Phase Transitions in the Theory of Lattice Gases

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Preface

Mathematical physics is the discipline of people who try to reach a deep understanding of physical phenomena by following the rigorous style and method of pure mathematics. It is a discipline that lies on the border between physics and mathematics. The purpose of mathematical physicists is not to calculate phenomena quantitatively but to understand them qualitatively. They work with theorems and proofs, not with numbers and computers. Their aim is to clarify with mathematical precision the meaning of the concepts upon which physical theories are built.

Freeman Dyson,
From Eros to Gaia, 2013 [220, Section 14.2]

The object of mathematical rigor is to sanction and legitimize the conquests of intuition.

Jacques Hadamard,
quoted by Borel *in Leçons sur la theorie des fonctions, 1898 [91]*

Classical Statistical Mechanics, as formulated by Boltzmann, Gibbs and Maxwell is one of the great pieces of physics of the 19th century. It seeks to explain how the underlying classical mechanics can lead to bulk matter described by a few parameters and how the laws of thermodynamics can come from a microscopic picture.

In its most naïve form, it leaves a great mystery. The fundamental rules of statistical mechanics for describing a system lead to functions which are real analytic in all bulk parameters. But bulk matter exhibits sharp transitions like the start of the melting of a solid or discontinuous change in the magnetization of a ferromagnet as the magnetic field is varied through 0. Sometimes these phase transitions are connected with a symmetry breaking, making their elucidation relevant to a wide swath of particle physics.

Eventually, physicists realized that one needed to consider the infinite volume limit where analyticity can fail. Of course, it is a big leap from saying discontinuities are possible and proving that they actually occur. The primary goal of this book is to explore the occurrence of phase transitions in a set of especially simple models where the finite volume objects have only a finite set of states because space is replaced by a lattice and the objects that live on the lattice have an elementary structure, most simply two values, either ± 1 or 0 and 1. Of course, to study continuous symmetries, we'll have to go beyond a finite state space and our "spins" will take values in spheres.

Much of the work (but far from all) described in this book is from the period 1960–1980, a kind of golden era of the study of lattice gases. It is surprising that in the over 40 years since those results, a comprehensive book hasn't appeared that discusses these results in spite of a number of graduate courses that includes at least some of this subject. That said, there are some excellent monographs that deal (sometimes, on certain topics, more extensively than we do) with some of the topics discussed here. I mention Friedli–Velenik [257], Georgii [286], Sinai [658] (Ruelle's classics [598, 604] and Israel [377] overlap much more with the topics in [644]).

I wrote an earlier book, *The Statistical Mechanics of Lattice Gases*, Princeton University Press, 1993 [644] that describes the framework of classical and quantum lattice gases and I have gotten many requests to write a book like the current one. That earlier work should be viewed as a companion to the present monograph. Chapter 1 of this book summarizes that framework with a focus on the results that we'll need later. We will refer to that other book, which we'll call SMLG, for some number of facts we need later.

Chapters 2 and 3 describe two general tools: correlation inequalities (which go back to Griffiths in 1967) and Lee-Yang theorems (going back to their 1952 paper). Examples of results that follow include the fact that the plus boundary condition state of ferromagnetic Ising models maximizes the expectation of a single spin. This implies that this expectation is the right hand derivative of the free energy in magnetic field and Chapter 3 yields the fact that for the Ising model in any dimension, the free energy is real analytic in the region $h \neq 0$.

In SMLG (see also Section 1.4 below), it is proven in great generality that one dimensional lattice gases with finite range interaction have no phase transitions (although, in Sections 2.4 and 7.3 of the current book we'll show certain very long range 1D interactions do have phase transitions). It is a general feature that most 2D systems with discrete symmetries have phase transitions and this is the subject of Chapter 4. Because we've included a potpourri of results and methods relevant to discrete models in that chapter, it is the third longest one only bested by the ones on correlation inequalities and on geometric methods.

The next two chapters discuss spin systems with continuous symmetry like the classical Heisenberg model. Here the lesson is that there are no phase transitions in 2D (Chapter 5) but there can be in three or more dimensions (Chapter 6). Chapter 8 will deal with something weaker than multiple phases which does sometimes happen in 2D with continuous symmetries. As mentioned, the bulk of the techniques discussed in this book predate 1980. On the other hand, Chapter 7 discusses mainly post 1980 work (while the FK representation and a precursor of random currents were introduced around 1970, their impressive usefulness was only realized after 1980). The two ways of rewriting Ising models will make accessible results like the absence of non-translation invariant equilibrium states in 2D and continuity of various important quantities at the critical point that don't seem to be within the earlier techniques. Chapter 9 summarizes the main results for the nearest neighbor Ising and classical Heisenberg models and lists some open problems.

My original expectation was to write a book of about 350 pages written in three or so years with complete proofs included of all the results on the models that I discussed. In June 2022, four years and 450 pages into the project, Michael Aizenman helped convince me to put in a full chapter on geometric methods, which is Chapter 7 (interestingly enough, several weeks layer Hugo Duminil-Copin won the Fields Medal for his part in that work) and my expectations ran into limitation of both space and time. Space in that I really wanted to keep the book in one volume and time in that I really hoped to finish no later than some time in 2024. So I made the decision that there could be a small number of exceptions to the complete proofs rule (I'd already decided to follow this strategy with the Main Theorem of Piogov–Sinai where I explained the framework and meaning of the result and gave lots of examples). In general, I skip proofs where those in the literature are fairly clear and where I didn't feel I could add much by providing all the technical details. Where possible, I explained at least some of the machinery and or other ideas behind the proofs. As far as plans of mice and men, I note that three years and 350 pages turned into six and half years and 650 pages!

As with SMLG, we will mainly discuss classical lattice systems but we will say something about the quantum case.

We will use some simple notions of topology (like compact spaces), measures and basic analysis and some simple ideas in complex analysis (and a few more advanced notions like the Vitali convergence theorem). We will also occasionally need some deep result (mainly Choquet theory) from the area of convex analysis. The reader needing more information can consult my books [648, 649, 650, 651, 652, 653] for more details; for some of the more advanced notions, I will give explicit references from them. Understandably, I refer to my own books for background mathematics but, of course, the reader should use their preferred references. We will use some simple, and very occasionally, more sophisticated graph theory throughout this book. While most of it is fairly intuitive, the reader desiring more can consult some of the monographs on the subject such as Bollobás [82], Diestel [169], Harary [348] or Wilson [755].

Books are demarcated by what they don't cover as much as what they do. For every aspect of this subject, the literature is vast, so I can't hope for my discussion to be complete, but I made a decision to try to deal with multiple aspects of many topics and to do so with considerable details, so the material I discuss is extensive. That said, in some parts SMLG was close to encyclopedic and I would not use that word to describe any part of the present book. For that reason, to avoid excessive length, I focus almost entirely on Ising models and their multidimensional classical analogs (rotor, classical Heisenberg, ...). I do have some material on quantum lattice systems but much less than on Ising (our discussion is limited to the "standard" models and does not, for example, discuss in detail discrete symmetry breaking in any example of quantum lattice systems). In addition, I do not discuss spin glasses (Bovier [102] is a book on the subject of spin glasses and random field Ising models) nor lattice gauge theories nor (except for one brief glimpse: see Section 2.8) quantum field theory applications. There is also almost no discussion of models with constraints (e.g. hard core lattice gases and 6- and 8- vertex models) although six-vertex models have a cameo in the discussion in Chapter 7 of first order transitions in Potts models in $2D$. My attitude was to always settle, if necessary, for spin $\frac{1}{2}$ (rather than more general a priori measure) with pair interactions. That said, dealing with more general a priori measures often is illuminating and very interesting, so I often do so. When a more general set of Hamiltonians than pair interactions is instructive or more natural, I do so, but I generally restrict to pair interactions. If the a priori measure is unbounded, there are issues of regularity of state as mentioned slightly in the Notes to Section 2.10, but we will normally not pay proper attention to this issue.

I made the decision not to discuss critical exponents extensively, in part, because the mathematically precise results are limited and those that exist should be accessible with the background of the material I do cover. Nor do I discuss some of the fascinating work, much of recent vintage on the behavior at the critical point - Hairer [339, Section 1] has a review of many of the high points of that work.

The presentation is also strongly impacted by my own taste and predilections. My ideal proof might have a short illuminating calculation but I try my best to avoid lengthy calculations (which is not to say that I always succeed) even if the alternative is to appeal to some big gun of abstract analysis. I am not a fan of expansions and so this presentation will not have some devices that a presentation by others might. That said about my predilections and intentions, I should mention that for example, some of our results in Section 2.9 (see Theorem 2.9.13) rely on a graphical expansion that goes back to an old paper of, er, Barry Simon! I'd emphasize though that this preference means this book is lacking any discussion of the important multiscale analysis pioneered by Fröhlich-Spencer although we are able to present the most important results originally obtained by this method by other means (see Section 7.3 and Chapter 8).

No doubt, currently active statistical physicists will say that with no discussion of the renormalization group nor of SLE ideas, I have skipped all the important stuff; but I made the decision to focus on mathematically precise results rather than subjects that are still under much serious development (although fascinating open problems remain) . So the book has modest scope – to expose the basics of a beautiful, fun subject in a way that I hope will be useful to both mathematicians and theoretical physicists.

Several software tools were useful in the preparation of this book: Winedt and MiKTeX for TeXing, Mathematica for some calculations and for some of the graphs, Inkscape for the production of the vector graphics and GIMP for raster graphics.

Some remarks on notations and personal quirks:

1. We will use some standard notation for mathematical objects such as

$\mathbb{C}$	The complex numbers
$\mathbb{C}_+$	The upper half plane, $\{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$
$\mathbb{N}$	The natural numbers, $0, 1, 2, \dots$
$\mathbb{Q}$	The rational numbers, $\{n/m \mid m \neq, m, n \in \mathbb{Z}0\}$
$\mathbb{R}$	The real numbers
$\mathbb{S}^k$	The k-dimensional sphere, i.e. the set of unit vectors (in the Euclidean norm) in $\mathbb{R}^\nu$ with $\nu = k + 1$
$\mathbb{Z}$	The integers, $0, \pm 1, \pm 2, \dots$
$\mathbb{Z}_+$	The strictly positive integers, $1, 2, 3, \dots$
$\mathbb{Z}^\nu$	The ν-dimensional integral lattice, i.e. the set of ν-tuples of integers

After this Preface is an extensive list of this and other notation. Author and subject indices as well as a long Reference List is at the end.

2. When speaking of real numbers, it is obvious what *non-negative* and *strictly positive* mean. If we just use *positive*, we normally mean strictly positive. For matrices, A, on $\mathbb{C}^\nu$ or $\mathbb{R}^\nu$ or operators on an inner product space, *non-negative* means each $\langle \varphi, A\varphi \rangle$ non-negative for all φ and *strictly positive* each $\langle \varphi, A\varphi \rangle$ strictly positive for all $\varphi \neq 0$. We say that A is *positive* is its is non-negative and not the 0 matrix or operator.

3. Even if we don't say so explicitly, unless we specifically say otherwise, any real or complex function on a space with a measure will be Baire measurable (see Simon [649, Section 4.3] for the definition). Similarly, unless we specifically say otherwise, all measures on $\mathbb{R}^\nu$ will be assumed to have finite moments.
4. Unless we specifically say otherwise, $\sum_{ij}$ means that we count each (unordered) pair only once (i.e. we don't separately count (ij) and (ji)). We only use such sums where the summand is symmetric under interchange of i and j.
5. We have chosen to alphabetize van Hove and von Neumann (and the like) under "v" even though one can make the case they should be under "H" and "N".
6. When we give the age that an author wrote a paper, we refer to the age the author reached in the year of publication of the paper.
7. I have been told that if a sentence ends with a set out equation, one should put a period in the equation. But I find dangling periods in equations confusing so I do not put them in although I am aware that makes me a sinner.
8. Two abbreviations that we will use are: *iff*=if and only if and *iidrv*= independent, identically distributed, random variable
9. Sometimes for emphasis, we will use boldface for vectors as in $\mathbf{e}$ and $\boldsymbol{\sigma}$, but sometimes when we hope there won't be confusion, we'll just use e and σ.
10. We have numbered most equations, even those we don't quote here because we want them to be quotable in citations by others.

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The subject of this book is important for our physical understanding and has deep and beautiful mathematics. It is surprising how simple and easy the concepts are. I hope the reader has as much fun reading this as I had writing it.

Barry Simon
Los Angeles and Jerusalem, January 2025