ON THE NUMBER OF BOUND STATES OF TWO BODY SCHRÖDINGER OPERATORS – A REVIEW

Barry Simon*

Given a measurable function V on R^n, consider the operator $-\Delta + V$ on $L^2(R^n)$. Under wide circumstances, this operator is known to be essentially self-adjoint on $C_0^\infty(R^n)$ (see [1] for a review) and under more general circumstances, it can be defined as a sum of quadratic forms [2, 3, 4]. Physically, it represents the Hamiltonian (energy) operator of the particles in nonrelativistic quantum mechanics after the center of mass motion has been removed. For this reason, $-\Delta + V$ is called a two-body Schrödinger operator. We will denote by $N(V)$ the dimension of the spectral projection for $-\Delta + V$ associated with $(-\infty, 0)$; physically the number of bound states. If V is spherically symmetric, we abuse notation and use V also as the symbol for the obvious function on $[0, \infty)$, i.e. the one with $V(x) = V(|x|)$. $n_\ell(V)$ for $\ell \geq 0$ will denote the number of bound states of the operator $-d^2/dx^2 + \ell(\ell+1)r^{-2} + V(r)$ on $L^2(0, \infty)$ (with the boundary condition $u(0) = 0$ if $\ell = 0$). Of course, for $n = 3$, one has the well-known partial wave expansion which yields

$$N(V) = \sum_{\ell=0}^{\infty} (2\ell+1)\, n_\ell(V) \; .$$

For $n > 3$, similar expansions exist but are associated with some nonnegative nonintegral ℓ. (For $n = 2$, $\ell = -1/2$ enters.) It is an interesting

*A. Sloan Fellow; research partially supported by USNSF under Grant GP-39048.

question to relate qualitative properties of V to N(V) and $n_\ell(V)$. Results of this kind go back to Jost-Pais [5] who proved that $N(V) = 0$ if $\int_0^\infty r|V(r)|dr < \infty$ and Bargmann [6] who proved the celebrated bound:

$$n_\ell(V) \leq (2\ell+1)^{-1} \int_0^\infty r|V(r)|dr \ .$$

Stimulated by Bargmann's paper, something of an industry has developed and we will review some of the results and methods that have emerged. Throughout we will be cavalier about self-adjointness questions, but we emphasize that these kind of details can easily be filled in by following e.g. [7].

§1. *The Methods and Bounds of Bargmann and Calegero*

As a common thread running through all work on the properties of N(V) is the min-max principle of Weyl, Fisher and Courant which takes the following general form:

THEOREM 1.1. *Let* A *be self-adjoint operator which is bounded below, and let* Q(A) *be its quadratic form domain. Let*

$$\mu_n(A) \equiv \max_{\psi_1,\dots,\psi_{n-1}} \left[\min_{\substack{\phi \in [\psi_1,\dots,\psi_{n-1}]^\perp;\ \|\phi\|=1 \\ \phi \in Q(A)}} (\phi, A\phi) \right] .$$

Then either:

(a) $\mu_n(A)$ *is the* nth *eigenvalue from bottom of the spectrum of* A *counting multiplicity and* A *has purely discrete spectrum in* $(-\infty, \mu_n(A))$

or

(b) μ_n *is the bottom of the essential spectrum of* A. *If* (b) *holds, then* A *has at most* n−1 *eigenvalue in* $(-\infty, \mu_n)$ *and* $\mu_n(A) = \mu_{n+1}(A) = \cdots$

For a proof and further discussion, see [7]. A major corollary of the min-max principle is the following:

COROLLARY 1.2 (Comparison Theorem). *Let* A *and* B *be self-adjoint operators with* $A \leq B$ *in the sense that* $Q(A) \supset Q(B)$ *and* $(\psi, A\psi) \leq (\psi, B\psi)$ *for all* $\psi \in Q(A)$. *Then* $\mu_n(A) \leq \mu_n(B)$ *for all* n *and, in particular,*

$$\dim P_{(-\infty,a)}(A) \leq \dim P_{(-\infty,a)}(B)$$

for all a.

The proof is immediate. Since one has the following (see e.g. [7]):

THEOREM 1.3. *Let* $V \in L^{n/2}(R^n) + L^\infty_\varepsilon(R^n)$ [*for any* ε, $V = V_1^{(\varepsilon)} + V_2^{(\varepsilon)}$ *with* $V_1^{(\varepsilon)} \in L^{n/2}$ *and* $\|V_2^{(\varepsilon)}\|_\infty < \varepsilon$]. *Then* $\sigma_{ess}(-\Delta + V) = [0, \infty)$ $(n \geq 3)$.

One can apply Corollary 1.2 to Schrödinger operators:

THEOREM 1.4.

(a) *Let* $V \in L^{n/2} + L^\infty_\varepsilon$ *and let* V_- *be its negative part, i.e.* $V_- = \max(-V, 0)$. *Then*

$$N(V) \leq N(-V_-)$$

$$n_\ell(V) \leq n_\ell(-V_-) \quad \textit{if } V \textit{ is central} .$$

(b) *Let* $V, W \in L^{n/2} + L^\infty_\varepsilon$ *with* $V \leq W$ *pointwise. Then*

$$N(W) \leq N(V)$$

$$n_\ell(W) \leq n_\ell(V) \quad \textit{if } V \textit{ and } W \textit{ are central.}$$

In addition to this result, the main input used by Bargmann is the following:

THEOREM 1.5. *Let* $V \in C_0^\infty(R^n)$ *be centrally symmetric. Fix* $\ell \geq 0$. *Let* u *be a solution of*

$$-u'' + \ell(\ell+1)r^{-2}u + Vu = 0; \quad u(0) = 0 \ .$$

Then $n_\ell(V)$ *is the number of zeroes of* u *on* $(0, \infty)$.

REMARKS.

1. This result is true under much greater generality than $V \in C_0^\infty$. However, for our purposes, this is enough. For a bound of the form $n_\ell(V) \leq (2\ell+1)^{-1} \int_0^\infty r|V(r)|\,dr$, once proven for $V \in C_0^\infty$ extends to all V by a simple limiting argument.

2. Theorem 1.5 follows by a simple min-max principle which exploits the Sturm comparison theorem; see [7]. Alternatively Theorem 1.5 can be proven by combining Levinson's theorem [8] with the method of variable phases [9]; see [7, 9].

Martin [10] has remarked on an interesting "local" comparison theorem:

THEOREM 1.6 (Martin's local comparison theorem). *Let* u *be any solution of* $-u'' + \ell(\ell+1)r^{-2}u + Vu = 0$. *Suppose that* $V \geq W$ *on* (a,b) *and that* u *has* n *zeroes in* (a,b). *Then* $n_\ell(W) \geq n-1$.

REMARKS.

1. The proof is simple. By a Sturm comparison theorem, any other solution of $-u'' + \ell(\ell+1)r^{-2}u + Vu = 0$ has at least $n-1$ zeroes in (a,b) and therefore by another Sturm argument, any solution of $-u'' + \ell(\ell+1)r^{-2} + Wu = 0$ has at least $n-1$ zeroes in (a,b).

2. As a typical application of this result, we note that so long as W is strictly negative in some interval (a,b), $\varliminf_{\lambda\to\infty} \lambda^{-1/2} n_\ell(\lambda W) > 0$, for compare with a square wall.

3. One can use Martin's principle to prove [52]: If $V(r)$ is a continuous function on $(0,\infty)$ and $\ell_{max}(\lambda)$ is the largest angular momentum for which $-\Delta + \lambda V$ has bound states on R^3, then

$$\lim_{\lambda\to\infty} \ell_{max}(\lambda V)/\lambda^{1/2} = [-\min(r^2 V)]^{1/2} .$$

Calegero [11] invented a very elegant method for exploiting Theorem 1.5. In case $\ell = 0$, it goes as follows: Let u solve $-u'' + Vu = 0$. Define $a(r)$ by

$$(a(r)+r)u'(r) = u(r) . \tag{1}$$

Then, $a(r)$ obeys the Riccati equation

$$a'(r) = -V(r)[r+a(r)]^2 . \tag{2}$$

Now, by (1), $a(r) \to 0$ as $r \to 0$, in fact, $a(r) = o(r)$. Moreover, if $V \leq 0$, (2) says that a is monotone increasing. A simple geometric argument (Figure 1) shows that the number of zeroes of u is identical to the number of "poles" of a. The idea is to introduce an auxiliary function which is a function of $a(r)$, use (2) to get a differential inequality which can be integrated.

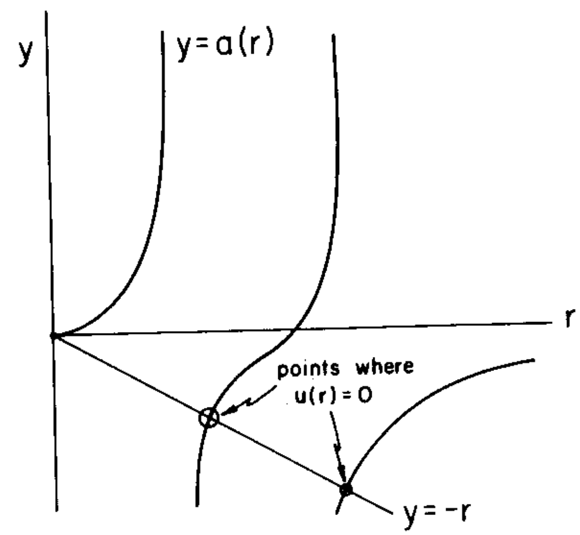

Fig. 1

EXAMPLE 1. Let $b(r) = r^{-1}a(r)$. Then

$$b'(r) = -rV(r)[1+b(r)]^2 - r^{-1}b(r) . \tag{3}$$

If $n_0(V) = n$, then $b(r)$ has poles at $p_1, \cdots, p_n$ and zeroes at $z_1 = 0$, $z_2, \cdots, z_n$ (and perhaps at a z_{n+1}) with $z_1 < p_1 < z_2 < p_2 < \cdots < p_n$. On

(z_i, p_i), b is positive, so by (3)

$$b'(r) \leq r|V(r)|\,(b(r)+1)^2$$

so that

$$1 = \int_{z_i}^{p_i} -\frac{d}{dr}\left(\frac{1}{b+1}\right) dr \leq \int_{z_i}^{p_i} r|V(r)|\, dr\ .$$

Summing over i, we get the $\ell = 0$ Bargmann bound [6]:

$$n_0(V) \leq \int_0^\infty r|V(r)|\, dr\ .$$

EXAMPLE 2. Suppose that $V \leq 0$ is smooth and $V'(r) \geq 0$. Define $\nu(r)$ by

$$\tan \nu(r) = (-V(r))^{1/2}\,(a(r)+r)\ .$$

Then ν obeys:

$$\nu'(r) = |V(r)|^{1/2} - \tfrac{1}{2}\,(V'(r)/|V(r)|)\,(\cos^2 \nu(r) \tan \nu(r))\ .$$

Now, if $n_0(V) = n$, $(a(r)+r)$ has zeroes $z_1 = 0,\ z_2, \cdots, z_n$ and poles, $p_1, \cdots, p_n$ with $z_1 < p_1 < z_2 < \cdots < p_n$. In (z_i, p_i), $\tan \nu > 0$, so $\nu'(r) \leq |V(r)|^{1/2}$. Thus

$$\frac{\pi}{2} = \int_{z_i}^{p_i} \nu'(r)\, dr \leq \int_{z_i}^{p_i} |V(r)|^{1/2}\, dr\ .$$

Summing over i, we get Calegero's bound [11]:

$$n_0(V) \leq \frac{2}{\pi}\int_0^\infty |V(r)|^{1/2}\ .$$

REMARKS.

1. For $\ell \neq 0$, one defines a_ℓ by:

$$u'(r)\,[r^{\ell+1} + a_\ell(r)\, r^{-\ell}] = u(r)\,[(\ell+1)\, r^{\ell} - \ell a_\ell(r)\, r^{-\ell-1}]\ .$$

Then a_ℓ obeys the Riccatti equation:

$$a_\ell' = -(2\ell+1)^{-1} V(r)\,[r^{2\ell+1} + a_\ell(r)]^2\ .$$

Bargmann's bound is proven by using $b_\ell = r^{-2\ell-1} a_\ell$; see [9], pp. 182-184.

2. There is a connection between a(r) and the scattering length; in particular, $\lim_{r\to\infty} a(r)$ is the scattering length [11, 12].

We close this section by stating formally some of the bounds on $n_\ell(V)$:

THEOREM 1.7 (Bargmann [6])

$$(2\ell+1)\, n_\ell(V) \leq \int_0^\infty r|V_-(r)|\, dr\ .$$

Calegero [9, 11, 13] has proven a variety of bounds on $n_\ell(V)$ among which we mention:

THEOREM 1.8 (Calegero [11], Cohn [14]). *Suppose that* V *is negative and monotone increasing. Then*:

$$n_\ell(V) \leq \frac{2}{\pi}\int_0^\infty |V(r)|^{1/2}\, dr\ .$$

THEOREM 1.9 (Calegero [11]). *Let* $I_\rho = \int_0^\infty dr\, r^\rho |V(r)|$. *Then*:

$$n_\ell(V) \leq \frac{1}{2} + \frac{2}{\pi}\,(I_0 I_2)^{1/2}$$

$$n_\ell(V) \leq 1 + \frac{2}{\pi}\,(I_0 I_2 - I_1^2)^{1/2}\ .$$

Glaser et al., have proven:

THEOREM 1.10 ([15]). *For* $1 < p < \infty$:

$$(2\ell+1)^{2p-1} n_\ell(V) \le \frac{(p-1)^{p-1}}{p^p} \frac{\Gamma(2p)}{[\Gamma(p)]^2} \int_0^\infty r^{2p-1} |V(r)|^p \, dr \, .$$

Notice that as $p \downarrow 1$, this bound goes over to Bargmann's bound.

§2. *The Method of Birman and Schwinger*

In 1961, the Russian mathematician, M. Birman, and the American physicist, J. Schwinger, independently published almost identical proofs of the following theorem:

THEOREM 2.1 (Birman [16] - Schwinger [17]). *On* R^3:

$$N(V) \le \frac{1}{(4\pi)^2} \int dx\, dy\, |x-y|^{-2} |V(x)| \, |V(y)| \, . \tag{4}$$

The first step in the proof is to note that:

LEMMA 2.2. E *is an eigenvalue of* $-\Delta+\lambda V$ *with* $V \le 0$, $\lambda > 0$, $E < 0$, *if and only if* λ^{-1} *is an eigenvalue of* $K_E = |V|^{1/2}(-\Delta-E)^{-1}|V|^{1/2}$ *and the multiplicities are equal.*

REMARK. *Formally* $(-\Delta+\lambda V)\psi = E\psi$ if and only if $K_E\phi = \lambda^{-1}\phi$ where $\phi = |V|^{-1/2}V\psi$. For a careful proof, see [3].

The second step is the simple but elegant:

LEMMA 2.3. *The number of eigenvalues of* $-\Delta + V$ *less than* $E < 0$, *is the number of* $\lambda \in (0,1]$ *for which* E *is an eigenvalue of* $-\Delta+\lambda V$.

PROOF. Let $\mu_n(\lambda)$ be given by the min-max principle for $-\Delta+\lambda V$. Then $\mu_n(\lambda) \le 0$ for all λ and decreases as λ increases. Moreover, $\mu_n(\lambda) \uparrow 0$ as $\lambda \downarrow 0$. Thus the number of $\mu_n(1)$ less than E is identical to the number of μ_n for which $\mu_n(\lambda) = E$ for some $\lambda \in (0,1]$.

The two lemmas immediately imply the following basic "Birman-Schwinger" principle.

THEOREM 2.4. *Let* $V \le 0$, $E < 0$. *The number of eigenvalues of* $-\Delta+V$ *in* $(-\infty, E]$ *is the same as the number of eigenvalues of* $K_E = |V|^{1/2}(-\Delta-E)^{-1}|V|^{1/2}$ *in* $[1,\infty)$ *(counting multiplicity).*

PROOF OF THEOREM 2.1. The number of eigenvalues of K_E larger than 1 is clearly dominated by the sum of the squares of the eigenvalues which equals $\mathrm{Tr}(K_E^2)$. Since $(-\Delta-E)^{-1}$ has an integral Kernel $(4\pi)^{-1}|x-y|^{-1} \exp(-\sqrt{-E}|x-y|)$, we see that:

$$N_E(V) \le \frac{1}{(4\pi)^2} \int dx\, dy\, |x-y|^{-2} |V(x)| \, |V(y)| e^{-2k|x-y|} \tag{5}$$

where $E = -k^2$ and $N_E(V)$ is the number of eigenvalues in $(-\infty, E]$; (5) is also a bound of Birman-Schwinger. Taking $E \uparrow 0$, (4) results.

REMARKS.

1. By a classical inequality of Sobolev (see e.g. [1]), $\int dx\, dy\, |x-y|^{-2} |V(x)| \, |V(y)| \le C\|V\|_{3/2}^2$ where $\|V\|_p = (\int |V|^p dx)^{1/2}$. Thus

$$N(V) \le C\|V\|_{3/2}^2 \tag{6}$$

We return to this in Section 3C below.

2. See Ghirardi-Rimini [18], Fonda-Ghirardi [19] and Konno-Kuroda [20] for modification of the Birman-Schwinger bound.

§3. *Further Applications of the Birman-Schwinger Principle*

A. *Recovery of Bargmann's Bound* [16, 17]. Let h_0 be the operator $-\frac{d^2}{dr^2} + r^{-2}\ell(\ell+1)$ on $L^2(0,\infty)$ (with boundary condition $u(0) = 0$ if $\ell = 0$).

Then, as in the proof of Theorem 2.1,

$$n_\ell(V) \le \lim_{E\uparrow 0} \operatorname{Tr}(|V_-|^{1/2}(h_0-E)^{-1}|V_-|^{1/2})$$
$$= \operatorname{Tr}(|V_-|^{1/2}h_0^{-1}|V_-|^{1/2}) \ . \qquad (7)$$

Now h_0^{-1} is an (unbounded) integral operator with kernel $(2\ell+1)^{-1}[\min(x,y)]^{\ell+1}[\max(x,y)]^{-\ell}$, so the trace in (7) is $\int_0^\infty (2\ell+1)^{-1}x|V_-(x)$ which gives Bargmann's bound.

B. *Low-Dimensions*. Students in a first quantum mechanics course, learn that if V is a negative spherical square wall in R^n, then $-\Delta+\lambda V$ has bound states for all positive λ if $n=1$ and has no bound states for small λ if $n=3$. What about $n=2$? There is some confusion about this question in the published and preprint literature – we first learned the correct answer from M. Kac. The Birman-Schwinger principle is ideal for studying this question:

THEOREM 3.1. *Consider* $-\Delta+\lambda V$ *on* $L^2(R^n)$ *for* $n=1$ *or* 2. *Suppose* $V\le 0$ *and* $V\in L^p+L^q$ *with* $1<q<\infty$ *and* $p>1$ *if* $n=2$, $p=1$ *if* $n=1$ [*in this case* $-\Delta+\lambda V$ *can be defined as a sum of forms,* K_E *is a bounded, compact operator and* $\sigma_{ess}(-\Delta+\lambda V)=[0,\infty)$]. *Then* $N(\lambda V)>0$ *for all* $\lambda>0$.

PROOF. By the Birman-Schwinger principle, we must show that for any $\lambda>0$, there is $E<0$, so that K_E has an eigenvalue larger than λ^{-1}. Since K_E is positive and compact, it clearly suffices to prove that $\lim_{E\uparrow 0}\|K_E\|=\infty$. This follows if we prove that $\lim_{E\uparrow 0}(\eta,K_E\eta)=\infty$ for some fixed $\eta\in L^2$. Let η be the characteristic function of some bounded set on which V obeys $a<V(x)<a^{-1}$ for some $a>0$. Let $f=|V|^{1/2}\eta\in L^1\cap L^2$, $f\ge 0$ so $\hat f$ is nonvanishing near 0. Thus

$$\lim_{E\uparrow 0}(\eta,K_E\eta) = \lim_{E\uparrow 0}\int |\hat f(p)|^2(p^2-E)^{-1}d^np$$
$$= \int |\hat f(p)|^2 p^{-2}d^np$$

diverges if $n=1$ or 2.

REMARKS.

1. As we shall see in Section 5, if $V\in L^{n/2}(R^n)$ and $n\ge 3$, then $N(\lambda V)=0$ if λ is small.

2. If $V\in C_0^\infty(R^n)$, $n=1,2$ with $V\le 0$, then $N(\lambda V)=1$ for λ small by the following argument: Place Neumann boundary conditions on a sphere, S, containing suppV. This can only increase $\dim P_{(-\infty,0)}$ (see, e.g. [7]). But S breaks R^n into a ball B and an exterior E. $-\Delta_N$ is positive on $L^2(E)$ and since $-\Delta_N$ on $L^2(B)$ has an isolated simple eigenvalue at 0, $-\Delta_N+\lambda V$ can have at most one negative eigenvalue for λ small.

3. Let $\|-\|$ be a translation invariant norm on $C_0^\infty(R^n)$ $(n=1,2)$. Then, given any m, ε, we can find $V\in C_0^\infty$ with $\|V\|<\varepsilon$ and $N(V)\ge m$. For pick any $f\in C_0^\infty$ with $f\le 0$ and $\|f\|\le\varepsilon/m$. Since $-\Delta+f$ has at least one eigenvalue, $-\Delta+V$ will have at least m if V is the sum of m translates of f all sufficiently far from one another. Thus, there is no bound if $n=1,2$ of the form $N(V)\le$ function of a translation invariant norm. The situation is very different if $n\ge 3$ (see Section 3C).

4. If $n=1$, we have the bound

$$N(V) \le 2+\int_{-\infty}^{\infty}|x|\,|V_-(x)|dx \qquad (8)$$

for, let $-\Delta_D$ be the operator with Dirichlet boundary conditions at $x=0$. Then, by Bargmann's bounds in each half-space, $-\Delta_D+V$ has at most $\int_{-\infty}^{\infty}|x|\,|V_-(x)|dx$ eigenvalues in $(-\infty,0)$. Thus, since $-\Delta_D+V$ and $-\Delta+V$ are self-adjoint extensions of a common operator with deficiency indices (2,2), (8) follows.

5. Theorem 3.1 illustrates that Calegero's bound, Theorem 1.8 does not hold for all V.

6. It is false that if V is negative somewhere on $\mathbb{R}$, then $-d^2/dx^2+V$ has negative bound states. For example, if $V(x) = -1$ for $|x| < \pi/4$, $V(x) = 4$ for $\pi/4 \leq |x| \leq \pi/4+2$; then $-d^2/dx^2+V$ has no bound states. One can show [21] if $n = 1,2$ and $\int|x|^2|V(x)|dx < \infty$ if $n = 1$ ($\int(1+|x|^2)^\delta|V(x)|d^2x < \infty$ and $\int|V(x)|^{1+\delta}dx < \infty$ if $n = 2$), then $-\Delta+\lambda V$ has a bound state for all small λ if and only if $\int V(x)d^nx \leq 0$.

C. *Quasi-Classical and Almost Quasi-Classical Bounds.* The basic principle of the quasi-classical limit to quantum mechanics is that each bound state requires a volume h^3 in phase space. Thus, in units with $h = 1 = 2m$, on $\mathbb{R}^n$

$$N_{c\ell}(V) = (2\pi)^{-n}\tau_n \int (V_-(x))^{n/2}\,dx \tag{9}$$

where τ_n is the volume of the unit ball in $\mathbb{R}^n$. $\tau_n\int(V_-(x))^{n/2}dx$ is the volume of phase space $\langle p,x\rangle$ where $p^2+V(x) < 0$. As we discuss in Section 4, there is a sense in which $N(V)$ and $N_{c\ell}(V)$ become equal when V is large. There is a general conjecture which has been made by Glaser et al., [15], Simon [22] and E. Lieb [23]:

CONJECTURE. *Let* $n \geq 3$. *There is a constant* C_n *so that*

$$N(V) \leq C_n N_{c\ell}(V) \tag{10}$$

for all $V \in L^{n/2}$.

REMARKS.

1. In fact, Glaser et al., [15] suggest that $C_3 = 8/\sqrt{3}$ and prove (10) for $n = 3$ whenever $N(V) \leq 2$.
2. For $n = 1,2$, (10) fails by our remarks in 3B.
3. As we shall see, (10) holds as $N(V) \to \infty$, in the sense that $\lim_{\lambda\to\infty} N(\lambda V)/N_{c\ell}(\lambda V) = 1$.

4. In general, for suitable C_n, (10) holds with $N(V) = 1$ (see Section 5) and then, by an argument of Glaser et al., [15], for $N(V) \leq 2$.
5. Simon [22] has proven (10) is equivalent to a natural conjecture involving "weak trace ideals."
6. By a limiting argument, (10) need only be proven for $V \in C_0^\infty$.

(10) says $N(V) \leq C_n' \|V_-\|_{n/2}^{n/2}$ where $\|V_-\|_p^p = \int|V_-|^p\,dx$. Using the Birman-Schwinger principle and interpolation theory for weak trace ideals (developed in [22]), one can prove:

THEOREM 3.2 ([22]). *Let* $n \geq 3$, $\varepsilon > 0$. *Then, there exists a constant* D_n, *so that*

$$N(V) \leq D_{n,\varepsilon}(\|V\|_{\frac{1}{2}n+\varepsilon} + \|V\|_{\frac{1}{2}n-\varepsilon})^{n/2} \;. \tag{11}$$

REMARK. As we discuss in Section 4, this estimate, unlike those of Bargmann and Birman-Schwinger, has the proper large coupling constant behavior.

D. *The Lieb-Thirring Bound.* In their beautiful paper on the stability of matter, Lieb and Thirring use the Birman-Schwinger bound (5), to prove:

THEOREM 3.3 ([24]). *Let* $n = 3$. *Let* $V \in L^{5/2}(\mathbb{R}^3)$. *Let* $e_1(V) \leq e_2(V) \leq \cdots$ *be the negative eigenvalues of* $-\Delta+V$. *Then*

$$\sum_i |e_i(V)| \leq \frac{4}{15\pi}\int V^{5/2}(x)\,d^3x \;. \tag{12}$$

SKETCH OF PROOF. By (5) and the comparison theorem:

$$\begin{aligned} N_E(V) &\leq N_{E/2}((V-E/2)_-) \\ &\leq (4\pi)^{-2}\int (V-E/2)_-(x)(V-E/2)_-(y)\,|x-y|^{-2}\,e^{-\sqrt{2E}|x-y|} \\ &\leq (4\pi\sqrt{2E})^{-1}\int |(V-\tfrac{1}{2}E)_-(x)|^2\,dx \end{aligned}$$

by Young's inequality. Now:

$$\sum_i |e_i(V)| = \int_{-\infty}^{0} |E|\, dN_E$$

$$= \int_{-\infty}^{0} N_E\, dE$$

$$\leq (4\pi\sqrt{2})^{-1} \int dx \int_0^{\infty} \alpha^{-1/2} |(V+1/2\alpha)\_(x)|^2\, d\alpha$$

which yields (12).

REMARKS.

1. Similar results hold for sums $\sum |e_i(V)|^{\nu}$ for other ν and for n different from 3, see [25].

2. This theorem is especially interesting since the quasi-classical value for $\sum |e_i(V)|$ is $(15\pi^2)^{-1} \int |V\_(x)|^{5/2}\, d^3x$.

§4. *Large Coupling Constant: The Quasi-Classical Limit*

The number of bound states of $-\Delta + \lambda V$ is the same as that for $-\lambda^{-1}\Delta + V$, so that large λ is the same as small $\hbar$. Thus one expects the quasi-classical approximation to be good. Martin [26] has proven:

THEOREM 4.1 ([26]). *If* V *is a Hölder continuous function of compact support, then,*

$$\lim_{\lambda\to\infty} N(\lambda V)/N_{c\ell}(\lambda V) = 1 \ . \tag{13}$$

Martin uses the method of Dirichlet-Neumann bracketing [27, 7]. Independently of Martin, Tamura [28] proved (13) for a wider class of V.

Since $N_{c\ell}(\lambda V) = \lambda^{n/2} N_{c\ell}(V)$, (13) gives the large λ behavior of $N(\lambda V)$. It shows that for λ large, the Birman-Schwinger bound which $\sim c\lambda^2$ is not good. The advantage of the bound (11) is that it gives the proper large λ behavior. Using (11), one can prove:

THEOREM 4.2 ([22]). *Let* $n \geq 3$. *Let* $V \in L^{n+\varepsilon} \cap L^{n-\varepsilon}$. *Then* (13) *holds.*

§5. *Small Coupling Constant; When is* $N = 0$?

The question of when $N = 0$ was first asked by Jost and Pais [5]; more recently, it has been deeply studied by Glaser, Martin, Grösse and Thirring [15]. In the one-dimensional case, special interest is connected with this problem because of the following remark of Glaser et al., [15]:

THEOREM 5.1. *Let* $a(V) = \int_0^{\infty} f(x)|V(x)|^p$ *for* $f \geq 0$. *If* $a(V) < 1$ *implies that* $0 = N(V)$, *then, for any* V,

$$N(V) \leq a(V) \ .$$

PROOF. Let $n = N(V)$. Let u be the zero energy solution of the Schrodinger equation. Let $x_0 = 0$, $x_1, \cdots, x_n$ be its zeroes. Let $V_i = V\chi_i$ where χ_i is the characteristic function of (x_{i-1}, x_i). Then $u_i \equiv u\chi_i \in Q(-d^2/dx^2 + V_i)$, $(u_i, (-d^2/dx^2 + V_i)u_i) = 0$, so $N(V_i) \geq 1$. Thus $a(V_i) \geq 1$, so $a(V) \geq \sum_i a(V_i) \geq n$.

Thus the Jost-Pais result [5] implies Bargmann's bound [6]. The bounds of Glaser et al., [15] quoted as Theorem 1.10 are proven by using Theorem 5.1 and the results below.

THEOREM 5.2. *If* $\| |V|^{1/2}(-\Delta)^{-1}|V|^{1/2} \| < 1$, *then* $N(V) = 0$.

PROOF. This follows from the Birman-Schwinger principle, but also by an alternate proof [15]: If $\| |V|^{1/2}(-\Delta)^{-1}|V|^{1/2} \| < 1$, then $|V| \leq -\Delta$ so $-\Delta + V \geq 0$ which implies $N(V) = 0$.

THEOREM 5.3. *Let* $n \geq 3$ *and* $p \geq n/2$. *Then, there is a constant* $C_{n,p}$ *so that if*

$$C_{n,p} \int |x|^{2p-n} |V\_(x)|^p\, d^n x < 1 \ , \tag{14}$$

then $N(V) = 0$.

PROOF. It is well known that $r^{-2} \le C_n p^2$ if $n \ge 3$. Thus $r^{-1}p^{-2}r^{-1}$ is bounded from L^2 to L^2. By Sobolev's inequality p^{-2} is bounded from L^{q_0} to $L^{q_0'}$ where $q_0^{-1} = 1/2 + 1/n$ and $p' = (1-p^{-1})^{-1}$. Thus, by the Stein interpolation theorem, $r^{-\alpha}p^{-2}r^{-\alpha}$ is bounded from L^{q_α} to $L^{q_\alpha'}$ where $q_\alpha^{-1} = 1/2 + n^{-1}(1-\alpha)$. As a result, if $r^{-\alpha}V^{1/2} \in L^{n/(1-\alpha)}$, $V^{1/2}(-\Delta)^{-1}V^{1/2}$ is bounded and if $\|r^{-\alpha}V^{1/2}\|_{n/1-\alpha} \le D_{n,\alpha}$, then $N(V) = 0$. This is just (14).

REMARKS.

1. For $n = 3$, this result is independently due to Glaser et al., [15].
2. That $r^{-\alpha}p^{-2}r^{-\alpha}$ is bounded from L^{q_α} to $L^{q_\alpha'}$ is a result of Strichartz [29].
3. One can use the same argument if $n = 1$, $p \ge 1$ if we deal with the operator $-d^2/dx^2$ on $L^2(0,\infty)$ with boundary condition $u(0) = 0$: one interpolates between $r^{-1}p^{-2}r^{-1}$ bounded from L^2 to L^2 and that $r^{-1/2}p^{-2}r^{-1/2}$ is bounded from L^1 to L^∞: The later follows from the explicit kernel $x^{-1/2}\min(x,y)y^{-1/2}$ of the integral operator $r^{-1/2}p^{-2}r^{-1/2}$.

The above theorem leaves open the question of the best value of the constant $C_{n,p}$. Glaser et al., [15] proves:

THEOREM 5.4. $C_{3,p} = (p-1)^{p-1}\Gamma(2p)/4\pi p^p[\Gamma(p)]^2$. *For* $n = 1$ *with Dirichlet boundary conditions*:

$$\tilde{C}_{1,p} = (p-1)^{p-1}\Gamma(2p)/p^p[\Gamma(p)]^2 .$$

In the above, we have considered the question of when $N(V)$ is zero. If $N(V)$ is zero, there is some possibility that $-\Delta$ and $-\Delta + V$ are unitarily equivalent. This problem is discussed by Kato [30] (small coupling) and Lavine [31] (repulsive interactions).

§6. *Other Results*

We want to briefly describe some further results about $N(V)$ giving references to additional literature. The really interesting open questions involve n-body problems. There are no known general bounds on the number of eigenvalues below the continuum limit: As we shall see (C and D below), there are various pathologies which make obtaining such bounds difficult.

A. *How does* $N_E(V)$ *approach infinity if* $N(V) = \infty$? We have already seen that if $V \in L^{n/2}$ $(n \ge 3)$, then $N(V) < \infty$. As a complimentary result, one has the following:

THEOREM 6.1. *If* $V(r) \le -C r^{-2+\varepsilon}$ *for* $|r| > R_0$ *for some* $C, \varepsilon > 0$, *then* $\mu_n(-\Delta+V) < 0$ *for all* n. *In particular*, $N(V) = \infty$.

REMARKS.

1. Of course, if $V \ge -C r^{-2-\varepsilon}$, then $N(V) < \infty$ by the $L^{n/2}$-bound.
2. One can actually show $V \le -(c_n+\varepsilon)r^{-2}$ implies $N(V) = \infty$ and $V \ge -(c_n-\varepsilon)r^{-2}$ implies $N(V) < \infty$ for suitable c_n. $c_n = 1/4$ if $n = 1,3$.
3. For more details of the proof, see [27, 7].

SKETCH. Let ψ be a fixed function with support in $\{x \mid 1 \le |x| \le 2\}$. Let $\psi_n(x) = \psi(2^{-n}x)$. Then, for n large, $(\psi_n, (-\Delta+V)\psi_n) \le c_1 n^{-2} - c_2 n^{-2+\varepsilon} < 0$ so one can find an infinite orthonormal set with $(\phi_n, H\phi_m) = 0$ if $n \ne m$ and $(\phi_n, H\phi_n) < 0$. It follows that $\mu_n(H) < 0$ for all n.

Suppose now that $\sigma_{ess}(-\Delta+V)$ is $[0,\infty)$. Then $N_E(V) \uparrow \infty$ as $E \uparrow 0$. One can ask how. As one might expect, this limit is one where quasi-classical (i.e. phase space) consideration predominate; see Brownell-Clark [32], McLeod [33], Tamura [34] or Reed-Simon [7].

B. N-*Body; Small Coupling*. For small coupling N-Body systems, one can show that $N(V)$ is zero by the general principle:

THEOREM 6.2. *Let* V_{ij} $(1 \le i < j \le n)$ *be potentials on* R^3 *with* $N(\frac{1}{2}(n-1)V_{ij}) = 0$ *for all* i, j. *Then*

$$H = -\sum_{i=1}^{n} \Delta_i + \sum_{i<j} V_{ij}(r_i - r_j)$$

has spectrum contained in $[0,\infty)$.

PROOF. Since $-\Delta_i - \Delta_j = -1/2\Delta_{[ij]} - 2\Delta_{(ij)}$ where $[ij]$ is associated with $1/2(r_i + r_j)$ and (ij) with $r_j - r_i$. Thus

$$\sum -\Delta_i + \sum V_{ij} \geq \sum_{i<j} \frac{2}{n-1}(-\Delta_{(ij)} + V_{ij})$$

$$= \sum_{i<j} \frac{2}{n-1}\left(-\Delta_{(ij)} + \frac{n-1}{2} V_{ij}\right) \geq 0$$

by hypothesis.

For results on when $-\sum_i \Delta_i$ and $-\sum \Delta_i + \sum V_{ij}$ are unitarily equivalent, see Orio-O'Carroll [35] (small coupling) and Lavine [31] (repulsive potentials).

C. *The Effimov Effect.* Effimov [36] has suggested the following: Let V be a fixed short range spherically symmetric potential. Let

$$\tilde{H}(\lambda) = -\Delta_1 - \Delta_2 - \Delta_3 + \lambda[V(r_{21}) + V(r_{31}) + V(r_{23})]$$

and let $H(\lambda)$ be $\tilde{H}(\lambda)$ with the center of mass motion removed. Fix λ to be the coupling constant at which $-2\Delta + \lambda V$ has its first s-wave resonance. At this value of λ, it is claimed that $H(\lambda)$ has infinitely many bound states.

The point of this prediction is that the occurrence of an s-wave resonance sets up an effective long range force. The occurrence of such an effect shows the difficulty of establishing bounds on the number of bound states in the N-body case.

For further discussion of this effect, see Amado-Noble [37] and Yafeev [38].

D. *Two-Body Continuum Limit.* It is a basic theorem of the spectral theory of multiparticle Schrödinger operators that for a large class of potentials, the continuous spectrum of the Schrödinger operator is $[\Sigma,\infty)$ where Σ is determined as follows: let $C_1,\cdots,C_k$ be a breakup of the n-particles into clusters. Let $E_1,\cdots,E_k$ be eigenvalues of the Hamiltonian associated to clusters $C_1,\cdots,C_k$ with their center of mass motion removed (if $\#(C_i) = 1$, E_i must be 0). Then $\Sigma = \min(E_1 + \cdots + E_k)$ where the minimum is over all breakups into clusters and all choices of eigenvalues. This is a result of Hunziker [39], Van Winter [40] and Zhislin [41]; see also Jörgons-Weidmann [42] or Reed-Simon [7].

There is considerable information available about $N(V)$ in case Σ is determined by a breakup into two clusters. An example of this is the case of atoms where one has the result of Zhislin [41].

THEOREM 6.3. *Atoms have an infinite number of bound states.*

REMARK. For Helium with an infinitely heavy nucleus, this is a result of Kato [43].

In case Σ is determined by a two cluster breakup, one has the following *intuition* [44]: If the sum of the potentials between the cluster is a long range two-body potential (i.e. $r^{-2+\varepsilon}$ at infinity), then the number of bound states is infinite. If this sum is a short range (i.e. $r^{-2-\varepsilon}$ at infinity), then there are only finitely many eigenvalues in $(-\infty,\Sigma)$. Under suitable technical hypothesis, the former was proven by Simon [44] and the latter by Combes [45]. One result of this is the following [44]:

THEOREM 6.4. *There exist three two body potentials* V_1, V_2, V_{12} *so that*

$$H(\lambda) = -\Delta_1 - \Delta_2 + \lambda V_1(r_1) + \lambda V_2(r_2) + \lambda V_{12}(r_1 - r_2)$$

has the following property. There are $0 < \lambda_1 < \lambda_2 < \cdots$ *so that if* $\lambda \in [0,\lambda_1) \cup (\lambda_2,\lambda_3) \cup \cdots \cup (\lambda_{2n},\lambda_{2n+1}) \cdots$ $H(\lambda)$ *has finitely many eigen-*

values in $(-\infty, \Sigma(\lambda))$ *and if* $\lambda \epsilon [\lambda_1, \lambda_2] \cup \cdots \cup [\lambda_{2n-1}, \lambda_{2n}] \cup \cdots$ *then* $H(\lambda)$ *has infinitely many eigenvalues in* $(-\infty, \Sigma(\lambda))$.

For additional information on bound states of N-body systems, see [46-51].

ADDED NOTE.

During the production of this book, three interesting new papers on N(V) appeared. A. Martin (CERN preprint) has proven that

$$N(V) \leq (2\pi)^{-1} (\|V\|_1^{1/2} \|V\|_2)$$

in three dimensions. E. Lieb (Princeton Preprint) and M. Cwikel (IAS Preprint) have proven the general conjecture discussed in the text, that for $n \geq 3$

$$N(V) \leq c_n \|V\|_{n/2}^{n/2} .$$

Both bounds have the correct large coupling constant behavior.

DEPARTMENTS OF MATHEMATICS AND PHYSICS
PRINCETON UNIVERSITY

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REPRINTED FROM

Studies in Mathematical Physics

Essays in Honor of Valentine Bargmann

edited by
E. H. Lieb, B. Simon, and A. S. Wightman

Princeton Series in Physics

Princeton, New Jersey 1976