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MAXIMAL AND MINIMAL SCHRÖDINGER FORMS

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§1. INTRODUCTION

Let $V \geqslant 0$, $V \in L^1_{\mathrm{loc}}$, $\boldsymbol{a} \in L^2_{\mathrm{loc}}$. There are a priori two natural quadratic forms associated to $(-\mathrm{i}\nabla - \boldsymbol{a})^2 + V$. Namely, ($\nabla\varphi$ = distributional gradient), let

$$Q(h_{\max}) = \{\varphi \in L^2 |\ (\nabla - \mathrm{i}a)\varphi \in L^2;\ V^{1/2}\varphi \in L^2\} \tag{1.1}$$

and

$$h_{\max}(\varphi, \psi) = \sum_{j=1}^{\nu} ((\partial_j - \mathrm{i}a_j)\varphi, (\partial_j - \mathrm{i}a_j)\psi) + (V^{1/2}\varphi, V^{1/2}\psi). \tag{1.2}$$

Define $h_{\min}$ to be the form closure of $h_{\max}$ restricted to $C_0^\infty \times C_0^\infty$. Our main result in this note is that $h_{\min} = h_{\max}$ under the most general conditions considered above.

Surprisingly, this natural question appears to have been asked only recently by Kato [15]. One can phrase the extensively studied question of whether $-\Delta + V$ is essentially self-adjoint on C_0^∞ in virtually identical terms; namely let $V \in L^2_{\mathrm{loc}}$ and let

$$D(H_{\max}) = \{\varphi \in L^2 \mid -\Delta\varphi + V\varphi \in L^2\}$$

with

$$H_{\max}\varphi = -\Delta\varphi + V\varphi.$$

Let $H_{\min}$ be the operator closure of $H_{\max}$ restricted to C_0^∞. Since $H_{\max} = H_{\min}^*$, the operator equality is one of essential self-adjointness and is a necessary preliminary to the existence of a self-adjoint realization of $-\Delta + V$. The form equality is only one of uniqueness since general theorems associate self-adjoint operators to any closed form [10], [16]. This is probably the reason that the question has only recently been raised. We also note that in finite volume problems, $h_{\min}$ and $h_{\max}$ do differ; one corresponds to Dirichlet boundary conditions, the other to Neumann conditions (see e.g. ref. [18]).

Our main tool in this note will be certain semigroup ideas which can be traced back to developments in constructive quantum field theory, especially the $P(\varphi_2)$-Hamiltonian self-adjointness proof of Rosen [19] and Segal [22], abstracted in ref. [31]. These ideas were useful in the study of Schrödinger operators [23], [6], [8] and, in particular, led to the proof of the fact that $-\Delta + V$ is essentially self-adjoint on C_0^∞ if $V \geqslant 0$ and $V \in L^2(R^\nu, e^{-x^2}\, dx)$ [23]. They appear to have been abandoned as a self-adjointness tool, in part because of the success of a different method of Kato [11], who in particular removed the global $L^2(R^\nu, e^{-x^2}\, dx)$ restriction and replaced it by a local L^2_{loc} condition (we recover this result in Theorem 3.1 below). Kato based his proofs on an ingenious *distributional* inequality

$$\Delta|\varphi| \geqslant \operatorname{Re}((\operatorname{sgn} \varphi)\, \Delta\varphi) \tag{1.3}$$

($\operatorname{sgn} \varphi = \lim_{\varepsilon \downarrow 0} \varphi^*/(|\varphi|^2 + \varepsilon^2)^{1/2}$) for $\varphi \in L^1_{loc}$ with $\Delta\varphi \in L^1_{loc}$. Our return to the semigroup methods is motivated in part to the realization that (1.3) is "essentially" a semigroup statement: indeed [26], for nice φ, it is equivalent to the assertion that

$$|e^{t\Delta}\varphi| \leq e^{t\Delta}|\varphi|. \tag{1.4}$$

We emphasize that this note should be viewed as our continuation of an approach studied roughly five years ago: for example, after looking at the proof of Theorem 3.1, one might be led to a similar proof of the main perturbation theorem in the theory of hypercontractive semigroups: this proof would just be that of Faris [17]!

One point that is made by our results here concerns the need for "Sobolev restrictions" in the study of Schrödinger operators, i.e. inequalities on p in L^p_{loc} hypothesis that can be traced back to the use of a Sobolev inequality. Such inequalities occur in the negative singularities of V and the $-|x|^{-2}$ example (see e.g. ref. [23]) shows they are really necessary there. Our point is they should not be necessary anywhere else with the sole probable exception that if one allowed complex-valued a's, then Sobolev restrictions would almost surely be required on Im a. For the positive part of V, this was our point already in 1972 [23]. As regards forms, this note shows no such restriction is needed on $\boldsymbol{a}$: Kato [15] was only able to show $h_{min} = h_{max}$ under the Sobolev hypothesis $\boldsymbol{a} \in L^p_{loc}$, $p > \nu$. As regards the self-adjointness question for $(-i\nabla - \mathbf{a})^2$, Sobolev restrictions were made, in the work of Schechter [20], [21] and Simon [24] (we recover this result in § 3). While we can improve their L^p_{loc} restrictions form $p > \nu$ to $p \geq 6\nu/(\nu + 2)$ for $\nu \geq 5$, we still get a restriction whose origin is in a Sobolev inequality. It is an interesting open question to prove that

Conjecture. If $V \in L^2_{loc}$, $V \geqslant 0$, $\boldsymbol{a} \in L^4_{loc} \nabla \cdot a \in L^2_{loc}$, *then* $(-i\nabla - \boldsymbol{a})^2 + V$ *is essentially self-adjoint on* C_0^∞.

Similarly, in his recent work on complex valued V, Kato [14] had a Sobolev restriction on Im V; in their extension of this work, Brézis-Kato [4] had a similar condition, but following a suggestion of the present author that the philosophy and methods of the present paper suggest Im $V \in L^1_{loc}$ should suffice, Sobolev restrictions were removed from that case.

To the extent that we rely largely on (1.4) (and the related (2.3) below), our methods can be viewed as exploiting a "semigroup facet" of Kato's inequality. An advantage of this aspect is that Sobolev restrictions can often be avoided. Two advantages of the original distributional version are the following: (1) it works easily to answer certain natural distributional questions [12]; for example, I do see how to obtain Theorem X.32 of ref. [17] by just using the methods below. (2) It does not require an a priori construction of H by quadratic forms as we do below and thus it can accommodate complex V[14], [4]; our methods below only seem to work if V has values in a sector $|\arg V(x)| \leqslant \frac{\pi}{2} - \varepsilon$ (some $\varepsilon > 0$).

In §2, we prove $h_{\min} = h_{\max}$; in §3, we consider when $H_{\min} = H_{\max}$ and in §4, we prove a convergence result following ideas in Kato [15].

§2. EQUALITY OF THE MINIMAL AND MAXIMAL SCHRÖDINGER FORMS

To illustrate the main ideas, we begin with the $a = 0$ result, one which also follows from ideas in ref. [12].

THEOREM 2.1. *Let $V \in L^1_{loc}$, $V \geqslant 0$ and let $h_{\max}$ be the form* (1.2) (*with $a = 0$*) *on the form domain* (1.1). *Then C_0^∞ is dense in $Q(h_{\max})$ in the norm*

$$\|\varphi\|_{+1} = [h_{\max}(\varphi, \varphi) + (\varphi, \varphi)]^{1/2}. \tag{2.1}$$

Proof. Let H be the operator associated to $h_{\max}$. We first claim that

$$|e^{-tH} \varphi| \leqslant e^{+t\Delta}|\varphi| \tag{2.2}$$

for all $\varphi \in L^2$. For let $V_n = \max(V, n)$ and let $H_n = -\Delta + V_n$. Then $(H_n + i)^{-1}$ converges strongly to $(H + i)^{-1}$ by the monotone convergence theorem for forms [10], [27], [28], so by the continuity of the functional calculus [16], it suffices to prove (2.2) with H replaced by H_n. But, by the Trotter product formula in its original form [33], $e^{-tH_n} = \text{s-}\lim_{m\to\infty} (e^{-tH_n/m}\, e^{-tH_n/m})^m$ so that (2.2) follows from $|e^{+s\Delta} \varphi| \leqslant e^{s\Delta}|\varphi|$ (i.w. $e^{s\Delta}$ has a positive integral kernel) and $|e^{-sV_n}\varphi| \leqslant |\varphi|$.

(2.2) and the inequality

$$\|e^{t\Delta}\varphi\|_\infty \leqslant c_t\|\varphi\|_2 \tag{2.3}$$

($e^{t\Delta}$ is convolution with a function in L^2) imply that

$$\text{Ran}\,(e^{-H}) \subset L^\infty. \tag{2.4}$$

Recall that $X \subset L^2$ is called a form core for H if and only if $X \subset Q(h_{\max})$ and X is dense in $Q(h_{\max})$ in $\|\cdot\|_{+1}$. Since $\mathrm{Ran}(e^{-H})$ is a form core for H by the spectral theorem, (2.4) implies that $L^\infty \cap Q(h_{\max})$ is a form core for H.

Now let $\psi \in C_0^\infty(R)$ and let $\varphi \in Q(h_{\max}) = D(\nabla) \cap D(V^{1/2})$. Then $\psi\ \varphi \in D(V^{1/2})$ and

$$\nabla(\psi\ \varphi) = (\nabla\psi)\varphi + \psi\nabla\varphi \tag{2.5}$$

so $\psi\ \ \varphi \in D(\nabla)$, i.e. $\psi\varphi \in Q(h_{\max})$. From (2.5), we conclude that if $\eta \in C_0^\infty$ with $\|\eta\|_\infty = 1$ and $\eta(x) = 1$ for x then $\varphi_n = \eta(\cdot/n)\varphi$ converges to φ in $\|\cdot\|_{+1}$. Thus $S = \{\varphi \in L^\infty \mid \text{supp } \varphi \text{ compact}\} \cap Q(h_{\max})$ is a form core for H.

Let j_δ be an approximate identity (i.e. $j_\delta(x) = \delta^{-\nu} j(x/\delta)$ where $j \in C_0^\infty$, $\int j(x)\,dx = 1$ and $0 \leq j$) and let $\varphi \in S$, the set just defined. Let

$$\varphi_\delta = j_\delta *, \varphi.$$

Then since $\varphi \in Q(h_{\max}) \in D(\nabla)$, $\nabla\varphi_\delta \to \nabla\varphi$ in L^2; since $\varphi \in L^2$, $\varphi_\delta \to \varphi$ in L^2 and since $\underline{\varphi \in L^\infty}$, and $V^{1/2} \in L^2$, $V^{1/2}\varphi_\delta \to V^{1/2}\varphi$ in L^2. We conclude that $\varphi_\delta \to \varphi$ in $\|\cdot\|_{+1}$. Since $\varphi_\delta \in C_0^\infty$, we have shown that C_0^∞ is a form core for H. ▨

Remarks. 1. The usefulness of (2.2) and (2.3) in tandem has been noted in a different context by Davies [6].

2. If we use Kato's strong Trotter formula [13] discussed below, we could avoid the use of the monotone convergence theorem for forms. Kato's strong form will be essential below.

The main result of this paper is

THEOREM 2.2. *Let $V \in L^1_{\text{loc}}$, $\boldsymbol{a} \in L^2{}_{\text{loc}}$, $V \geq 0$ and let $h_{\max}$ be the form* (1.2) *on the form domain* (1.1). *Then C_0^∞ is dense in the norm $\|\cdot\|_{+1}$ of* (2.1).

We first reduce this result to

THEOREM 2.3. *Let H be the operator associated to the form $h_{\max}$ of Theorem* 2.2. *Then*

$$|e^{-tH}\varphi| \leq e^{t\Delta}|\varphi|. \tag{2.6}$$

Proof of Theorem 2.2 *given Theorem* 2.3. One need only follow the proof of Theorem 2.1. (2.6) and (2.3) imply that $L^\infty \cap Q(h_{\max})$ is a form core for H. Replacing (2.5) by

$$(\nabla - ia)(\psi\varphi) = (\nabla\psi)\varphi + \psi(\nabla - ia)\varphi \tag{2.5'}$$

we see that S is form core for H and noting that since $\varphi \in L^\infty$, $a(\varphi_\delta - \varphi) \to 0$ we have that C_0^∞ is a form core for H. ▨

Our proof of Theorem 2.3 depends on the fact that "in one dimension, magnetic vector potentials can be removed by a guage transformation" (this is just a physicist's expression for (2.8) below).

LEMMA 2.4. *Suppose that* $f \in L^p_{loc}(R^\nu)$, $\partial_1 f \in L^1_{loc}(R^\nu)$ ($\partial_1 =$ *distributional* $\partial/\partial x_1$), $g \in L^1_{loc}(R^\nu)$ *real-valued and* $\partial_1 g \in L^q_{loc}(R^\nu)$ *with* $p^{-1} + q^{-1} = 1$. *Then*

$$\partial_1(e^{ig}f) = e^{ig}(\partial_1 f) + i\, f\, e^{ig}(\partial_1 g). \tag{2.7}$$

Proof. If f and g are C^∞, this is trivial. The hypotheses are just such to allow one to prove distributional convergence as $\delta \to 0$ of both sides of (2.7) when f, g are replaced by f_δ, g_δ. ▨

LEMMA 2.5. *Let* $a_1 \in L^2_{loc}(R^\nu)$. *Then* $-i\partial_1 - a_1$ *is essentially self-adjoint on* $C_0^\infty(R^\nu)$ *and its closure* $-iD_1$ *obeys*

$$-iD_1 = e^{i\lambda_1}(-i\partial_1)e^{-i\lambda_1} \tag{2.8}$$

for a real-valued function λ_1 *in* L^2_{loc}. ((2.8) *is intended in the precise sense that the unitary operator* $U =$ *multiplication by* $e^{-i\lambda_1}$ *is a map of the domain of* $-iD_1$ *onto the domain of* $-i\partial_1$ and $(-i\partial_1)\, U\varphi = U(-iD_1)\varphi$). *Moreover, the domain of* D_1 *is*

$$\mathscr{D}_1 = \{\varphi \in L^2 \mid (\partial_1 - ia_1)\varphi(\text{dist. sense}) \in L^2\}. \tag{2.9}$$

Proof. Let $\lambda_1(x_1, \ldots, x_\nu) = \int_0^{x_1} a_1(y, x_2, \ldots, x_\nu)\, dy$ so that $\lambda_1 \in L^2_{loc}$ by the Schwarz inequality. Suppose that $f \in L^2$ and $(\partial_1 - ia_1)f = f$ (distributional sense). Then $\partial_1 f \in L^1_{loc}$, so we can apply Lemma 2.4 with $g = \lambda_1$. We see that $\partial_1(e^{-i\lambda_1}f) = e^{-i\lambda_1}(\partial_1 - ia_1)f = e^{-i\lambda_1}f$. Since $i\partial_1$ is essentially self-adjoint on C_0^∞ (by the Fourier transform), $e^{-i\lambda_1} f = 0$. Thus $f = 0$. Since $(\partial_1 - ia_1)f = \pm f$ has no distributional solutions, $-iD_1 \pm i$ has L^2 as its range, so it is self-adjoint. (2.8) follows from (2.7) and the fact that the above shows that the domain of D_1 is $\mathscr{D}_1$. ▨

Proof of Theorem 2.3. Let D_j be the operator $\partial_j - ia_j$ on $\mathscr{D}_j$ (given by (2.9)). Then $Q(h_{\max}) = D(V^{1/2}) \cap \bigcap_j \mathscr{D}_j$ and $h_{\max}(\varphi, \psi) = \sum_j (D_j\varphi, D_j\psi) + (V^{1/2}\varphi, V^{1/2}\psi)$; that is H is the form sum $(-D_1^2) + (-D_2^2) + \ldots + V$ in the precise sense used by Kato and Masuda [13] in their proof of the strong Trotter product formula so their result shows that

$$e^{-tH} = s\text{-}\lim_{m\to\infty} (e^{+tD_1^2/m} \ldots e^{+tD_\nu^2/m}\, e^{-tV/m})^m. \tag{2.10}$$

By (2.8), $e^{+sD_j^2} = e^{i\lambda_j}\, e^{s\partial_j^2}\, e^{-i\lambda_j}$ so that

$$|e^{sD_j^2}\varphi| \le e^{s\partial_j^2}|\varphi|. \tag{2.11}$$

(2.6) follows from (2.10), (2.11) and $|e^{-sV}| \le 1$. ▨

Remarks. 1. The proof of (2.11) shows that there is actually equality for a single j. But since $|e^{s\partial_j^2}\varphi|$ is in general only less than $e^{s\partial_j^2}|\varphi|$ (equality only if arg φ is constant), one does not get equality in (2.6) even if $V=0$.

2. For smooth a and V, (2.2) first appears in Simon [26] who was led to conjecture it on the basis of Kato's inequality in magnetic fields [11] and one of its applications [25]. Simon quotes a proof sketched to him by E. Nelson based on a Feynman-Kac formula and Ito stochastic integrals. The details of this proof can be found in ref. [30]. Subsequently, a proof (again for regular a and V) was found directly from Kato's inequality. This proof is based on an abstract result conjectured in ref. [26] and then proven independently by Hess et al [9] and Simon [29]. Kato [15] found an approximation argument to extend the proof to arbitrary $a \in L^2_{loc}$ if H is the operator associated to the *minimal* form. The above proof is the most "elementary" in many ways even for smooth a and V. It is closely related to the Feynman-Kac-Ito proof.

3. The inequality (2.6) for smooth a and V has been useful in analysing the operators H when $a \neq 0$, see Avron et al [1], [2], [3] and Combes et al [5].

We close this section with a consequence of Theorem 2.2 which will be useful in the next section:

COROLARY 2.6. *Suppose that* $V \geq 0$ *and that* (1.1) *holds. Then*

$$D(H) = \{\varphi \in Q(h_{\max}) | \widetilde{H}_{\text{dist}}\varphi \in L^2\}$$

where $\widetilde{H}_{\text{dist}}$ *is given by*

$$\widetilde{H}_{\text{dist}}\varphi = -\Delta\varphi + 2i\nabla\cdot(a\varphi) + (-i\nabla\cdot a + a^2 + V)\varphi. \tag{2.12}$$

Proof. By construction of H [10, 16]

$$D(H) = \{\varphi \in Q(h_{\max}) | \exists\, \eta \in L^2 \text{ with } h_{\max}(\varphi, \psi) = (\eta, \psi) = (\eta, \psi) \text{ for all } \psi \in Q(h_{\max})\}.$$

Since $\varphi \in Q(h_{\max})$, one can replace $\psi \in Q(h_{\max})$ by $\psi \in$ some form core of H. By Theorem 2.2, we can take $\psi \in C_0^\infty$. But for $\psi \in C_0^\infty$,

$$h_{\max}(\varphi, \psi) = \int (\widetilde{H}_{\text{dist}}\varphi)(x)\, \psi(x)\, dx.$$

§ 3. EQUALITY OF THE MINIMAL AND MAXIMAL SCHRÖDINGER OPERATORS

We now recover a result of Kato [11].

THEOREM 3.1. *Let* $V \geq 0$, $V \in L^2_{loc}$. *Then* $-\Delta + V$ *is essentially self-adjoint on* C_0^∞ *and its closure equals the form sum* H.

Proof. We just run through the proof of Theorems 2.1 and 2.2 using the extra information gained from Corollary 2.6. By (2.6)

$$L^\infty \cap D(H) \supset e^{-H}[L^2]$$

is an operator core for H. Let $\varphi \in D(H)$ and $\psi \in C_0^\infty$. Then, by (2.5) $\psi\varphi \in Q(h_{\max})$ and by (2.12)

$$\widetilde{H}_{\text{dist}}(\psi\varphi) = \psi(\widetilde{H}_{\text{dist}}\varphi) - 2\nabla\psi \cdot \nabla\varphi - \varphi\Delta\psi. \tag{3.1}$$

Since $Q(h_{\max}) \subset D(\nabla)$, (3.1) implies that $\widetilde{H}_{\text{dist}}(\psi\varphi) \in L^2$ so that $\psi\varphi \in D(H)$ and $\eta(\cdot/n)\ \varphi \to \varphi$ in H-graph norm for $\eta \in C_0^\infty$ with $\eta \equiv 1$ near $x = 0$. Thus

$$S' = \{\varphi \in L^\infty \cap D(H) \mid \text{supp}\ \varphi \text{ is compact}\}$$

is an operator core for H, Let $\varphi \in S'$ and $\varphi_\delta = j_\delta * \varphi$. Then $-\Delta\varphi + V\varphi \in L^2$ and $V\varphi \in L^2$ so $-\Delta\varphi \in L^2$. Thus $-\Delta\varphi_\delta \to -\Delta\varphi$ in L^2. Similarly since $\varphi \in L^\infty$, and supp φ is compact, $V\varphi_\delta \to V\varphi$ in L^2. ▨

The problem in extending the above proof to general a's with $\boldsymbol{a} \in L^4_{\text{loc}}$, and $\nabla \cdot \boldsymbol{a} \in L^2_{\text{loc}}$ is that at the point above where one concludes $-\Delta\varphi \in L^2$, one can only conclude that $-\Delta\varphi + 2ia\cdot\nabla\varphi \in L^2$ and $\nabla\varphi \in L^2$. To get that $-\Delta\varphi \in L^2$ and $a\cdot\nabla\varphi \to a\cdot\nabla\varphi$ seems to require a Sobolev estimate (the result one uses in the form case is that $\varphi \in \text{Ran}\ (e^{-H})$ implies $\varphi \in L^\infty$; the analog one would need is that $\varphi \in \text{Ran}\ (e^{-H})$ implies $(\nabla - ia)\varphi \in L^\infty$—this analog would yield the conjecture in section 1, but we don't see how to prove it). We begin by recovering the result of Simon [24] which used some subtle refinements of Kato's inequality:

THEOREM 3.2. *Let $V \geq 0$, $V \in L^2_{\text{loc}}$, $\boldsymbol{a} \in L^p_{\text{loc}}$ with $p \geq 4$, $p > \nu$ and $\nabla \cdot a \in L^2_{\text{loc}}$. Then $(-i\nabla - a)^2 + V$ is essentially self-adjoint on $C_0^\infty(R^\nu)$.*

Proof. By following the proof of Theorem 3.1 using

$$\widetilde{H}_{\text{dist}}(\psi\varphi) = \psi(\widetilde{H}_{\text{dist}}\varphi) - 2\nabla\psi\cdot(\nabla - ia)\varphi - \varphi\nabla\psi$$

in place of (3.1) and $Q(h_{\max}) \subset D(\nabla - ia)$, one sees that S' is an operator core for H. Let $\varphi \in S'$. Since $(a^2 + V + i\nabla\cdot a)\varphi \in L^2$, we have that $(-\nabla + 2ia\cdot\nabla)\varphi \in L^2$. Since $(\nabla - ia)\varphi \in L^2$ and $a\varphi \in L^p_{\text{loc}} \subset L^2_{\text{loc}}$, we have that $\nabla\varphi \in L^2$ (since φ has compact support). We want to improve this to $\nabla\varphi \in L^q$ with $q^{-1} = \dfrac{1}{2} - \dfrac{1}{p}$ by the following bootstrap argument: Suppose we know that $\nabla\varphi \in L^r$. Then $a\cdot\nabla\varphi \in L^s$ with $s^{-1} = r^{-1} + p^{-1}$ so that $-\Delta\varphi \in L^{\min(s,2)}$. If $\nu \geq 2$, $\partial_i\Delta^{-1}$ is an integral operator with a kernel dominated by $|x - y|^{-(\nu-1)}$ so by a Sobolev inequality (see eg. ref. [17]) $\partial_i\varphi \in L^t$ where $t^{-1} = [\min(s, 2)]^{-1} - \dfrac{1}{\nu}$. Thus, so long as

$r \leq q$, $\nabla\varphi \in L^r$ implies that $\nabla\varphi \in L^t$ with $t^{-1} = r^{-1} + (p^{-1} - \nu^{-1})$. Since $p > \nu$, in a finite number of steps we obtain $\nabla\varphi \in L^q$. (In $\nu = 1$ dimension, $\nabla(\Delta + 1)^{-1}$ is bounded from L^1 to an L^p, $p < \infty$). Thus if $\varphi_\delta = j_\delta * \varphi$, $a \cdot \nabla\varphi_\delta \to a \cdot \nabla\varphi$ in L^2 and since $\Delta\varphi \in L^1$, $\Delta\varphi_\delta \to \Delta\varphi$ in L^2. It follows that $\varphi_\delta \to \varphi$ in H-graph norm, so C_0^∞ is an operator core for H. ▨

By a slightly different argument, we obtain a result which is considerably better than the above result as allows singularities of a (the p involved is always less than 6 so for large ν it is much better; for $\nu = 4$, it is a strict improvement since $p = 4$ is allowed).

THEOREM 3.3. *Let $\nu \geq 4$ and $p = 6\nu/(\nu + 2)$. Suppose that $\boldsymbol{a} \in L^p_{\text{loc}}$, V, $\nabla \cdot \boldsymbol{a} \in L^{p/2}_{\text{loc}}$. Then $(-i\nabla - \boldsymbol{a})^2 + V$ is essentially self-adjoint on $C_0^\infty(R^\nu)$.*

Proof. As above, we need only show that for $\varphi \in S'$, we have that $\nabla\varphi \in L^r$ where $r^{-1} = \frac{1}{2} - p^{-1} = \left(\frac{p}{2}\right)^{-1} - \nu^{-1}$ (for the p above). Since $\varphi \in S'$ and $(\mathbf{a}^2 + V - i\nabla \cdot \boldsymbol{a}) \in L^{p/2}_{\text{loc}}$, we have that

$$-\Delta\varphi + 2i\nabla \cdot (a\varphi) \equiv \psi \in L^{p/2}. \tag{3.2}$$

Now

$$\partial_i\varphi = -\partial_i\Delta^{-1}\psi + 2i\Sigma\partial_i\Delta^{-1}\partial_j(a_j\varphi). \tag{3.3}$$

The first term in (3.3) is in L^r by (3.2) and a Sobolev estimate as in the proof of Theorem 3.2. Since $\partial_i\nabla^{-1}\partial_j$ is a bounded operator on L^q(each $q \neq 1, \infty$) [32] and $a_j\varphi \in L^p$ and so in L^r (since $r \geq p$ for $\nu \geq 4$ and $a_j\varphi$ has compact support) we have that the second term in (3.3) lies in L^r. Thus $\nabla\varphi \in L^r$ as desired. ▨

§4. CONTINUITY OF H IN a AND V

Let $H(a, V)$ be the operator associated to the form $h_{\max}$. The following is essentially an argument of Kato [15] but with several simplifications made possible by Theorem 2.2 and Lemma 2.5.

THEOREM 4.1. *Let $a_n, a \in L^2_{\text{loc}}$, $V_n, V \in L^1_{\text{loc}}$, $V_n, V \geq 0$ and suppose that $\boldsymbol{a}_n \to \boldsymbol{a}$ in L^2_{loc} and $V_n \to V$ in L^1_{loc}. Then $H_n \equiv H(a_n, V_n)$ converges to $H(a, V) \equiv H$ in the strong resolvent sense.*

Proof. Let $f \in L^\infty \cap L^2$ and let

$$\varphi_n = (H_n + i + 1)^{-1} f.$$

Then, clearly ($h_n = h_{\max}$ for (a_n, V_n))

$$\|\varphi_n\|_2 \leq \|f\|_2, \|(\nabla - ia_n)\varphi_n\|^2 + \|V_n^{1/2}\varphi_n\|^2 = h_n(\varphi_n, \varphi_n) = (H_n\varphi_n, \varphi_n) \leq \|f\|_2. \tag{4.1}$$

Let φ be a weak-limit point of φ_n. By (4.1) and the weak compactness of balls, we may pass to a subsequence and suppose that $\psi_n = (\nabla - \mathrm{i}a_n)\varphi_n \to \psi$ and $\eta_n \equiv$ $\equiv V_n^{1/2}\varphi_n \to \eta$. Let $g \in C_0^\infty$ so that $(\nabla - \mathrm{i}a_n)g \to (\nabla - \mathrm{i}a)g$ *strongly*. Thus

$$((\nabla - \mathrm{i}a)g, \varphi) = \lim((\nabla - \mathrm{i}a_n)g, \varphi_n) = -\lim(g, \psi_n) = -(g, \psi)$$

so, using Lemma 2.5, $\varphi \in D((\nabla - \mathrm{i}a))$ and

$$(\nabla - \mathrm{i}a_n)\varphi_n \overset{w}{\to} (\nabla - \mathrm{i}a)\varphi. \tag{4.2a}$$

Similarly $\varphi \in D(V^{1/2})$ (so $\varphi \in D(h)$) and

$$V_n^{1/2}\varphi_n \overset{w}{\to} V^{1/2}\varphi. \tag{4.2b}$$

By definition of φ_n, for $g \in C_0^\infty$

$$(g,f) = \sum_j ((\partial_j - \mathrm{i}(a_n)_j)g,\ (\partial_j - \mathrm{i}(a_n)_j)\varphi_n) + (V_n^{1/2}g,\ V_n^{1/2}\varphi) + (\mathrm{i} + 1)\,(g,\ \varphi_n).$$

Using (4.2) and the strong convergence for g, we conclude that

$$(g,f) = h(g, \varphi) + (\mathrm{i} + 1)\,(g, \varphi).$$

It follows that $\varphi \in D(H)$ and $(H + \mathrm{i} + 1)\varphi = f$. Thus, any weak limit point of φ_n is $(H + 1 + \mathrm{i})^{-1}f$. By the compactness of the unit ball we see that $(H_n + \mathrm{i} + 1)^{-1}$ converges weakly to $(H + \mathrm{i} + 1)^{-1}$. Similarly $(H_n - \mathrm{i} + 1)^{-1}$ converges weakly to $(H - \mathrm{i} + 1)^{-1}$ so by the resolvent formula, $\|(H_n + \mathrm{i} + 1)^{-1}f\|^2 = \frac{1}{2}\mathrm{i}(f, (H_n + \mathrm{i} + 1)^{-1} -$ $- (H_n - \mathrm{i} + 1)^{-1}f) \to \|(H + \mathrm{i} + 1)^{-1}f\|^2$ and thus the resolvents converge strongly. ▨

One consequence of Theorem 4.1 is the proof of the Feynman-Kac-Ito formula for e^{-tH}:

$$(f, \mathrm{e}^{-tH}g) = \int \mathrm{d}x\, f(x) \int \mathrm{d}\mu(b) g(x + b(t) \exp(-A(b)) \tag{4.3}$$

$$A(b) = \int_0^t V(x + b(s)\mathrm{d}s + \mathrm{i}\left[\frac{1}{2}\int (\nabla \cdot a)\,(x + b(s))\,\mathrm{d}s + \int a(b(s) + x)\cdot \mathrm{d}b\right]$$

for arbitrary $a \in L^2_{\mathrm{loc}}$. In (4.3), $\mathrm{d}\mu$ is Wiener measure (normalized Brownan motion) and $\mathrm{d}b$ is an Ito-stochastic integral. For (4.3) can be established for smooth a and V and the right side can be shown to be continuous in a, V (see e.g. ref. [30]). Theorem 4.1 implies that the left side is continuous.

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