p + eA$, *the generator of* $U(\alpha)$, *is a constant of motion,*

(c) $U(\alpha)\, xU(\alpha)^{-1} = x + \alpha$,

(d) *k has the commutation relation*

$$k \times k = -ieB.$$

Remarks. (1) $U(\alpha)$ is interpreted as skewed phase space translations:

$$T_\alpha(x, p) = [x + \alpha, p + (e/2)B \times \alpha]. \tag{2}$$

(2) The explicit formula in (b) for k is gauge dependent ($A = \frac{1}{2}B \times x$).

(3) We use $k_\perp$ throughout for the component of k perpendicular to B, and $k_\parallel = k \cdot B/|B|$.

(4) $k_\perp = e(c \times B)$; $c_\perp = (eB^2)^{-1}(B \times k)$.

Proof.

$$\begin{aligned} U(\alpha)^{-1}(p - eA)\, U(\alpha) &= e^{-ip\alpha}[p - eA - (e/2)\, B \times \alpha]\, e^{ip\alpha} \\ &= p - eA. \end{aligned} \tag{3}$$

Part (b) follows by differentiating $U(\alpha)$ and the Heisenberg equations of motion. Parts (c) and (d) are immediate. k is the pseudomomentum. ▌

The analogy of the usual momentum is quite transparent. $\{U(\alpha)\}$ realizes the Abelian group of translations $\{T_\alpha\}$ as a projective representation with $\omega(\alpha, \alpha') = e^{ieB\times\alpha\cdot\alpha'}$ as multiplier, i.e.,

$$U(\alpha)\, U(\alpha') = \omega(\alpha, \alpha')\, U(\alpha + \alpha'). \tag{4}$$

$\omega(\alpha, \alpha')$ is determined by the flux through the parallelogram (α, α').

From a foundational point of view Eq. (4) is quite interesting. Let $B = (0, 0, B)$. The family $\{U(\alpha) \mid \alpha = (\alpha_x, \alpha_y, 0)\}$ together with rotations about the three axe, and the projection valued measure associated to $(x, y, 0)$, form a two-dimensional "position operator" in the sense of Mackey [21]. What distinguishes this situation from the usual one is that the representation of the Euclidean group has a multiplier which is nontrivial on the translations. One can show that this cannot happen in dimension larger than 2! For this reason, Mackey's proof that only the Schrödinger representation occurs is *not* applicable. There is yet another way of looking at the pseudomomentum which is useful because it gives a natural tensor product decomposition of the Hilbert space.

PROPOSITION 2.3. *Let* $\beta = |eB|/2$, $|B| = B \cdot \hat{z}$ *and* $\mathscr{H}_a = \oplus |n\rangle\langle n|$ *with* $|n\rangle = (a^\dagger)^n |0\rangle$, $a^\dagger$ *creation operator where*

$$a = \tfrac{1}{2}\{\beta^{-1/2}(p_x + ip_y) - i\beta^{1/2}(x + iy)\}, \tag{5}$$
$$b = \tfrac{1}{2}\{\beta^{-1/2}(p_y + ip_x) - i\beta^{1/2}(y + ix)\} \tag{6}$$

and $|0\rangle$ *is the state with* $a|0\rangle = 0$. *Then*:

(a) $\mathscr{H} = \mathscr{H}_a \otimes \mathscr{H}_b \otimes L^2(dz)$.

(b) $H_0 = h_0 \otimes I \otimes I + I \otimes I \otimes p_z{}^2/2m$,

$$h_0 = (|eB|/m)(a^\dagger a + 1/2).$$

(c) $b = \beta^{1/2}(c_x - ic_y)$.

(d) $L_z = a^\dagger a - b^\dagger b$.

Remarks. (1) This proposition also exhibits the spectral properties of H_0.

(2) The tensor product decomposition shows that the spectrum has a countably infinite degeneracy.

(3) The natural "reduced" Hamiltonian is $H_0 \upharpoonright_{\mathscr{H}_a \otimes |0\rangle \otimes L^2(dz)}$.

We say more on this in Section 5.

Proof. Since H_0 is quadratic in x and p it is clearly a "harmonic oscillator in disguise." The content of the proposition is a Bogoliubov transformation to the canonical variables a and b.

Alternatively, it is instructive to write

$$H_0 = -\frac{1}{2m}\Delta + \frac{e^2B^2}{8m}(x^2 + y^2) + \frac{eB}{2m}L_z$$

and notice that the harmonic oscillator part of H_0 is just $(eB/2m)(a^\dagger a + b^\dagger b + 1)$ and that the L_z term is $(eB/2m)(a^\dagger a - b^\dagger b)$. The b's cancel out. ∎

The pseudomomentum carries over to multiparticle Hamiltonians in a way which is analogous to the momentum.

THEOREM 2.4. *Let H be the self-adjoint multiparticle Hamiltonian in a magnetic field B given formally by*

$$H = \sum_{i=1}^{n} \frac{1}{2m_i}(p_i - e_iA_i)^2 + \sum_{i<j} V_{ij}(x_i - x_j). \tag{7}$$

Then:

(a) $k = \sum_{i=1}^{n} k_i$ *is a constant of motion, i.e.,* $[k, H] = 0$,

(b) $k \times k = -iBQ$ *where* $Q = \sum_{i=1}^{n} e_i$.

Remarks. (1) We systematically use A_i for $A(x_i) = \frac{1}{2}B \times x_i$.

(2) The main interest is for atomic physics, and so $V(x) \to 0$, $|x| \to \infty$, in some sense, is intended. We have kept the theorem general partly because $V(x) = \omega^2x^2$ is completely soluble and so of some interest.

Proof. By Proposition 2.2 all we have to verify is the commutativity of

$$U(\alpha) = e^{-(i/2)(\Sigma e_i x_i)\cdot B\times\alpha}\, e^{i\Sigma_i p_i\cdot\alpha}$$

with V_{ij}. $\sum_{i=1}^{n} p_i$ commutes by translation invariance and $\sum_{i=1}^{n} e_iA_i$ is a multiplication operator.

Part (b) follows from

$$p_i \times A_j = -i\epsilon_{ijk}B_k. \tag{8}$$ ∎

In summary: The multiparticle Hamiltonian is invariant under a family of unitary transformations corresponding to an E^3 translation group. From this 2.4(b) it follows that the projective representation $\{U(\alpha)\}$ is:

(a) A unitary representation of the Euclidean translation group in the zero charge sector, $Q = 0$.

(b) A unitary representation of the Heisenberg group (with $\hbar = B \cdot Q$) in the plane crossed with the translation group of the line in the $Q \neq 0$ sector.

The zero charge sector is discussed in Section 4 and the nonzero charge sector in 5. Concluding this section, let us recall:

PROPOSITION 2.5. *Let*

$$H_0 = \sum_{i=1}^{n} \frac{1}{2m_i} (p_i - e_i A_i)^2. \tag{11}$$

(a) *For N three-dimensional particles, the spectrum of H_0 is absolutely continuous*:

$$\sigma(H_0) = \left[\sum \frac{|e_i B|}{2m_i}, \infty\right]. \tag{12}$$

(b) *For N two-dimensional particles, ($A = (B/2)(-y, x)$), $\sigma(H_0)$ is a countable set $0 < \lambda_1 < \lambda_2 \cdots$ with $\lambda_1 = \sum_i |e_i B|/2m_i$ and $\lambda_n \to \infty$. Each eigenvalue is of infinite multiplicity.*

Proof. Part (b) follows from

$$H_0 = h_1 \otimes I \otimes \cdots \otimes I + I \otimes h_2 \otimes I \otimes \cdots \otimes I + \cdots$$

and Proposition 2.3(b) since

$$\sigma(h_i) = \left\{\frac{|Be_i|}{m_i}\left(n_i + \frac{1}{2}\right) \middle| n_i = 0, 1, 2, \ldots\right\}. \tag{14}$$

This simple result, together with the invariance of the essential spectrum, will enable us to determine the spectral properties of the two-body problem. ∎

3. INTERACTION WITH RADIATION

Here we prove that interaction with the radiation field conserves the total momentum—the sum of the quasi-momentum and the photon momentum. For simplicity we restrict ourselves to a single charged particle. The same applies to multiparticle Hamiltonians.

A model Hamiltonian describing photon emission and absorption without pair creation is [9]:

$$H = (p - eA - \mathscr{A})^2 + \int d^3\kappa \, |\kappa| \, a_\kappa^{\dagger} a_\kappa. \tag{15}$$

$\mathscr{A}$ is the radiation field:

$$\mathscr{A}(x) = (2\pi)^{-3/2} \int \frac{d^3\kappa}{(2|\kappa|)^{1/2}} (a^{\dagger}(\kappa)\, e^{-i\kappa x} + a(\kappa)\, e^{i\kappa x}). \tag{16}$$

$a^{\dagger}(\kappa)$ is the photon creation operator and polarization indices have been suppressed (nothing changes in the argumentations).

We have also suppressed an ultraviolet cutoff in $\mathscr{A}$ which is necessary for finiteness of various quantities.

PROPOSITION 3.1. *Let* $U(\alpha) \equiv e^{-i(e/2)x\cdot B\times\alpha}e^{ip\cdot\alpha}e^{i\mathscr{K}\cdot\alpha}$ *where* $\mathscr{K}$ *is the radiation field momentum, i.e.,*

$$e^{i\mathscr{K}\cdot\alpha}a^{\dagger}(\kappa)\, e^{-i\mathscr{K}\cdot\alpha} = a^{\dagger}(\kappa)\, e^{i\kappa\cdot\alpha}. \tag{17}$$

Then

$$U^{-1}(\alpha)\, HU(\alpha) = H, \qquad \alpha \in \mathbb{R}^3. \tag{18}$$

Proof. All we have to check is

$$U^{-1}(\alpha)\, \mathscr{A}(x)\, U(\alpha) = \mathscr{A}(x). \tag{19}$$

Equation (19) follows from the cancellation of the phase shift produced by translating x and by (17), explicitly:

$$e^{i(p+eA+\mathscr{K})\cdot\alpha}\mathscr{A}(x)\, e^{-i(p+eA+\mathscr{K})\cdot\alpha} = e^{i(p+eA)\cdot\alpha}\mathscr{A}(x-a)\, e^{-i(p+eA)\cdot\alpha} = \mathscr{A}(x). \tag{20}$$

Remark. The recoil described by Proposition 3.1 is reminiscent of phonon and photon assisted transitions in the band theory of solids [16, 39]. We shall say more on the implication of this for the hydrogen spectrum in Section 4.

4. THE ZERO CHARGE SECTOR

By Theorem 2.4(b), all the components of the total pseudomomentum k commute with each other in the zero charge sector. They are, therefore, simultaneously diagonalizable, as is the case for the momentum when $B = 0$ (Q arbitrary). The Hamiltonian is invariant under a phase-space translation group $U(\alpha)$, $\alpha \in \mathbb{R}^3$, i.e., $U(\alpha)\, U(\beta) = U(\alpha + \beta)$.

The precise statement of the simultaneous diagonalizability of k can be expressed in terms of direct integrals [21].

THEOREM 4.1.

$$\mathscr{H} = \int_{\mathbb{R}^3}^{\oplus} d^3k\mathscr{H}_k\,,$$
$$H = \int_{\mathbb{R}^3}^{\oplus} d^3k\; H(k), \tag{21}$$

where

$$U(\alpha)\,\mathscr{H} = \int_{\mathbb{R}^3}^{\oplus} d^3k\; e^{ik\cdot\alpha}\,\mathscr{H}_k \tag{22}$$

and the $\mathscr{H}_k$*'s are unitarily equivalent.*

Remark. That $\mathscr{H}_k$ is equivalent to $\mathscr{H}_{k'}$ and that d^3k occur are not consequences of the general theory of direct integrals but rather of the special circumstances here.

When $B = 0$,

$$H(k) = k^2/2m + h. \tag{23}$$

Equivalently:

$$\mathscr{H} = L^2(d^3k) \otimes \mathscr{H}_0, \tag{24}$$

$$H = k^2/2m \otimes I + I \otimes h. \tag{25}$$

This trivial c-number and parabolic dependence on k (Eq. (23)) is special to $B = 0$ and the k dependence of the reduced Hamiltonians is more complicated when $B \neq 0$. Proposition 4.2 describes a special circumstance for $B \neq 0$ where the k dependence trivializes, and one has a tensor decomposition somewhat analogous to (25).

The following result depends on Lemma 5.2 and Theorem 5.3 and should be returned to after they are studied:

PROPOSITION 4.2. *Let C_1 and C_2 be noninteracting clusters with $Q_1 + Q_2 = 0$, $Q_1 \neq 0$, i.e.,*

$$H = H_{C_1} \otimes I + I \otimes H_{C_2}$$

on $\mathscr{H} = L^2(\mathbb{R}^{3|C_1|}) \otimes L^2(\mathbb{R}^{3|C_2|})$. Then the reduced Hamiltonians $H(k)$ are unitarily equivalent for two k's with $k'_\parallel = k_\parallel$. Moreover, for suitable coordinates x_i (defined in Remark 3, below)

$$\mathscr{H} \cong L^2(d^4x) \otimes \mathscr{H}_0$$

with

$$H = (x_1^2/2m_1 + x_2^2/2m_2) \otimes I + I \otimes h_r,$$

where $k_\parallel = x_1 + x_2$, $k_\perp = (x_3, x_4)$, and $m_i = \sum_{\alpha \in C_1} m_\alpha$.

Remarks. (1) Letting $q = \mu(x_1 m_1^{-1} - x_2 m_2^{-1})$ with $\mu = (m_1^{-1} + m_2^{-1})^{-1}$, the reduced mass, we see that

$$H = k_\parallel^2/2M + q^2/2\mu + h_r$$

so that $H(k)$ is only a function of $k_\parallel$ and the $k_\parallel$ dependence is a c-number and separates out.

(2) The independence of $k_\perp$ is an expression of the fact that H is independent of the distance between the charge centers of the two clusters.

(3) If a (resp. b) are the k's for cluster 1 and 2, then $x_1 = a_\parallel$, $x_2 = b_\parallel$, and $(x_3, x_4) = a_\perp + b_\perp$.

Proof. An immediate consequence of Lemma 5.2 and Theorem 5.3. ∎

The two-particle interacting case can be worked out explicitly. Without loss of generality, we set $e_1 = -e_2 = 1$; let μ be the reduced mass, $M = m_1 + m_2$, r the relative coordinate, and x the center of mass coordinate. p and P_T are the respective conjugate momenta.

Warning. P_T is distinct from k.

As we shall see later, the reduced kinetic energy at zero total pseudomomentum is

$$H_0 = (1/2m_1)(p - A)^2 + (1/2m_2)(p + A)^2. \tag{32}$$

H_0 "ties down" the particle in the following sense:

LEMMA 4.3.

$$\frac{1}{m_<}(p^2 + A^2) \geqslant H_0 \geqslant \frac{1}{m_>}(p^2 + A^2)$$

where $m_> = \max\{m_1, m_2\}$, $m_< = \min\{m_1, m_2\}$.

Remark. This is related to Lemma 6.4 for multiparticles.

Proof. By the Schwarz inequality

$$\pm(p \cdot A + A \cdot p) \geqslant -(p^2 + A^2).$$

Thus

$$\begin{aligned}\alpha(p + A)^2 + \beta(p - A)^2 &\geqslant (\alpha + \beta)(p^2 + A^2) - |\alpha - \beta|(p^2 + A^2)\\ &= 2\min(\alpha, \beta)(p^2 + A^2).\end{aligned}$$

Remarks. (1) Thus, the H_0 of (32) has a compact resolvent in two dimensions.

(2) Remark 1 and the inequality of Lemma 4.1 is clarified if one expands and notes that

$$H_0 = -\frac{1}{2\mu}\Delta + \frac{e^2B^2}{8\mu}(x^2 + y^2) + \frac{eB}{2}\left(\frac{1}{m_1} - \frac{1}{m_2}\right)L_z. \tag{32'}$$

As a result, in the oscillator realization of Proposition 2.3 there is only partial cancellation between the oscillator piece and the L_z term, i.e.,

$$H_0 = -\frac{1}{2\mu}\frac{d^2}{dz^2} + \frac{eB}{m_1}\left(a^\dagger a + \frac{1}{2}\right) + \frac{eB}{m_2}\left(b^\dagger b + \frac{1}{2}\right)).$$

(3) From (32′) we conclude that $H_0 \upharpoonright (L_z = 0)$ is only dependent on μ, the reduced mass, and not on the mass ratio m_1/m_2.

There are several unitarily equivalent representations of the reduced two-body Hamiltonian $H(k)$. This is the subject of the next two theorems.

THEOREM 4.4. *Let $k \in \mathbb{R}^3$. Then the reduced Hamiltonian $H(k)$: $L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is unitarily equivalent to*:

$$H^{\#}(k) = \frac{1}{2m_1}\left(\frac{m_1}{M}k + p - A\right)^2 + \frac{1}{2m_2}\left(\frac{m_2}{M}k - p - A\right)^2 + V(r) \tag{33}$$

where $A = \frac{1}{2}B \times r$.

Proof. Let $\varphi = \frac{1}{2}(B \times r) \cdot x$, $U = e^{i\varphi}$. Then

$$U^{-1}P_T U = P_T - A, \tag{34}$$

$$U^{-1}pU = p + B/2 \times x. \tag{35}$$

Using $p_1 = p - (m_1/M)P_T$, $p_2 = -p + (m_2/M)P_T$ one gets

$$\tilde{H} \equiv U^{-1}HU = \frac{1}{2m_1}\left(\frac{m_1}{M}P_T - p - A\right)^2 + \frac{1}{2m_2}\left(\frac{m_2}{M}P_T - p - A\right)^2 + V(r). \tag{36}$$

Moreover $U^{-1}kU = P_T$. The lemma below proves the theorem. ∎

LEMMA 4.5. *Let $\mathscr{H} = \int^{\oplus} \mathscr{H}_k\, d^3k$, let $K = \int^{\oplus} (k\mathbb{1})\, d^3k$ and $H = \int^{\oplus} H(k)\, d^3k$. Let U be a unitary operator from $\mathscr{H}$ to $\mathscr{H}'$ and let $K' = UKU^{-1}$. Then $\mathscr{H}'$ has a direct integral decomposition $\mathscr{H}' = \int^{\oplus} \mathscr{H}'_k\, d^3k$ so that $K' = \int^{\oplus} (k\mathbb{1})\, d^3k$ with respect to this decomposition and so that $H' = UHU^{-1} = \int^{\oplus} H'(k)\, d^3k$. Moreover, $U = \int^{\oplus} U(k)\, d^3k$ with $U(k)$: $\mathscr{H}_k \to \mathscr{H}'_k$ and $H'(k) = U(k)\, H(k)\, U(k)^{-1}$.*

This lemma is just an expression of the invariant character of direct integral decompositions.

Remark. The unitary transformation U above appears in [35].

EXAMPLE. A particularly simple situation arises for harmonic $V(x)$ with $m_1 = m_2$:

$$H(k) = \tfrac{1}{2}(\tfrac{1}{2}k + p - A)^2 + \tfrac{1}{2}(\tfrac{1}{2}k - p - A)^2 - (\omega^2/2)x^2. \tag{37}$$

The equations of motion are

$$\dot{x} = p,$$
$$\ddot{x} = -(\omega^2 + B^2/2)x + \tfrac{1}{2}k \times B + \tfrac{1}{2}B(B \cdot x).$$

In the B direction the characteristic frequency is ω. In the perpendicular plane the "distance between the particles" rotates around

$$\tfrac{1}{2}(k \times B)/(\omega^2 + B^2/2)$$

with the frequency $(\omega^2 + B^2/2)^{1/2}$.

Theorem 4.4 regards k as a momentum. The next theorem gives k a coordinate flavor:

THEOREM 4.6. *Let $\beta = k \times B/B^2$. Then $H(k)$ is unitarily equivalent to*

$$H^{\flat}(k) = \frac{1}{2m_1}(p - A)^2 + \frac{1}{2m_2}(p + A)^2 + \frac{k_{\parallel}^2}{2M} + V(r - \beta). \tag{38}$$

Proof. Let $W = e^{i(\alpha\cdot x + \beta\cdot p)}$, $\alpha = [(m_2 - m_1)/2M]k$. Then $W^{-1}P_T W = P_T$, $W^{-1}pW = p + \alpha$, $W^{-1}AW = A - \frac{1}{2}B \times \beta$, and $W^{-1}rW = r - \beta$. Applying W to Eq. (36) gives Eq. (38). Note that W depends only on relative coordinates and so operates on a single k fiber. ▮

Let us turn to spectral properties of the reduced Hamiltonians $H(k)$.

PROPOSITION 4.7. *Let $V \in L^2 + L_\epsilon^\infty$. Then the essential spectrum of the reduced two-body Hamiltonian $H(k)$ is*

$$\sigma_{\text{ess}}(H(k)) = [|B|/2\mu + k_{\parallel}{}^2/2m, \infty]. \tag{39}$$

Proof. It is enough to consider $V \in L^2$ and the theorem follows by a closure argument for compacts. Such V's are form Hilbert–Schmidt relative to the Laplacian and so:

$$\begin{aligned}
&\| V^{1/2}(H_0 + 1)^{-1} V^{1/2} \|_2 \\
&\quad \leqslant \| V^{1/2}(-\Delta + 1)^{-1/2} \|_4^2 \\
&\quad \times \|(-\Delta + 1)^{1/2}(-\Delta + x^2 + 1)^{-1/2}\|^2 \, \|(-\Delta + x^2 + 1)^{1/2}(H_0 + 1)^{-1/2}\|^2 < \infty,
\end{aligned}$$

where

$$\| A \|_{2p} = \{\mathrm{Tr}(AA^+)^p\}^{1/2p}$$

and Lemma 4.3 has been used in the last step. Hence $V(r - \beta)$ in Eq. (39) is compact relative to the kinetic energy for all β. The essential spectrum of $H_0(k)$ is $[|B|/2\mu + (k \cdot B)^2/2M, \infty)$. The result then follows by the stability of the essential spectrum under relatively compact perturbations [15, 23]. ▮

Remark. $\psi \in L^2 + L_\epsilon^\infty$ if $\forall \epsilon > 0$, $\exists \psi_{1\epsilon}, \psi_{2\epsilon}$ such that $\psi = \psi_{1\epsilon} + \psi_{2\epsilon}$ with $\sup |\psi_{1\epsilon}(x)| < \epsilon$, $\| \psi_{2,\epsilon} \|_2 < \infty$.

In [2–4] we have discussed enhanced binding in magnetic fields. The following result is typical.

PROPOSITION 4.8. *Let $V \leqslant 0$ pointwise everywhere and as in Proposition 4.6. Then $H(k)$ has at least one discrete eigenvalue.*

Remark. There is an important difference between the single-particle Hamiltonian treated in [4] and the reduced Hamiltonian above. In the single-particle case, $V \not\equiv 0$, $V \leqslant 0$, and spherical symmetry imply an infinite number of discrete eigenvalues, with no additional assumption on the range of the potential. This is not the case here even under the same assumption on V. The reason for the difference can be understood from Lemma 4.3. Let us call the bottom of the essential spectrum of H_0 $(k = 0)$, restricted to a fixed L_z subspace the L_z threshold. For the single-particle case, the bottom of the essential spectrum is an L_z threshold for an infinite number of L_z's, but this is not so for Eq. (32); the L_z thresholds accumulate at infinity. See Remark 2 following Lemma 4.3. The eigenvalues associated with these thresholds are therefore not discrete in the total spectrum. Thus if we add a smooth negative noncentral potential to V, we *expect* that H $(k = 0)$ will have only finitely many eigenvalues, whereas in the one-body case, H will continue to have an infinity of eigenvalues. In particular, for $k \neq 0$, we expect only finitely many eigenvalues for $H(k)$ (discrete or nondiscrete) even if V is central.

Let $E_n(k)$ denote the discrete eigenvalue function of $H(k)$. $E_n(k)$ depends trivially on $k_\parallel$, i.e.,

$$E_n(k) = (k_\parallel)^2/2M + E_n(k_\perp). \tag{40}$$

That the functions $E_n(k)$ are nice follows from

PROPOSITION 4.9. *Let V be as in Proposition 4.6 and real. Furthermore, let $k = \lambda\hat{e}$ with $\hat{e}$ a fixed unit vector. Then $H^{\#}(\lambda\hat{e})$ in the realization of Theorem 4.4 is holomorphic of type (B) in λ [15]. In particular, $E_n(\lambda)$ are real analytic as long as $E_n(\lambda)$ stays away from the essential spectrum.*

Proof. $p \pm A$ is form bounded relative to $(p \pm A)^2$. This remains invariant under the addition of V to $(p \pm A)^2$ for relatively compact V. The basic Rellich criterion for type (B) can be applied to Eq. (33) to yield the result. ∎

The $|k_\perp| \to \infty$ behavior is determined by:

PROPOSITION 4.10. *Let V be as in the previous proposition and $H^\flat(k)$ in the realization of Theorem 4.7. Then $H^\flat(k) \to H_0{}^\flat(k)$ in norm resolvent sense as $|k_\perp| \to \infty$. ($H_0(k)$ is independent of $k_\perp$.) Consequently*

$$\operatorname{Inf Spec}\{H(k)\} \to \operatorname{Inf Spec}\{H_0(k)\} = \frac{|B|}{2\mu} + \frac{(k_\parallel)^2}{2M}$$

as $|k_\perp| \to \infty$.

Proof. By an approximation argument, we may assume, that $V \in C_0^\infty$. By the resolvent equation

$$\begin{aligned}\| R - R_0 \| &\leqslant \| R \| \left\| V \frac{1}{(1 + x^2)^{1/2}} \right\| \|(1 + x^2)^{1/2} R_0 \| \\ &\leqslant c \| R \| \left\| V \frac{1}{(1 + x^2)^{1/2}} \right\|\end{aligned}$$

and Lemma 4.3 was used in the last step. The result follows now from

$$\sup_x \left| \frac{V(x + \beta)}{(1 + x^2)^{1/2}} \right| \to 0 \qquad \text{as } |k_\perp| \to \infty. \ ∎$$

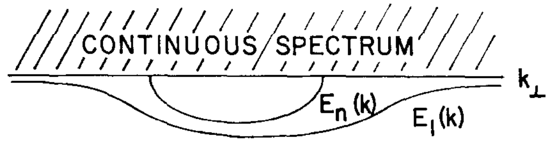

FIG. 1. Schematic energy curves for the reduced two-body Hamiltonian.

The qualitative behavior of the functions $E_n(k)$ is shown in Fig. 1. This should be contrasted with the parabolas of the $B = 0$ case.

Remark. It seems a little surprising that for an infinitesimal field, the parabolas change so drastically, especially since the (local in time) physics is clearly continuous as $B \to 0$. The point is that spectral information is global in time and that globally things are very different (on a classical level periodic in time) no matter how small the field is. When $m_1 = m_2 = 1$ and $V = 0$ and $k = k_0$ we can get a state of minimal energy if $B = 0$ by giving both particles velocity $k_0/2$, so $E = k_0^2/2$ (the parabola). If $B \neq 0$, so $k = p_1 + p_2 + \frac{1}{2}B \times r$, we take $v_1 = v_2 = 0$ and get k by choosing r large (i.e., as $| k | B^{-1}$) whence $E = \frac{1}{2}(v_1^2 + v\mathrm{J}^2) = 0$. Notice that as $B \to 0$, these states have no limit in the physical Hilbert space.

We can say something about the curvature of the ground state function $E_1(k)$:

PROPOSITION 4.11. *$E_1(k) - k^2/2M$ is concave downward.*

This follows from Theorem 4.4 and a general concavity of the ground state of $A + kB$ (a consequence of min–max).

In addition, if $V(x)$ is spherically symmetric, rotations around the B axis give:

PROPOSITION 4.12. *$E_n(k_\perp)$ is only a function of $| k_\perp |$.*

For Coulomb potentials we have an additional result:

THEOREM 4.13. *The ground state for the attractive two-body Coulomb Hamiltonian (hydrogen, positronium) is $k = 0$ more precisely*

$$E_1(0) \leqslant E_1(k). \tag{41}$$

Proof. The special property of the Coulomb potential is that the ground state for the corresponding one-body problem (infinitely massive proton) has $L_z = 0$. (See [1, 4] for further discussion.) We shall use this in our proof. Let

$$E(\nu_1, \nu_2, k) = E_1(k) \tag{42}$$

where $\nu_i m_i = 1$. Then:

$$\begin{aligned} E(\nu_1, \nu_2, k) &\geqslant \frac{\nu_2}{\nu_1 + \nu_2} E(0, \nu_1 + \nu_2, k) + \frac{\nu_1}{\nu_1 + \nu_2} E(\nu_1 + \nu_2, 0, k) \\ &= E(0, \nu_1 + \nu_2, k) = E(0, \nu_1 + \nu_2, 0) = E(\nu_1, \nu_2, 0). \end{aligned} \tag{43}$$

The first inequality comes from the convexity of the ground state in ν_i. The first equality comes from complex conjugation which implies $E(\nu_1, \nu_2, k) = E(\nu_2, \nu_1, k)$. The following equality comes from the fact that $H(k)$ and $H(0)$ are unitarily equivalent when $m_2^{-1} = 0$; this is evident from (33) and a gauge transformation. The last equality comes from the third remark following Lemma 4.3 and the fact that the ground state of the Coulomb problem has $L_z = 0$ [4]. ∎

Remark. There are central V's for which the ground state is not $L_z = 0$ and presumably, there are also V's for which the ground state does not have $k = 0$. The exact Coulomb force is not essential in the above. What is needed is that V is central, $V'(x) \geqslant 0$ (see [4]).

Let us conclude with a few remarks on possible spectroscopic implications. The simple models we have been treating, dubbed hydrogen and positronium for illustrative purposes, have, in the absence of a magnetic field a fourfold degeneracy $2S - 2P$. Part of this degeneracy is removed, in lowest order in B, but the $2S$ is still degenerate with $2P$ ($L_z = 0$) in this order in B. This degeneracy is removed by the center of mass and for $k \neq 0$ there is no degeneracy. Since the recoil due to photon absorption or emission will change k, a removal of degeneracy is predicted for the model.

We have remarked in another paper of the series that for spherically symmetric attractive V, the one-body Hamiltonians (in particular the spinless hydrogen with an infinitely massive proton) have bound states, embedded in the continuous spectrum at arbitrarily high energies [4].

The existence of these bound states is a consequence of rotation invariance around the B axis which prevents their coupling to the background continuum. Once the symmetry is broken, one expects these states to become resonances [27]. For $k = 0$ the reduced Hamiltonian is invariant under rotations around the B axis and so has high bound states. This is not the case for $k \neq 0$ and so probably there are no such high bound states. Hence, the center of mass motion couples the high bound states to the continuum to cause their decay.

5. The $Q \neq 0$ Sector

The zero charge sector produced a situation which differs from the standard reduction of center of mass in important details, but not in a fundamental way. The nonzero charge sector, on the other hand, leads to a different state of affairs; one cannot diagonalize the full symmetry group so there is no reduced center of mass Hamiltonian in the conventional sense. We shall, nevertheless, be able to analyze the problem. Most striking is the existence of an underlying infinite degeneracy, *that is, the well-known Landau degeneracy, is a feature of any* $Q \neq 0$ *system in constant field.* In addition, there is a notion of reduced Hamiltonians that is applicable also for this case.

PROPOSITION 5.1. *k generates the Heisenberg group in* 2-dim *perpendicular to B (with $\hbar = QB$) crossed with the translation group.*

Remarks. (1) Identify k_x with p and k_y with q.

(2) The Heisenberg group in 2-dim is:

$$g\{\omega_1, (q_1, p_1)\}\, g\{\omega_2, (q_2, p_2)\} = g\{\omega_1 + \omega_2 - (\hbar/2)(q_1 p_2 - q_2 p_1), (q_1 + q_2, p_1 + p_2)\}, \tag{44}$$

$(q, p) \in \mathbb{R}^2$, $\omega \in \mathbb{R}$.

It is easy to check that $g(\alpha) = e^{ik\cdot\alpha}$ satisfies (44) by Theorem 2.4 (use the Baker–Hausdorf formula).

Separation of the c.m. motion in the direction parallel to B is trivial:

LEMMA 5.2. *Let $t = k_\parallel$. Then*

$$\mathscr{H} = L^2(dt) \otimes \mathscr{H}_\perp,$$
$$H = t^2/2M \otimes I + I \otimes H_\perp.$$

In the following we shall proceed to analyze $H_\perp$. Here it proves somewhat more convenient to take a multiple of $k_\perp$, namely, $c = \sum_{i=1}^n e_i(c_i)_\perp$, with $c \times c = -iBQ/|B|^2$.

THEOREM 5.3. *$H_\perp$ is a tensor product, i.e., for suitable h_r*

$$\mathscr{H}_\perp = L^2(dy) \otimes \mathscr{H}_0,$$
$$H_\perp = I \otimes h_r,$$
$$c_x = -(iBQ/|B|^2)(d/dy) \otimes I,$$
$$c_y = y \otimes I.$$

COROLLARY 5.4. *The spectrum of $H_\perp$ is of infinite multiplicity* (*so that* $\sigma_{\text{disc}}(H_\perp) = \varnothing$).

The natural notion of c.m. reduction is the restriction of $H_\perp$ to the subspace $f \otimes \mathscr{H}_0$, $f \in L^2(dy)$. In fact a natural distinguished basis is the eigenstate for the number operator $C^*C = c_x^2 + c_y^2$. The reduced subspaces are therefore labeled by a positive integer in contradistinction with the $Q = 0$, where the labeling is by a two-dimensional vector.

Proof of Theorem 5.3. c_x and c_y satisfy the canonical commutation relations ($\hbar = Q\,|B|$). It is well known that the Heisenberg group has the property that any representation with $(\omega, x = 0, p = 0) \to e^{i\omega}\mathbb{1}$ is a direct sum of the unique irreducible representation. Thus, the von Neumann algebra $\mathfrak{a}$ generated by $\{e^{ic\cdot\alpha}\}$ has the form $B \otimes I$ with B the von Neuman algebra of all bounded operators on $L^2(dc)$. $e^{iH_\perp t}$ is in the commutant of $\mathfrak{a}$ by Theorem 2.4. By an abstract result, $e^{iH_\perp t} = I \otimes e^{iht}$ which proves the theorem. ▮

$\xi^\dagger = (2Q)^{-1/2}(c_x - ic_y)$ is creation operator:

$$L^2(dc) = \bigoplus_n | n\rangle = \bigoplus_n (\xi^\dagger)^n | 0\rangle. \tag{45}$$

This decomposition is natural for center of mass reduction, as the next result shows (Proposition 5.5 is the analog of Proposition 4.7):

PROPOSITION 5.5. *Consider the two-body Hamiltonian. Let $V \in L^2 + L_\epsilon^\infty$, then V is compact relative to h_r.*

Proof. Suppose we show that $V(H_\perp + (e^2B^2/2M) | c |^2 + 1)^{-1}$ is compact on $\mathscr{H}_\perp$. Then restricting to the space with $c^2 = \text{const}$, we see that $V(h_r + \text{const} + 1)^{-1}$ is compact on $\mathscr{H}_0$.

In $\mathscr{H}_\perp$ we can use the five coordinates $r = x_1 - x_2$ in $\mathbb{R}^3$ and $B \times R$ in $\mathbb{R}^2$ where $R = e_1x_1 + e_2x_2$ in $\mathbb{R}^3$ (it is here that $e_1 + e_2 \neq 0$ enters for it implies that R is independent of r). Let A be the operator ($M = m_1 + m_2$)

$$A = \frac{1}{2m_1} P_1^2 + \frac{1}{2m_2} P_2^2 - \frac{t^2}{2M} + \frac{1}{M}(B \times R)^2 + \frac{1}{2M}(P_1 + P_2)^2.$$

We will prove in Lemma 6.4 that

$$[H_\perp + ((eB)^2/2M)| c |^2 + 1]^{-1} \mathrel{\underset{\cdot}{\leqslant}} (A + 1)^{-1}$$

where $C \mathrel{\underset{\cdot}{\leqslant}} D$ means that $| C\psi | \leqslant D | \psi |$ pointwise. It follows by general principles (see [3]) that it is sufficient to prove that

$$V(A + 1)^{-1} \in \mathscr{I}_p$$

for some $p < \infty$ where $\mathscr{I}_p = \{C \mid \text{tr}(| C |^p) < \infty\}$. Let $\rho = (B \times R)$. Let

$$A_0 = -\Delta_r - \Delta_\rho + \rho^2 \equiv A_r + A_\rho .$$

Then it is easy to see [23, Vol. II] that $D(A) = D(A_0) = D(-\Delta_r) \cap D(-\Delta_\rho) \cap D(\rho^2))$ and thus $(A_0 + 1)(A + 1)^{-1}$ is bounded (by the closed graph theorem). It thus suffices to prove that $V(A_0 + 1)^{-1} \in \mathscr{I}_p$. Since A_r and A_ρ commute, $(A_r + 1)^{1/2}(A_\rho + 1)^{1/2}(A_0 + 1)^{-1}$ is bounded and thus we need only show that $V(A_r + 1)^{-1\ 2}(A_\rho + 1)^{-1\ 2}$ is in $\mathscr{I}_p$. For V in C_0^∞, $V(A_r + 1)^{-1/2}$ is certainly in $\mathscr{I}_p$ in the r variables for $p > 3$ and $(A_\rho + 1)^{-1/2}$ is in $\mathscr{I}_p$ in the ρ variable for $p > 4$, so we are done. ∎

6. THE HVZ THEOREM

In Sections 4 and 5 we have described a special form of the HVZ theorem, i.e., for the two particle problem. We shall now consider the more general case.

The two-body problem has a trivial aspect about it which is that the essential spectrum starts at $|B|/2\mu + (k_{\parallel})^2/2\mu$, i.e., it has an explicit k dependence. This is not true in general and the bottom of the essential spectrum will have a k dependence that depends on details of the interaction.

We shall follow [24] in our proof of HVZ.

THEOREM 6.1. *Let H_r be an N particle Hamiltonian with* c.m. *removed, i.e., $H_r = H(k)$ if $Q = 0$ and $H_r = H_{|0>\otimes\mathscr{H}_0}$ if $Q \neq 0$. Let α be a partitioning into disjoint, nonempty clusters C_1^α, C_2^α and*

$$H_r = H_1^\alpha \otimes I + I \otimes H_2^\alpha + V^\alpha$$

with the obvious meaning, e.g., V^α = interaction between C_1^α and C_2^α. Then

$$\inf\{\sigma_{\text{ess}}(H_r)\} = \inf_\alpha\{\sigma(H_1^\alpha \otimes I + I \otimes H_2^\alpha)\}.$$

Before getting into the proof, let us consider some applications: There is a natural concept of atoms, ions, and molecules in this framework with point particle nuclei.

For atoms and ions one has:

PROPOSITION 6.2. *The $\inf_\alpha$ in Theorem 6.1 is obtained for α_0 with one electron removed.*

This is an easy application of HVZ together with the fact that the electron–electron interaction is purely repulsive.

A consequence of Proposition 6.2, Theorem 6.1, and Proposition 4.2 is

COROLLARY 6.3. *Let $H(k)$ be the reduced Hamiltonian for an N particle, neutral atom. Then*

$$\inf[\sigma_{\text{ess}}\{H(k)\}] = c + k_{\parallel}^2/2M$$

where c = min *energy of ion in magnetic field.*

For molecules one expects, in general, a nontrivial dependence on k. This means that the "kinematic" conservation law of energy does not follow a universal curve (parabola) but depends on the details of the interaction. For example, the disassociation of the hydrogen molecule into two hydrogen atoms involves the nontrivial function

$$\inf_{k'}\{E_1(k'_\perp) + k'^2_{\parallel}/2M + E_1(k - k'_\perp) + (k - k')^2_{\parallel}/2M\}$$

where $E_1(k_\perp)$ is the ground state energy function of hydrogen (Propositions 4.12 and 4.13).

Pointwise bounds on the resolvent played an important role in Avron *et al.* [3]. These are convenient for proving compactness of certain operators. Since HVZ is basically a compactness result, they enter naturally also here. In [26, 36] a sufficient condition for the pointwise inequality $(H+1)^{-1} \leqslant (A+1)^{-1}$ was formulated which is of the form $\mathrm{Re}[(\operatorname{sgn}\psi)\, H\psi] \geqslant A\,|\psi|$, pointwise, with e^{-tA} positivity preserving and $\operatorname{sgn}\psi = \bar{\psi}/|\psi|$ if $|\psi| \neq 0$ and 0 otherwise.

LEMMA 6.4. *Let* $M = \sum_{i=1}^{N} m_i$, $R = \sum e_i x_i$,

$$H = \sum_i \frac{1}{2m_i}(p_i - e_i A_i)^2 + \frac{1}{2M} k^2,$$

$$A = \sum_i \frac{1}{2m_i} p_i^2 + \frac{1}{M}(B \times R)^2 + \frac{1}{2M}\Big(\sum_i p_i\Big)^2.$$

Then

$$\mathrm{Re}[(\operatorname{sgn}\psi)\, H\psi] \geqslant A\,|\psi|, \qquad \psi \in D(H).$$

Proof. Kato's inequality in magnetic field (see, e.g., [23, Vol. II]) asserts that

$$\mathrm{Re}[(\operatorname{sgn}\psi)[t(p_i - \alpha_i)\psi]] \geqslant t(p_i)\,|\psi|$$

for any positive definite quadratic form t and any smooth α_i. The idea is to rewrite H as $t(p_i)$ + function of r_i. By finding the minimizing point of

$$f(\theta) = \sum_i \frac{1}{2m_i}(\theta_i - eA_i)^2 + \frac{1}{2M}\Big(\sum_i (\theta_i + e_i A_i)\Big)^2$$

one finds that

$$f(\theta) = \sum_i \frac{1}{2m_i}(\theta_i - \alpha_i)^2 + \frac{1}{2M}\Big(\sum_i (\theta_i - \alpha_i)\Big)^2 + \frac{1}{M}(B \times R)^2$$

with

$$\alpha_i = \frac{-m_i}{M}(B \times R) + e_i A_i .$$

Thus

$$H = \sum_i \frac{1}{2m_i}(p_i - \alpha_i)^2 + \frac{1}{2M}\Big(\sum_i (p_i - \alpha_i)\Big)^2 + \frac{1}{M}(B \times R)^2$$

whence Kato's inequality implies the lemma. ∎

By the result of [26, 36], we have the following (used already to prove Proposition 5.5):

COROLLARY 6.5. *Let H, A be as in Lemma* 6.4. *Then*

$$(H+1)^{-1} \leqslant (A+1)^{-1}.$$

Proof of Theorem 6.2. Let $|x|^2 = \sum_{i\leqslant j}(x_i - x_j)^2$, $\varphi(x) \in C_0^\infty$ such that $\varphi = 1$ if $|x| \leqslant 1$ and 0 if $|x| \geqslant 2$ and $J_{\leqslant n}(x) = \varphi(x/n)$. Let H_0 be the Hamiltonian with no pair interaction and c.m. motion in the B direction removed and let

$$A = (H_0 + 1)^{-1} J_{\leqslant n}.$$

Since J is translation invariant, A is reducible. By [24] the proof of the "hard direction" of HVZ is reduced to proving compactness of A with c.m. removed. In the zero charge sector, $Q = 0$, this means $A(k)$ is compact. In the nonzero charge sector if is sufficient to show that the nonreduced

$$A' = (H_0 + k^2 + 1)^{-1} J_{\leqslant n}$$

is compact. For the $Q = 0$ sector (resp. $Q \neq 0$) this just follows the proof of Proposition 4.7 (resp. Proposition 5.5). ∎

The easy direction, $\sigma(H_r - V^\alpha) \subset \sigma(H_r)$, follows Hunziker's proof [12]: Given any $u \in D(H_r)$, $\lim_{s\to\infty} \|(H_r - E) W_s^\alpha u\| = \|(H_r - V^\alpha - E)u\|$ where W_s^α translates the clusters C_1^α, C_2^α relative to one other in the B direction by an amount s. This and

$$\mathrm{spec}(A) = \{E \mid \forall\epsilon, \exists u \|u\| = 1, \|(A - E)u\| < \epsilon\}$$

imply that $\sigma(H_r - V^\alpha) \subset \sigma(H_r)$. ∎

Remark. The proof of the "hard direction" works for two-dimensional particles but the easy direction depends on the possibility of translating parallel to the fields. Indeed, the continuous spectrum is only present for this reason and will not be present in two dimensions.

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