

QUANTUM DYNAMICS:  
FROM AUTOMORPHISM TO HAMILTONIAN

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We describe the mathematical arguments involved in passing from a one-parameter measurable group of automorphisms of the basic quantum structures to the Schrödinger equation.

§1. *Introduction*

Every student of quantum mechanics learns in a first course that quantum dynamics is governed by the Schrödinger equation  $i\hbar\dot{\psi} = H\psi$ . However, even professional quantum mechanics who have delved into the axiomatic foundations of quantum theory are sometimes unaware of the full chain of argument leading from the primitive version of dynamics as a one-parameter continuous (or measurable) group of automorphisms of the axiomatic structure to the Schrödinger equation. Our goal in this note is to put down in one place this full chain of argument. We expect the experts will find nothing new here and we do not intend a review of the literature. This note will have served its purpose if the student of the foundations of quantum theory is able to find here, in one place, things which would have formerly taken him into five or six research articles. The traditional route from continuous automorphisms takes the following steps:

(1) *Wigner's Theorem*. Every automorphism is induced by a unitary or antiunitary, uniquely determined up to a phase.

(2) *Bargmann-Wigner Theorem*. Given a one-parameter continuous group of automorphisms, the phases of step (1) can be chosen so that the corresponding family of unitaries depends continuously on the parameter.

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(3) *Multiplier Problem for R.* Steps (1) and (2) lead to a family of unitaries  $U$  with  $U(a)U(b) = \omega(a, b)U(a+b)$ . There is a function  $\lambda(a)$  with  $\omega(a, b) = \lambda(a+b)\lambda(a)^{-1}\lambda(b)^{-1}$  so that  $\tilde{U}(a) = U(a)\lambda(a)$  obeys  $\tilde{U}(a)\tilde{U}(b) = \tilde{U}(a+b)$ .

(4) *Stone's Theorem.* Every one-parameter strongly continuous group of unitaries is of the form  $U(a) = \exp(-iaA)$  for some self-adjoint operator  $A$ .

We intend to follow a slightly longer route to prove a result slightly stronger in two ways. First we wish to consider several different meanings of automorphism which *a priori* might be very different. Step (1) is then several theorems including results of Wigner [1] and Kadison [2]. Secondly, we wish to assume *a priori* only that the automorphisms are (Borel) measurable so that steps (2) and (3) take on a slightly different content. Before step (4) we must insert

(3½) *Von Neumann's Theorem.* Every weakly measurable unitary representation of  $R$  is strongly continuous.

§2. *What is a Quantum Automorphism?*

Let  $\mathcal{H}$  be a separable Hilbert space. Depending on which axiom scheme one adopts, one is led to various *a priori* notions of automorphism:

(1) *Wigner automorphism:* Let  $P(\mathcal{H})$  be the complex projective space for  $\mathcal{H}$ , i.e. identify  $\psi, \eta \in \mathcal{H}$  with  $\|\psi\| = \|\eta\| = 1$  if  $\psi = a\eta$  for some  $a = e^{i\theta} \in \mathbb{C}$ .  $P(\mathcal{H})$  is the family of equivalence classes under this relation. By definition,  $\langle [\psi], [\eta] \rangle = |\langle \psi, \eta \rangle|$ . A *Wigner automorphism* is a bijection  $\alpha : P(\mathcal{H}) \rightarrow P(\mathcal{H})$  such that  $\langle \alpha[\psi], \alpha[\eta] \rangle = \langle [\psi], [\eta] \rangle \cdot t \rightarrow \alpha_t$  is called *measurable* if  $t \rightarrow \langle \alpha_t[\psi], [\eta] \rangle$  is measurable for all  $[\psi], [\eta] \in P(\mathcal{H})$ . An alternative way of describing  $P(\mathcal{H})$  is as the space of all projections  $P_\psi$  of rank one. Then  $\langle [\psi], [\eta] \rangle = \text{Tr}(P_\psi P_\eta)^{1/2}$ .

(2) *Kadison automorphism.* Let  $\mathcal{S}(\mathcal{H})$  denote the set of density matrices on  $\mathcal{H}$ , i.e. trace class operators  $\rho$  with  $\text{Tr}(\rho) = 1, \rho \geq 0$ .  $\mathcal{S}(\mathcal{H})$  is a convex set and a *Kadison automorphism* is a map  $\beta : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H})$  which is a bijection and which is affine, i.e.

$$\beta(t\rho_1 + (1-t)\rho_2) = t\beta(\rho_1) + (1-t)\beta(\rho_2)$$

all  $\rho_1, \rho_2 \in \mathcal{S}(\mathcal{H}), 0 \leq t \leq 1$ . A family  $t \rightarrow \beta_t$  is called measurable if and only if  $t \rightarrow \text{Tr}(\beta_t(\rho)A)$  is measurable for all  $\rho \in \mathcal{S}, A \in \mathcal{B}(\mathcal{H})$ , the bounded operators on  $\mathcal{H}$ .

(3) *Segal automorphism.* Let  $\mathcal{B}_r(\mathcal{H})$  denote the bounded self-adjoint operators on  $\mathcal{H}$  endowed with the natural linear structure and the Jordan product:

$$A \circ B = 1/2(AB + BA) .$$

A Segal automorphism is a bijection  $\gamma : \mathcal{B}_r(\mathcal{H}) \rightarrow \mathcal{B}_r(\mathcal{H})$  so that  $\gamma$  is linear and

$$\gamma(A \circ B) = \gamma(A) \circ \gamma(B) .$$

If this condition is only assumed for commuting  $A$  and  $B$ , we call  $\gamma$  a weak Segal automorphism.  $t \rightarrow \gamma_t$  is called measurable, if and only if  $t \rightarrow \langle \psi, \gamma_t(A)\psi \rangle$  is measurable for all  $A \in \mathcal{B}_r(\mathcal{H})$  and  $\psi \in \mathcal{H}$ .

Wigner automorphisms arise in a framework of analyzing quantum mechanics in terms generalized Stern-Gerlach experiments and overlap probabilities (see Ax [3] for a recent treatment). Kadison automorphisms arise in a framework describing general states and observables (see von Neumann [4] or Mackey [5]). Segal automorphisms arise in a framework that emphasizes the structure of observables (see e.g. Segal [6]).

*A priori* the four types of automorphisms appear very different although each does seem to capture an important property of a symmetry. Wigner automorphisms preserve the basic objects of the theory as viewed from an overlap probability point of view. Kadison automorphisms are based on the interpretation of convex combinations of states as statistical mixtures. Weak Segal automorphisms are based on the notion of expectations of commuting observables. I see no simple physical reason why these families of automorphisms are essentially the same; in fact, if "automorphism" is replaced by "endomorphism" (bijection no longer required), then the families are no longer the same! Nevertheless, one has:

**THEOREM 2.1 (Wigner's Theorem).** Every Wigner automorphism is of the form

$$a[\psi] = [U\psi]$$

where  $U$  is either unitary or antiunitary and is uniquely determined up to one overall phase, i.e. if  $a[\psi] = [U'\psi]$ , then  $U' = aU$  with  $a = e^{i\theta}$ .

**THEOREM 2.2 (Kadison's Theorem).** Every Kadison automorphism is of the form:

$$\beta(\rho) = U\rho U^*$$

for a unitary or antiunitary map  $U$  uniquely determined up to a phase.

**THEOREM 2.3.** Every weak Segal automorphism is a strong Segal automorphism and is of the form

$$\gamma(A) = U^*AU$$

for a unitary or antiunitary map  $U$  uniquely determined up to a phase.

We remark that Wigner's Theorem looks more like the others if we write it in terms of projections:

$$P_{a(\psi)} = UP_\psi U^*$$

We write Theorems 2.2 and 2.3 as  $U\rho U^*$  and  $U^*AU$  so that  $\text{Tr}(\beta(\rho)A) = \text{Tr}(\rho\gamma(A))$ , i.e. the  $\beta$  and  $\gamma$  associated to a fixed  $U$  are related by the distinction between the Schrödinger and Heisenberg pictures.

These theorems have been compared (e.g. [5]) to the fundamental theorem of projective geometry [7]; this theorem says that given any map  $\alpha: P(V) \rightarrow P(V)$ , the projective space of a finite dimensional vector space  $V$  over a field  $F$ , which takes lines in  $P(V)$  into lines we can find  $U: V \rightarrow V$  so that  $\alpha([\psi]) = [U\psi]$ .  $U$  will be additive and obey  $U(a\psi) = m(a)U(\psi)$  for some automorphism  $m$  of  $F$ . Galois theory yields many automorphisms of  $\mathbb{C}$  other than the identity and complex conjugation, so the theorems, while related, are certainly not equivalent; we discuss this further in Section 4.

The main "super theorem" of this note is the following:

**THEOREM 2.4.** Let  $t \rightarrow a_t$  (resp.  $\beta_t, \gamma_t$ ) be a map from  $\mathbb{R}$  to the Wigner (resp. Kadison, Segal, weak Segal) automorphisms which is measurable and obeys  $a_t a_s = a_{t+s}$  (resp. for  $\beta$  or  $\gamma$ ). Then, there exists a self-adjoint operator  $H$  (not necessarily bounded) unique up to an overall additive constant so that

$$a_t[\psi] = [e^{-iHt}\psi]$$

$$\text{(resp. } \beta_t(\rho) = e^{-iHt}\rho e^{iHt}; \gamma_t(A) = e^{iHt}A e^{-iHt}\text{)} .$$

### §3. The Two-Dimensional Case

The two-dimensional case of Theorems 2.1-3 is not only a preliminary step in the general case, but provides a simple insight into the theorems: the unitary vs. antiunitary choice is related to the fact that isometries of Euclidean three space are either orientation preserving ("pure rotations") or orientation reversing ("reflections"). This remark and much of our discussion in this section follows Hunziker [8].

All three types of automorphisms involve self-adjoint operators, so we will exploit the fact that the matrices

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

are a basis for  $\mathcal{B}_1(\mathbb{C}^2)$ , as a real vector space. If

$$A = a + \vec{a} \cdot \vec{\sigma}$$

then

$$UAU^* = a + \overline{R(U)}\vec{a} \cdot \vec{\sigma}$$

where  $R(U)$  is an orientation preserving (resp. reversing) isometry if  $U$  is unitary (resp. antiunitary). The map  $R$  is onto all isometries and  $R(U) = R(U')$  if and only if  $U = e^{i\theta}U'$ . We will not prove all these facts from scratch, but assume the reader is familiar with the two-to-one map of  $SU(2)$  onto  $SO(3)$ . Given this, the general unitary case follows from the fact that any unitary  $U$  is of the form  $e^{i\theta}\bar{U}$  with  $\bar{U} \in SU(2)$  and  $e^{i\theta}$  determined up to  $\pm 1$ . The antiunitary case follows by noting that if  $U$  is the natural complex conjugation on  $\mathbb{C}^2$ , then  $R(U)$  is just reflection in the plane, orthogonal to the 2 axis and that any antiunitary is the product of a unitary and the complex conjugation.

(1) *Wigner Automorphisms.* The rank one projections are precisely of the form

$$P(\vec{a}) = 1/2(1 + \vec{a} \cdot \vec{\sigma}); \quad |\vec{a}| = 1$$

so any Wigner automorphism is associated with a bijection of the unit sphere in  $\mathbb{R}^3$ . Since

$$2\text{Tr}(P(\vec{a})P(\vec{b})) = 1 + \vec{a} \cdot \vec{b},$$

this map is an isometry of the sphere and hence a "pure rotation" or "relection."

(2) *Kadison Automorphisms.* The  $\mathcal{S}(\mathbb{C}^2)$  operators are precisely of the form

$$\frac{1}{2}(1 + \vec{a} \cdot \vec{\sigma}); \quad |\vec{a}| \leq 1$$

so any Kadison automorphism is associated with an affine map of the unit sphere onto itself. Such a map is isometry of  $\mathbb{R}^3$ .

(3) *Segal Automorphisms.* Let  $\gamma$  be a weak-Segal automorphism. Then  $\gamma$  clearly takes the orthogonal projections onto themselves since it preserves  $P^2 = P$ . Since  $1$  is the unique projection of the form  $P_1 + P_2$  for non-zero projections  $P_1, P_2$ ;  $\gamma(1) = 1$ . Thus  $\gamma$  also induce in a map of the rank one projections to themselves and so of the unit sphere in  $\mathbb{R}^3$  to itself, i.e.,

$$\gamma(a + \vec{a} \cdot \vec{\sigma}) = a + \overline{M}\vec{a} \cdot \vec{\sigma}$$

where  $M$  is a linear map taking the unit sphere to itself.  $M$  is thus an isometry of  $\mathbb{R}^3$ . Since every weak-Segal automorphism is of the requisite form, a fortiori every Segal automorphism is of the requisite form.

§4. *Wigner's Theorem*

In this section we prove Theorem 2.1. We use the letters  $p, q, \dots$  to denote points in  $P(\mathcal{H})$ , i.e. "rays" in  $\mathcal{H}$ . The linear structure of  $\mathcal{H}$  induces a geometric structure on  $P(\mathcal{H})$ , as is well-known [7]; if  $M \subset \mathcal{H}$  is a subspace of  $\mathcal{H}$ , we denote by  $P(M)$ , those  $p \in P(\mathcal{H})$  which arise from rays in  $M$ .

LEMMA 4.1. *If  $\alpha$  is a Wigner automorphism and  $M \subset \mathcal{H}$  is a  $k$ -dimensional subspace, then there is an  $M'$  (denoted by  $\alpha(M)$ ) also of dimension  $k$  so that  $\alpha(p) \in M'$  if and only if  $p \in M$ .*

PROOF. Let  $\phi_1, \dots, \phi_k$  be an orthonormal basis for  $M$  (we write  $M = [\phi_1, \dots, \phi_k]$ ). Pick representatives  $\psi_1, \dots, \psi_k$  for  $\alpha([\phi_1]), \dots, \alpha([\phi_k])$  and let  $M'$  be their span. Since the  $\psi_i$  are orthonormal,  $M'$  has dimension  $k$ . Now  $p \in M$  if and only if  $\sum_{i=1}^k \langle p, \phi_i \rangle^2 = 1$  if and only if  $\sum_{i=1}^k \langle \alpha(p), \alpha[\phi_i] \rangle^2 = 1$  if and only if  $\alpha(p) \in M'$ . ■

REMARK. In the language of projective geometry [7], we have just proven that  $\alpha$  defines a collinealition. Below we reduce Wigner's Theorem to the three dimensional case. We could prove this three dimensional case by appealing to the fundamental theorem of projective geometry. This would leave open an arbitrary automorphism of  $\mathbb{C}$  which we could prove was either the identity or the conjugation by appealing to the two-dimensional case (Section 3).

LEMMA 4.2. *For any two dimensional  $M \subset \mathcal{H}$  and any Wigner automorphism,  $\alpha$ , there exists a unitary or antiunitary map  $U: M \rightarrow \alpha(M)$ , unique up to phase, so that*

$$\alpha[\phi] = [U\phi]$$

for all  $\phi \in M$ .

PROOF. Let  $\phi_1, \phi_2$  be an orthonormal basis for  $M$  and choose  $\psi_1, \psi_2$  so that  $a[\phi_i] = [\psi_i]$ . Define a unitary  $V: a(M) \rightarrow M$  by  $V\psi_i = \phi_i$ . Then  $\beta: P(M) \rightarrow P(M)$ , given by  $\beta = a_V \circ a$  with  $a_V[\psi] = [V\psi]$  is a Wigner automorphism. The lemma follows by appealing to the two dimensional case (Section 3). ■

LEMMA 4.3. *If the map of Lemma 4.2 is unitary (resp. antiunitary) for one  $M \subset \mathcal{H}$ , it is unitary (resp. antiunitary) for all two dimensional  $M' \subset \mathcal{H}$ .*

PROOF. There is a direct algebraic proof (see page 89 of [7]), but we prefer to develop a suggestion of Bargmann. Let us topologize  $P(\mathcal{H})$  by putting the metric

$$\rho(p, q) = 1 - \langle p, q \rangle$$

on it (to check it is a metric, we note for any  $\psi \in p$ , there is an  $\eta \in q$  so that  $\frac{1}{2} \|\psi - \eta\| = \rho(p, q)$  and use the triangle inequality on  $\mathcal{H}$ ). Clearly, any Wigner automorphism is an isometry on  $P(\mathcal{H})$  and so continuous.

Given  $p, q, r \in P(\mathcal{H})$ , following Bargmann [9], we define a number  $\chi(p, q, r)$  by choosing  $\psi \in p$ ,  $\eta \in q$ ,  $\gamma \in r$  and letting

$$\chi(p, q, r) = \langle \psi, \eta \rangle \langle \eta, \gamma \rangle \langle \gamma, \psi \rangle$$

and noting that  $\chi$  is independent of choice.  $\chi$  is jointly continuous, since given  $p_n \rightarrow p$ , we can find  $\psi_n \in p_n$ ,  $\psi \in p$  so that  $\|\psi_n - \psi\| \rightarrow 0$ .

Let  $\psi$  and  $\phi$  be orthonormal vectors and let  $p, q, r$  be given by

$$p = [\psi]; \quad q = [\eta], \quad \eta = \frac{1}{\sqrt{2}}(\psi - \phi); \quad r = [\gamma], \quad \gamma = \frac{1}{\sqrt{2}}(\psi - i\phi).$$

Then  $\chi(p, q, r) = \frac{1}{4}(1+i)$ . Let  $M'$  be the two dimensional space spanned by  $\psi, \phi$ . Then the map induced by  $a$  via Lemma 4.2 is unitary (resp. antiunitary) if and only if  $\chi(a(p), a(q), a(r)) = \frac{1}{4}(1+i)$  (resp.  $\frac{1}{4}(1-i)$ ). Given another  $M''$  we can continuously vary  $\psi$  and  $\phi$  to  $\psi', \phi'$  generating  $M''$  and so  $p, q, r$  to  $p', q', r'$  so by continuity  $\chi(a(p), a(q), a(r))$  will be  $\frac{1}{4}(1+i)$  if and only if  $\chi(a(p'), a(q'), a(r')) = \frac{1}{4}(1+i)$ . ■

LEMMA 4.4. *Let  $\mathcal{H}$  have dimension 3. Let  $a$  be a Wigner automorphism and  $\phi_1, \phi_2, \phi_3$  a basis for  $\mathcal{H}$ . If  $a$  leaves the subsets  $P([\phi_1, \phi_2])$  and  $P([\phi_1, \phi_3])$  pointwise fixed, then  $a$  is the identity.*

PROOF. Let  $\eta = a\phi_1 + b\phi_2 + c\phi_3$  with  $a \neq 0$  and real. We will first prove that  $a([\eta]) = [\eta]$ . Pick  $\eta' \in a[\eta]$  with  $(\phi_1, \eta') = a$  which fixes  $\eta'$ . Set  $b' = (\phi_2, \eta')$ . Now since  $([\eta], [\psi]) = ([\eta'], [\psi'])$  for  $\psi = \phi_2$ ,  $\frac{1}{\sqrt{2}}(\phi_1 + \phi_2)$  and  $\frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ , we have

$$|b| = |b'|; \quad |a+b| = |a+b'|; \quad |a-ib| = |a-ib'|.$$

For fixed  $b$ , the last three equations on  $b'$  require  $b'$  to lie on three circles which intersect only in one point. Thus  $b' = b$ . Similarly  $c' = c$  so  $\eta' = \eta$ . The  $a = 0$  case follows by continuity. ■

LEMMA 4.5. *Wigner's theorem holds in case  $\dim \mathcal{H} = 3$ .*

PROOF. We only prove existence; uniqueness follows as in the general proof below. Given  $U$  unitary or antiunitary, let  $a_U$  be the induced Wigner automorphism. Given  $a$ , a Wigner automorphism, it clearly suffices to find  $U_1, \dots, U_k$  so that  $a_{U_k} \circ \dots \circ a_{U_1} \circ a = \text{id}$  since then  $a = a_U$  with  $U = U_1^{-1} \dots U_k^{-1}$ . Pick a basis  $\phi_1, \phi_2, \phi_3$  for  $\mathcal{H}$  and  $\psi_1, \psi_2, \psi_3$  so that  $a([\phi_i]) = [\psi_i]$ . Define  $U_1$  to be the unique unitary (resp. antiunitary) with  $U_1\psi_i = \phi_i$  if  $a$  is unitary (resp. antiunitary) on two dimensional subspaces. Let  $a_1 = a_{U_1} \circ a$ . Then  $a_1([\phi_i]) = [\phi_i]$  and is unitary on two dimensional subspaces. Now, by the two dimensional theorem, we can find a unitary  $V_1: [\phi_1, \phi_2] \rightarrow [\phi_1, \phi_2]$  so that  $a_1 \upharpoonright [\phi_1, \phi_2] = a_{V_1}$ . Let  $U_2 = V_1^{-1}$  on  $[\phi_1, \phi_2]$  and the identity on  $[\phi_3]$ . Then  $a_2 = a_{U_2} \circ a_1$  is the identity on  $[\phi_1, \phi_2]$  and on  $[\phi_3]$ . Applying the two dimensional case to  $[\phi_1, \phi_3]$ , we can find  $V_2: [\phi_1, \phi_3] \rightarrow [\phi_1, \phi_3]$  so that  $a_2 \upharpoonright [\phi_1, \phi_3] = a_{V_2}$  and we can fix the phase so that  $V_2\phi_1 = \phi_1$ . Let  $U_3 = V_2^{-1}$  on  $[\phi_1, \phi_3]$  and the identity on  $[\phi_2]$  (and so on all of  $[\phi_1, \phi_2]$ ). Then  $a_{U_3} \circ a_2$  is the identity on  $[\phi_1, \phi_2]$ , and  $[\phi_1, \phi_3]$  and so the identity by Lemma 4.4. ■

LEMMA 4.6. For any three dimensional  $M \subset \mathcal{H}$  and Wigner automorphism  $\alpha$ , there is a unitary or antiunitary  $U: M \rightarrow \alpha(M)$  so that  $\alpha[\psi] = [U\psi]$  for all  $\psi \in M$ .

PROOF. As in Lemma 4.2.

PROOF OF THEOREM 2.1. Without loss, suppose  $\alpha$  is unitary on two-dimensional spaces. Fix  $\phi \in \mathcal{H}$  and  $\psi \in \alpha[\phi]$ . Given any  $\eta \in \mathcal{H}$ , let  $M$  be the span of  $\phi$  and  $\eta$ . By Lemma 4.2, find  $V_M: M \rightarrow \alpha(M)$  inducing  $\alpha$  on  $M$  with its phase determined by  $V_M\phi = \psi$ . Define  $U\eta$  to be  $V_M\eta$ . We must prove  $U$  is linear, so given  $\eta_1, \eta_2 \in \mathcal{H}$ , let  $N$  be the span of  $\phi, \eta_1, \eta_2$ . By Lemma 4.6 we can find a unitary  $W: N \rightarrow \alpha(N)$  (it can't be antiunitary since its restrictions to two dimensional spaces must be unitary!) inducing  $\alpha$  on  $N$ . By change of phase we can suppose  $W\phi = \psi$ . Since the restriction of  $W$  to any two dimensional  $M \subset N$  is unitary,  $W \upharpoonright M = V_M$  and so  $U \upharpoonright N = W$ . Since  $W(a\eta_1 + b\eta_2) = aW(\eta_1) + bW(\eta_2)$ ,  $U$  is linear. By construction  $U$  is norm preserving. In the above construction only the phase of  $\psi$  is arbitrary. ■

§5. Kadison's Theorem

Following Roberts-Roepstorff [10], we prove Theorem 2.2 by reducing it to Theorem 2.1. This reduction is essentially part of an argument of Hunziker [8]. For  $M \subset \mathcal{H}$ , a subspace,  $\mathcal{S}(M)$  is the subset of  $\mathcal{S}(\mathcal{H})$  of those  $\rho$  with  $\text{Ran } \rho \subset M$ .

LEMMA 5.1. Let  $\beta$  be a Kadison automorphism. Then for any two dimensional subspace  $M \subset \mathcal{H}$ , there is a two-dimensional subspace  $\beta(M) \subset \mathcal{H}$  so that  $\beta[\mathcal{S}(M)] = \mathcal{S}(\beta(M))$ .

PROOF.  $\mathcal{S}(M)$  is a face of the convex set  $\mathcal{S}(\mathcal{H})$  with the property: there exist two extreme points  $u, v \in \mathcal{S}(M)$  so that  $\mathcal{S}(M)$  is the smallest face containing  $u$  and  $v$ . Moreover any such face with more than one point is  $\mathcal{S}(M')$  for some  $M'$  (for if  $u = P_\phi, v = P_\psi$ , take  $M' = [\phi, \psi]$ ). Since  $\beta$

is a convex automorphism, it preserves the structure of faces and in particular  $\beta[\mathcal{S}(M)]$  is  $\mathcal{S}(M')$  for some  $M'$ . ■

REMARKS.

1. This proof extends to all finite-dimensional subspaces.
2. For an alternate proof, see Hunziker [8].

LEMMA 5.2. Given any Kadison automorphism  $\beta$ , there is a Wigner automorphism  $\alpha$  with  $\beta(P_\psi) = P_{\alpha(\psi)}$  for every one dimensional projection  $P_\psi \in \mathcal{S}(\mathcal{H})$ .

PROOF. Given  $\beta$ , we note that since the  $P_\psi$  are the extreme points of  $\mathcal{S}(\mathcal{H})$ ,  $\beta(P_\psi) = P_{\alpha(\psi)}$  for some map  $\alpha$  on the rays. We must prove that  $\alpha$  preserves the inner product on  $P(\mathcal{H})$ . Given  $\phi, \psi \in \mathcal{H}$ , let  $M$  be span of  $\phi$  and  $\psi$ . By composing  $\beta$  with the  $\beta$  induced by a unitary  $U$  which maps  $\beta(M)$  into  $M$ , we obtain  $\tilde{\beta} = \beta_u \circ \beta$  leaving  $\mathcal{S}(M)$  invariant. By the two dimensional case,  $\tilde{\beta}$  is induced by a unitary, so there is a unitary or antiunitary  $V: M \rightarrow \beta(M)$  so that  $\beta(\rho) = V\rho V^*$  for  $\rho \in \mathcal{S}(M)$  and thus  $\beta$  preserves  $\text{Tr}(P_\psi P_\phi)$ , i.e.  $(\alpha[\psi], \alpha[\phi]) = ([\psi], [\phi])$ . ■

LEMMA 5.3. If  $\beta$  is a Kadison automorphism which leaves each extreme point fixed, then  $\beta$  is the identity.

PROOF. We need only prove that  $\beta$  is continuous in trace-norm topology since any  $\rho \in \mathcal{S}(\mathcal{H})$  has an expansion  $\sum_{i=1}^{\infty} t_i P_{\psi_i}$  converging in trace norm and so is a limit of finite convex combinations of the  $P_{\psi_i}$ . Now  $\beta$  extends to the positive trace class operators by defining  $\beta_{\text{ext}}(A) = \text{Tr}(A)\beta(A/\text{Tr}(A))$ .  $\beta_{\text{ext}}$  obeys  $\beta_{\text{ext}}(A+B) = \beta_{\text{ext}}(A) + \beta_{\text{ext}}(B)$  (since  $\beta$  is affine),  $\beta_{\text{ext}}(\lambda A) = \lambda\beta_{\text{ext}}(A)$  for  $\lambda \geq 0$  and

$$\text{Tr}(\beta_{\text{ext}}(A)) = \text{Tr}(A)$$

We define  $\beta$  on all self-adjoint trace class operators by  $\tilde{\beta}(A) = \beta_{\text{ext}}(A_+) - \beta_{\text{ext}}(A_-)$  where  $A_+$  and  $A_-$  are the positive and negative parts of  $A$ .

Thus letting  $\|A\|_1 = \text{Tr}(|A|)$ :

$$\begin{aligned} \|\tilde{\beta}(A)\|_1 &\leq \|\beta_{\text{ext}(A_+)}\|_1 + \|\beta_{\text{ext}(A_-)}\|_1 \\ &= \text{Tr}(A_+) + \text{Tr}(A_-) = \|A\|_1. \end{aligned}$$

Since  $\tilde{\beta}$  is linear,  $\beta$  is continuous in  $\|\cdot\|_1$  and so  $\beta$  is continuous on  $\mathfrak{S}(\mathcal{H})$ . ■

**PROOF OF THEOREM 2.2.** By Wigner's theorem and Lemma 5.2, we can find  $U$ , unitary or antiunitary so that  $\beta_U \circ \beta$  is the identity on all  $P_\psi$ . Thus by Lemma 5.3,  $\beta_U \circ \beta = \text{id}$ , i.e.  $\beta = \beta_{U^{-1}}$ . ■

§6. *The Structure of Segal Automorphisms*

We will prove Theorem 2.3 by reducing it to Wigner's theorem. We first note:

**LEMMA 6.1.** *Let  $\gamma$  be a weak Segal automorphism. Then  $\gamma$  is order preserving (i.e.  $A \geq B$  implies  $\gamma(A) \geq \gamma(B)$ ),  $\gamma$  takes projections into projections,  $\gamma(1) = 1$  and  $\|\gamma(A)\| = \|A\|$ .*

**PROOF.** Since  $\gamma$  is linear, we need only prove  $C \geq 0$  implies  $\gamma(C) \geq 0$  to conclude that  $\gamma$  is order preserving. But  $\gamma(C) = \gamma(C^{1/2} \circ C^{1/2}) = \gamma(C^{1/2}) \circ \gamma(C^{1/2}) \geq 0$ .  $\gamma$  clearly takes projections into themselves since  $P^2 = P$  implies  $\gamma(P) = \gamma(P \circ P) = \gamma(P)^2$ . Since  $1$  is the unique maximal projection,  $\gamma(1) = 1$ . Finally  $1 \|A\| \pm A \geq 0$  implies  $\pm \gamma(A) + \|A\|1 \geq 0$  implies  $\|\gamma(A)\| \leq \|A\|$ . Since  $\gamma$  is invertible and  $\gamma^{-1}$  is a Segal automorphism  $\|A\| = \|\gamma^{-1}(\gamma(A))\| \leq \|\gamma(A)\|$ . ■

**LEMMA 6.2.** *Any weak Segal automorphism  $\gamma$  takes one dimensional projections onto one dimensional projections.  $\gamma$  thus induces a map  $\alpha : P(\mathcal{H}) \rightarrow P(\mathcal{H})$  so that  $\gamma(P[\psi]) = P_{\alpha[\psi]}$ .  $\alpha$  is a Wigner automorphism.*

**PROOF.** One dimensional projections are minimal (non-zero) projections, so, by Lemma 6.1,  $\gamma$  must take one into another. By a similar argument,  $\gamma$  must take two dimensional projections into themselves, so, as in Sections 4, 5, the two dimensional analysis of weak Segal automorphisms implies for any two dimensional  $M$ , there is a unitary or antiunitary map  $U : M \rightarrow \mathcal{K}$  so that  $\gamma(P[\psi]) = P_{[U\psi]}$  so that  $\alpha$  will preserve inner products. ■

**REMARK.** Given this theorem, one might expect that  $|\langle \psi, \phi \rangle|$  should be expressible in terms of  $P_\psi, P_\phi$  and the Jordan product. In fact:

$$P_\psi \circ (P_\psi \circ P_\phi) - \frac{1}{2} (P_\psi \circ P_\phi) = \frac{1}{2} |\langle \psi, \phi \rangle|^2 P_\psi.$$

**LEMMA 6.3.** *If  $\gamma$  is a weak Segal automorphism and  $\gamma(P_\psi) = P_\psi$  for all one dimensional projections, then  $\gamma = \text{identity}$ .*

**PROOF.** Let  $P$  be any projection. Since  $\psi \in \text{Ran } P$  if and only if  $P_\psi \leq P$  and  $\gamma$  is order preserving,  $\text{Ran } \gamma(P) = \text{Ran } P$ , i.e.  $\gamma$  leaves all projections invariant. Since  $\gamma$  is continuous in norm and any  $A \in \mathfrak{B}(\mathcal{H})$  is a norm limit of finite linear combinations of projections (by the Spectral Theorem),  $\gamma$  is the identity. ■

Given these lemmas, Theorem 2.3 follows from Wigner's theorem in just the way that Theorem 2.2 followed. ■

§7. *Lifting Measurability*

**DEFINITION.** A map  $t \rightarrow U(t)$  from the reals to the unitaries is called weakly measurable if  $t \rightarrow \langle \phi, U(t)\psi \rangle$  is measurable for all  $\phi, \psi \in \mathcal{K}$ . A map  $t \rightarrow \phi(t)$  from the reals to  $\mathcal{K}$  is called weakly measurable if  $\langle \psi, \phi(t) \rangle$  is measurable for each  $\psi \in \mathcal{K}$ .

In this section we prove:

**THEOREM 7.1.** *If  $t \rightarrow a_t$  is a measurable family of Wigner automorphisms obeying  $a_{t+s} = a_t a_s$ , then there is a family of unitaries  $U(t)$  so that  $a_t[\psi] = [U(t)\psi]$  and so that  $U(t)$  is weakly measurable.*

**REMARKS.**

1. We emphasize that we are dealing with everywhere defined functions, not merely almost everywhere defined functions.
2.  $a_{t+s} = a_t a_s$  plays no critical role in the proof. We include it for convenience.
3. That continuity lifts (i.e. the analog of Theorem 7.1 with continuity replacing measurability) is a result of Wigner [11], generalized (with local continuity only) by Bargmann [12] to more general groups. Given Section 8, Section 9, we actually prove their result.
4. Since  $\mathcal{H}$  is separable, weak measurability is equivalent to apparently stronger notions, see [13].
5. A similar result holds for  $\beta_t$  and  $\gamma_t$  given their relation to  $a$ 's.

**LEMMA 7.2.** *If  $U_1(t)$  and  $U_2(t)$  are weakly measurable, so is  $U_1(t)U_2(t)$ . If  $\psi(t)$  is a weakly measurable vector valued function, so is  $U_1(t)\psi(t)$ . If  $\psi(t)$  and  $\eta(t)$  are measurable, so is  $(\psi(t), \eta(t))$ .*

**PROOF.** Let  $\{\phi_n\}_{n=1}^\infty$  be an orthonormal basis. Since

$$(\psi, U_1(t)U_2(t)\psi) = \sum_{n=1}^\infty (\psi, U_1(t)\phi_n)(\phi_n, U_2(t)\psi),$$

it is measurable as the limit of measurable functions. The second and third statements follow similarly. ■

**LEMMA 7.3.** *There exists a unitary operator valued function  $U(\psi, \eta)$  defined on pairs of unit vectors so that:*

- (1) *If  $\phi$  is orthogonal to  $\psi$  and  $\eta$ , then  $U(\psi, \eta)\phi = \phi$*

- (2)  $U(\psi, \eta)\eta = \psi$

- (3) *If  $\psi(t)$  and  $\eta(t)$  are weakly measurable, then  $U(\psi(t), \eta(t))$  is weakly measurable.*

**PROOF.** If  $\psi = a\eta$  for some complex  $a$ , then define  $U(\psi, \eta) = 1 + (a-1)P_\eta$ . If  $\psi$  is orthogonal to  $\eta$ , define  $U(\psi, \eta)$  by

$$U(\psi, \eta)\phi = \phi + (\psi, \phi)(\eta - \psi) + (\eta, \phi)(\psi - \eta).$$

If  $(\psi, \eta) \neq 0$ , write  $(\psi, \eta) = |(\psi, \eta)|e^{i\theta}$ ,  $0 \leq \theta < 2\pi$  and  $U(\psi, \eta)$  so that (1) and (2) holds and  $U(\psi, \eta)\psi = e^{-2i\theta}\eta$ . Measurability (property (3)) is easy to check. ■

**LEMMA 7.4.** *If  $t \rightarrow a_t$  is a measurable family of Wigner automorphisms obeying  $a_{t+s} = a_t a_s$ , then each  $a_t$  is induced by a unitary. Moreover, for any  $\phi$ , we can choose  $\eta(t)$  weakly measurable so that  $a_t([\phi]) = [\eta(t)]$ .*

**PROOF.** Since  $a_t = (a_{t/2})^2$ ,  $a_t$  is induced by a unitary. Let  $\{\psi_m\}_{m=1}^\infty$  be an orthonormal basis. Let  $X_k = \{t \mid \langle [\psi_i], a_t[\phi] \rangle = 0, i=1, \dots, k-1; \langle [\psi_k], a_t[\phi] \rangle \neq 0\}$ . Each  $X_k$  is measurable, so we need only choose  $\eta(t)$  measurable on each  $X_k$ . Choose  $\eta(t)$  on  $X_k$  so that  $\langle \psi_k, \eta(t) \rangle > 0$  and  $a_t([\phi]) = [\eta(t)]$ . Let  $f_j(t) = \langle \psi_j, \eta(t) \rangle$ . We must show that each  $f_j(t)$  is measurable. As in Lemma 4.4,  $f_j$  is determined by  $|\langle \psi_j, \eta(t) \rangle|$ ,  $|\langle \psi_k + \psi_j, \eta(t) \rangle|/\sqrt{2}$  and  $|\langle \psi_k + i\psi_j, \eta(t) \rangle|/\sqrt{2}$ , so  $f_j$  is measurable. ■

**LEMMA 7.5.** *Let  $\dim \mathcal{H} = 2$  and  $\phi \in \mathcal{H}$  fixed. Let  $t \rightarrow a_t$  be measurable and induced by a unitary with  $a_t([\phi]) = [\phi]$  for all  $t$ . Choose  $U(t)$  inducing  $a_t$  so that  $U(t)\phi = \phi$ . Then  $U(t)$  is measurable in  $t$ .*

**PROOF.** Choose an isomorphism of  $\mathcal{H}$  and  $\mathbb{C}^2$  so that  $\phi$  corresponds to  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . In terms of Section 3,  $a_t$  corresponds to a rotation by angle  $\theta(t)$  in the 1-2 plane. Thus



$$U(t) = \begin{pmatrix} 1 & 0 \\ 0 & \exp(i\theta(t)) \end{pmatrix}$$

is measurable. ■

**PROOF OF THEOREM 7.1.** Choose  $\phi \in \mathcal{H}$ . By Lemma 7.4,  $\eta(t)$  measurable can be found so that  $[\eta(t)] = a_t([\phi])$ . Let  $\tilde{a}_t = \alpha_{U(\phi, \eta(t))} a_t$ . Then  $\tilde{a}_t$  is measurable and  $\tilde{a}_t[\phi] = [\phi]$ . Choose  $\tilde{U}(t)$  inducing  $\tilde{a}_t$  so that  $\tilde{U}(t)\phi = \phi$ . Since  $U(\phi, \eta(t))^{-1}\tilde{U}(t)$  induces  $a_t$ , we need only prove  $\tilde{U}(t)$  measurable. It suffices to show  $\tilde{U}(t)\psi$  is measurable for each  $\psi$  orthogonal to  $\phi$ . Choose  $\kappa(t)$  measurable so that  $\tilde{a}_t([\psi]) = [\kappa(t)]$  and let  $\beta_t = \alpha_{U(\psi, \kappa(t))} \tilde{a}_t$ . Then  $\beta_t$  is a measurable family leaving  $[\phi, \psi]$  invariant. Thus there is measurable  $V(t)$  on  $[\phi, \psi]$  so that  $V(t)\phi = \phi$ . Then, since  $U(\psi, \kappa(t))\tilde{U}(t)\phi = \phi$ , we conclude that  $\tilde{U}(t)\psi = U(\psi, \kappa(t))^{-1}V(t)\psi$ , so  $\tilde{U}(t)\psi$  is measurable. ■

The measurable choice  $t \rightarrow U(t)$  must obey  $U(t)U(s) = \omega(t, s)U(t+s)$  for some  $\omega(t, s) \in \mathbb{C}$  with modulus 1 on account of uniqueness up to phase and  $a_{t+s} = a_t a_s$ .

### §8. Multipliers for $R$

At this stage, we have a map  $a \mapsto U(a)$  from  $R$  to unitary operators which is weakly measurable and obeys

$$U(a)U(b) = \omega(a, b)U(a+b)$$

where  $\omega(a, b)$  is a measurable function from  $R \times R$  to  $\{a \in \mathbb{C} \mid |a| = 1\}$ . The associative law easily implies that  $\omega$  is a multiplier, where

**DEFINITION.** A (Borel) multiplier on  $R$  is a measurable map  $\omega : R \times R \rightarrow \{a \mid |a| = 1\}$  so that for all  $a, b, c \in R$

$$\omega(a, b)\omega(a+b, c) = \omega(a, b+c)\omega(b, c) .$$

**DEFINITION.** Given a measurable function  $\lambda : R \rightarrow \{a \mid |a| = 1\}$ , we define  $\partial\lambda$  by

$$\partial\lambda(a, b) = \lambda(a+b)\lambda(a)^{-1}\lambda(b)^{-1} .$$

Each  $\partial\lambda$  is easily seen to be a multiplier. The next step in the proof of Theorem 2.4 is:

**THEOREM 8.1.** Every Borel multiplier on  $R$  is of the form  $\omega = \partial\lambda$  for some  $\lambda$ .

The point of Theorem 8.1 is that  $\tilde{U}(a) = \lambda(a)U(a)$  is a weakly measurable map from  $R$  to the unitaries obeying  $\tilde{U}(a+b) = \tilde{U}(a)\tilde{U}(b)$ . The proof we give will basically follow Wigner [11] although we have been influenced by lectures of Mackey [14]. Before giving the proof, we note:

**REMARKS.**

1. The symbol  $\partial\lambda$  is motivated by the connection with cohomology of groups. Theorem 8.1 is essentially a statement that the two-dimensional (Borel) group cohomology of  $R$  with coefficients in the circle group is trivial.

2. Theorem 8.1 represents a subtle interplay between the group and measure structure of  $R$ . For  $R^2$  is isomorphic to  $R$  as a group (both are vector spaces over the rationals of the same dimension) and as a measure space (the Borel structures of any two complete separable metric spaces are isomorphic) but not as a group with a Borel structure. While  $R$  has no nontrivial multiplier,  $R^2$  does have nontrivial multipliers, e.g.

$$\omega(\langle a, b \rangle, \langle c, d \rangle) = \exp\left[\frac{1}{2}(ad - bc)\right]$$

the multiplier associated to the Heisenberg group  $\{\exp[i(aP - aQ)]\}$ .

3. For the structure of multipliers of more general groups, see Bargmann [12] or Mackey [15].

**LEMMA 8.2.** Without loss of generality, we may suppose that  $\omega(a, 0) = 1 = \omega(0, a)$ .

PROOF. Taking  $b = c = 0$  in the definition of a multiplier, we see that  $\omega(a, 0) = \omega(0, 0)$  for all  $a$  so  $\omega(a, 0)$  is a constant  $d$ . Similarly  $\omega(0, a) = \omega(0, 0)$ . Let  $\tilde{\omega} = (\omega) \partial \tilde{\lambda}$  where  $\tilde{\lambda}(a) = d$ . Then  $\tilde{\omega}$  is a multiplier with  $\tilde{\omega}(a, 0) = 1 = \tilde{\omega}(0, a)$ . If  $\tilde{\omega} = \partial \lambda'$ , then  $\omega = \partial[\lambda' \tilde{\lambda}^{-1}]$ . ■

Henceforth we suppose  $\omega(a, 0) = \omega(0, a) = 1$ .

LEMMA 8.3. *Without loss of generality, we may suppose that  $\omega(a, -a) = 1$  for all  $a$ .*

PROOF. First note that taking  $b = -a$ ,  $c = a$  in the definition of multiplier

$$\omega(a, -a)\omega(0, a) = \omega(a, 0)\omega(-a, a)$$

so that  $\omega(a, -a) = \omega(-a, a)$ . Define  $\lambda(a) = [\omega(a, -a)]^{1/2}$  where we take the square root with argument in  $[0, \pi)$ . Then

$$\partial \lambda(a, -a) = \lambda(0)\lambda(a)^{-1}\lambda(-a)^{-1} = \omega(a, -a)^{-1}$$

so that  $(\omega)(\partial \lambda) = \tilde{\omega}$  obeys  $\tilde{\omega}(a, -a) = 1$ . ■

Henceforth we suppose that  $\omega(a, -a) = 1$ .

LEMMA 8.4. *For any multiplier, there is a map  $a \rightarrow U(a)$  to the unitaries on some Hilbert space so that*

$$U(a)U(b) = \omega(a, b)U(a+b).$$

PROOF. Let  $\mathcal{H} = L^2(\mathbb{R}, d\mu)$  where  $\mu$  is Lebesgue measure. Define  $U(a)$  by

$$(U(a)f)(b) = \omega(b, a)f(a+b).$$

An immediate computation shows that

$$U(a)U(b) = \omega(a, b)U(a+b). \quad \blacksquare$$

LEMMA 8.5. *If  $\omega$  is a multiplier, then  $\omega(a, b) = \omega(b, a)$  for all  $a, b \in \mathbb{R}$ .*

PROOF. Let  $q(a, b) = \omega(a, b)/\omega(b, a)$  and let  $U$  be some  $\omega$ -representation. Then

$$U(a)U(b)U(a)^{-1} = q(a, b)U(b).$$

Moreover,  $\omega(a, b)\omega(-b, -a) = 1$  since by  $\omega(a, -a) = 1$ ,  $U(a)^{-1} = U(-a)$  whence  $(U(a)U(b))^{-1} = U(-b)U(-a) = \omega(-b, -a)U(-a-b)$  on the one hand and  $= (\omega(a, b)U(a+b))^{-1} = \omega(a, b)^{-1}U(-a-b)$  on the other hand. From the last two formulae, we conclude that  $q(a+b, c) = q(a, c)q(b, c)$  so that the measurability of  $q$  implies that  $q(a, b) = \exp(2\pi i a f(b))$  [to see this, just follow the arguments of Sections 9, 10 with  $\mathcal{H} = \mathbb{C}$ !]. Since  $q(a, a) = 1$ ,  $f(a) = a^{-1}n(a)$  where  $n(a)$  is an integer. Clearly  $q(a, b)q(b, a) = 1$  so  $ab^{-1}n(b) + ba^{-1}n(a) = n(a, b)$  for integers  $n(a)$ ,  $n(b)$ ,  $n(a, b)$ . Let  $b = a^3\sqrt{2}$ . Since  $3\sqrt{2}$ ,  $(3\sqrt{2})^{-1}$  and 1 are independent over  $\mathbb{Z}$ , we conclude  $n(a) = 0$  for all  $a$ , i.e.  $q(a, b) = 1$  for all  $a, b$ . ■

LEMMA 8.6. *For any multiplier, there is an irreducible family  $\{U(a)\}$  of unitaries on some Hilbert space so that  $U(a)U(b) = \omega(a, b)U(a+b)$ .*

PROOF. If  $\omega$  were continuous, we could form a locally compact group so that representations of the group with an additional property were in one-one correspondence to  $\omega$ -representations and then appeal to the theory of such representations. Since this is not available to us, we borrow the relevant argument!

Call a function  $\phi$  on  $\mathbb{R}$ ,  $\omega$ -positive definite if and only if  $\phi \in L^\infty(\mathbb{R})$  and

$$\int \overline{f(a)} f(b) \phi(b-a) \omega(-a, b) da db \geq 0$$

for all  $f \in L^1(\mathbb{R})$ . Given any  $\omega$ -rep,  $\phi(a) = (\psi, U(a)\psi)$  is  $\omega$ -positive definite, so by appealing to Lemma 8.4, there do exist  $\omega$ -positive definite functions.

Given  $\phi$   $\omega$ -positive definite, we put an inner product

$$(f, g) = \int \overline{f(a)} g(b) \phi(b-a) \omega(-a, b) da db$$

on  $L^1$  and form a Hilbert space in the obvious way. The maps  $(U(a)f)(b) = \omega(a, b-a)f(b-a)$  are easily seen to obey  $(U(a)U(b)f) = \omega(a, b)(U(a+b)f)$ ,  $U(0) = 1$  ( $f, U(a)g = (U(-a)f, g)$ ) so that  $U$  is an  $\omega$ -representation.

The set of  $\omega$ -positive definite functions is a compact convex subset of  $L^\infty$  in the weak  $*$  ( $L^1$ ) topology, so, by the Krein Millman theorem, there exist extreme points. Such extreme points are seen to lead to irreducible  $\omega$ -representations. ■

PROOF OF THEOREM 8.1. By Lemma 8.5,  $\{U(a)\}$  is a commuting family, so the irreducible representation of Lemma 8.6 is one dimensional by Schur's lemma. Thus, if this representation is multiplication by  $\lambda$ :

$$\lambda(a)\lambda(b) = \omega(a, b)\lambda(a+b)$$

i.e.  $\omega = \partial\lambda$ . ■

§9. Von Neumann's Theorem

In this section and the next, we complete the proof of Theorem 2.4 by proving that any weakly measurable family of unitary operators with  $U(a)U(b) = U(a+b)$  is of the form  $U(a) = e^{-iaH}$  for some self-adjoint  $H$ . Our two step proof consists in demonstrating two classical theorems. Our proofs follow those of Reed-Simon [13] to which the reader is referred for more information concerning the spectral theorem, self-adjointness, etc. Here we prove:

THEOREM 9.1 (von Neumann's Theorem [16]). Let  $t \rightarrow U(t)$  be a weakly measurable map from  $\mathbb{R}$  to the unitary operators on a separable Hilbert space,  $\mathcal{H}$ . Suppose  $U(t+s) = U(t)U(s)$ . Then  $U(t)$  is strongly continuous.

PROOF. Since the  $U(t)$ 's are uniformly bounded, it suffices to find a total set (subset whose finite linear combinations are dense) of  $\psi$ 's so

that  $t \rightarrow U(t)\psi$  is continuous. Pick an orthonormal subset  $\{\phi_n\}_{n=1}^\infty$  of  $\mathcal{H}$  and for  $a > 0$ , define  $\phi_n(a)$  as follows:

$$\eta \rightarrow \int_0^a (\eta, U(t)\phi_n) dt \equiv \ell_{n,a}(\eta)$$

defines a conjugate linear function with norm  $\leq a$ , so there is a vector  $\phi_n(a)$  with  $(\eta, \phi_n(a)) = \ell_{n,a}(\eta)$ . For obvious reasons we denote  $\phi_n(a)$  as

$$\int_0^a U(t)\phi_n dt .$$

A simple argument proves that

$$U(s) \int_0^a U(t)\phi_n dt = \int_s^{a+s} U(t)\phi_n dt$$

and thus that

$$\|(U(s) - U(s'))\phi_n(a)\| \leq \left\| \int_s^{s'} U(t)\phi_n dt \right\| + \left\| \int_{a+s}^{a+s'} U(t)\phi_n dt \right\| \leq 2|s-s'| .$$

It follows that  $t \rightarrow U(t)\phi_n(a)$  is continuous for all  $n$  and  $a$ , so we need only prove the  $\{\phi_n(a)\}$  total by the remarks above. Suppose  $\psi$  is orthogonal to all the  $\{\phi_n(a)\}$ . Then, for each  $n$ ,  $(\psi, U(t)\phi_n) = 0$  a.e. in  $t$ , so there must be a  $t_0$  with  $(\psi, U(t_0)\phi_n) = 0$  for all  $n$ . It follows that  $U(t_0)^{-1}\psi = 0$ , so  $\psi = 0$ . Thus the  $\{\phi_n(a)\}$  are total. ■

§10. Stone's Theorem

We complete the proof of Theorem 2.4 with:

**THEOREM 10.1** (Stone's Theorem [17]). *Let  $t \rightarrow U(t)$  be a strongly continuous map from  $\mathbb{R}$  to the unitaries operators so  $U(t+s) = U(t)U(s)$ . Then  $U(t) = e^{-iHt}$  for a unique self-adjoint operator  $H$ .*

**REMARKS.**

1. Since  $e^{-iHt}$  (which can be defined by the spectral theorem) is a strongly continuous unitary group, this sets up a one-one correspondence between such groups and *unbounded* self-adjoint operators.

2. If both  $U(t)$  and  $U'(t)$  generate the automorphisms  $\alpha(t)$ , then  $U(t)U'(t)^{-1}$  is a numerical representation of  $\mathbb{R}$  and so  $U(t) = U'(t)e^{i\alpha t}$  for some real  $\alpha$ , giving the uniqueness aspect of Theorem 2.4.

**PROOF.** Define an operator  $H$  with domain:

$$D(H) = \{\phi \mid U(t)\phi \text{ is differentiable at } t=0\}$$

and

$$H\phi = i \frac{d}{dt} [U(t)\phi] \Big|_{t=0} .$$

A simple argument shows that  $H$  is symmetric. Moreover, if  $U(t) = e^{-iAt}$  for some self-adjoint  $A$ , then  $H$  is closed and  $H = A$  (this yields uniqueness).

For any  $f \in C_0^\infty$ , the  $C^\infty$  functions of compact support and  $\phi \in \mathcal{H}$ , let  $\phi_f = \int f(t)U(t)\phi dt$  and let  $G$ , the Garding domain, be the finite span of the  $\phi_f$ . By an elementary computation  $\phi_f \in D(H)$  and  $H\phi_f = -i\phi_f'$  so  $G \subset D(H)$ . Moreover,  $U(s)\phi_f = \phi_{f(s)}$  where  $f^{(s)}(t) = f(t-s)$ , so  $G$  is left invariant by the  $U(t)$ 's. In addition,  $G$  is dense, since  $\phi_f \rightarrow \phi$  as  $f$  approaches a  $\delta$ -function in a suitable way (this uses the strong continuity!).

Suppose  $\psi$  is orthogonal to  $(H+i)[G]$ . Then, for any  $\eta \in G$ ,  $(\psi, U(t)\eta) = f(t)$  obeys:

$$\begin{aligned} \frac{d}{dt} f(t) &= (\psi, (-iH)U(t)\eta) && (G \text{ is invariant!}) \\ &= (i(H)^*\psi, U(t)\eta) \\ &= -f(t) . \end{aligned}$$

Thus  $f(t) = f(0)e^{-t}$ , since  $|f(t)| \leq \|\psi\| \|\eta\|$ ; taking  $t \rightarrow -\infty$ , we see that  $f(0) = 0$ , i.e.  $\psi$  is orthogonal to  $G$  and so zero. Similarly,  $(H-i)[G]$  is dense, so  $H$  is essentially self-adjoint on  $G$ .

Let  $A = \bar{H}$ . To complete the proof we need only show that  $U(t) = e^{-iAt}$ . Let  $\psi, \phi \in G$ . Then since  $\psi \in D(A)$  which is left invariant by  $e^{-iAt}$  and  $U(t)\phi \in G : \frac{d}{dt} (e^{-iAt}\psi, U(t)\phi) = 0$ , so  $U(t) = e^{-iAt}$ . ■

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