CHAPTER 22

NEW RIGOROUS EXISTENCE THEOREMS FOR PHASE TRANSITIONS IN MODEL SYSTEMS

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ABSTRACT

We describe a variety of new existence theorems for phase transitions in model systems including the classical Heisenberg and quantum spin 1/2 xy models in three or more dimensions. Two new tools are emphasized: the use of "chessboard estimates" to estimate contour probabilities in Peierls' type arguments and the use of "infrared bounds".

1. INTRODUCTION

In this note, I want to describe progress in the past two years in the rigorous theory of phase transitions involving some novel methods which have led to a variety of results including the first rigorous demonstration of a phase transition accompanied by a spontaneously broken continuous symmetry group. Two main themes are involved which, as we shall see, are intimately related. The first, involving a method for reducing estimates of contour probabilities in a Peierls' argument to bounding thermodynamic quantities, was developed by Glimm, Jaffe, and Spencer[1] to prove phase transitions in a model quantum field theory (equivalent to a classical statistical mechanics model) and extended to various lattice models including some quantum models by Fröhlich and Lieb.[2] The second, involving a new strategy of proving phase transitions was developed in the context of certain classical lattice gases and certain model quantum field theories by Fröhlich, Simon, and Spencer[3] and extended to certain quantum models by Dyson, Lieb, and Simon.[4] Extensions of both sets of ideas appear in a series of papers by Fröhlich, Israel, Lieb, and Simon.[5] For further discussion, the reader is referred to two lucid reviews of Fröhlich[6,7] and one of Fröhlich and Spencer.[8]

Both methods illustrate the rather close connection between constructive quantum field theory and rigorous statistical mechanics. The early years of this decade saw progress in the first subject due to the infusion of ideas from the second especially correlation inequalities[9,10] and high temperature[11] and related[12] expansions. More recently, there has been flux in the other direction. The particular high temperature expansion of Ref. 11 has been useful in some statistical mechanical models[13] and the methods discussed in this note owe a great deal to field theoretical intuition.

Models for which phase transitions were demonstrated for the first time with these methods include ($\nu \equiv$ number of space dimensions of lattice).

(a) Classical N-vector models (including the classical Heisenberg model: N=3) on nearest neighbor cubic lattices[3] and certain other lattices[5] if $\nu \geqslant 3$. The upper bound on the transition temperature in this case can be described by saying that with the normalization in which these models converge to the spherical model[14] as $N \to \infty$ [15], the upper bound is just the transition temperature of the spherical model. The bound is surprisingly accurate, e.g. if $N = \nu = 3$, the bound is off by only 9%.

(b) Quantum spin 1/2 nearest neighbor, simple cubic xy model in $\nu \geqslant 3$ dimensions.[4]

(c) Quantum nearest neighbor, simple cubic Heisenberg antiferromagnet for spin $s \geqslant 1$ and $\nu \geqslant 3$ and for $s = 1/2$ and ν sufficiently large.[4]

(d) Classical Heisenberg antiferromagnet with Fisher Coupling (i.e. nearest neighbor antiferromagnetic and next nearest neighbor ferromagnetic) in $\nu \geqslant 3$ dimensions and in non-zero external field.[5]

(e) $\nu = 2$ dimensional quantum anisotropic Heisenberg antiferromagnet in simple cubic nearest models[2] with values of the anisotropy parameter, ε, (defined by $zz + \varepsilon(xx + yy)$ interaction) much larger than those treated by Ginbre[16] and Robinson[16] (and, in particular with $\varepsilon \to 1$ as $s \to \infty$).

(f) $\nu = 2$ dimensional classical Heisenberg models[5] with long-range interactions $J(n) \sim n^{-\alpha}$ with $2 < \alpha < 4$.

(g) $\nu = 1$ or 2 dimensional quanum lattice models[5] with long-range interactions[17] $J(n) \sim n^{-\alpha}$ $1 < \alpha < 2$ $(\nu=1)$; $2 < \alpha < 4$ $(\nu=2)$.

(h) Certain quantum field theories including ϕ_2^4 (Ref. 1) ϕ_3^4 (Ref. 3) and Y_2 (Ref. 6).

In addition to these new results, insight has been gained on various older results:

(i) Dyson's result[18] that one-dimensional Ising models with long-range interactions have a phase transition; see Ref. 5 and 33.

(j) Malyshev's result[19] that the two-dimensional anisotropic classical Heisenberg model has a phase transistion for any anisotropy $\varepsilon < 1$; see Ref. 2.

(k) Dobrushin's results that the Ising antiferromagnet has a phase transition in an external field region $|\mu| \lesssim 1 - \tilde{\beta}T$, see Ref. 5.

We should also mention two results which were announced (in Refs. 4 and 2, respectively) whose proofs have developed gaps; it is our belief that the necessary infrared and chessboard estimates hold in these cases but some of the intermediate steps used in the existing proofs of these bounds for models (a) - (k) unfortunately fail (see Section 4).

($\sim$a) $\nu \geqslant 3$ dimensional quantum nearest neighbor simple cubic Heisenberg ferromagnet REMAINS OPEN

(~b) $\nu = 2$ dimensional quantum nearest neighbor simple cubic anisotropic Hesenberg antiferromagnet REMAINS OPEN, except for very small ε [16].

Having summarized the results, we deal with methods in the remainder of this note: in Section 2, we illustrate (following Ref. 2) the improved contour estimates by sketching (j) above; in Section 3, we illustrate (following Ref. 3,5) the use of infrared bounds in model (a); in Section 4, a variety of additional remarks can be found.

2. OS POSITIVITY, CHESSBOARD ESTIMATES AND THE PEIERLS' ARGUMENT

The first two notions in the title of this section are ideas developed in constructive quantum field theory. In their fundamental paper[21] on Euclidean region axiomatics, Osterwalder and Schrader emphasized a positivity condition which was the translation to the imaginary time region of the positivity of the inner product in the physical Hilbert space. This positivity condition, called OS positivity, physical positivity,[22] or reflection positivity turns out to be enough to reconstruct the Hamiltonian semigroup, e^{-tH}, an object which is known[9] to be the analog of the transfer matrix in lattice models of statistical mechanics. In some sense, there will be an analog of OS positivity in all lattice theories having a suitable kind of self-adjoint transfer matrix.[23]

In the development of $P(\phi)_2$ theories[24] various important bounds[25] were realized[26] to be proven quite easily using the Markov property of Nelson[27] and Symanzik,[27] a property which implies OS positivity. In the middle of 1975, many people[28] realized approximately simultaneously that OS positivity was sufficient for these bounds, that they were capable of generalization, and that many new results could be thereby obtained; in particular, Glimm, Jaffe, and Spencer,[1] realized the importance of these bounds to proofs of phase transitions. The bounds were further applied and abstracted by Fröhlich and Simon[29] who dubbed them "chessboard estimates". The relevance of these two notions to lattice gases was realized in Refs. 2 - 5, especially Refs. 2 and 5.

To describe the notions consider an Ising chain of $2n$ spins at sites $\pm 1/2, \pm 3/2, \ldots, \pm (n-1/2)$. Let $\langle\cdot\rangle_0$ denote the usual free expectation $2^{-2n} \sum_{\sigma_i=\pm 1} \cdot$ and for a function F of the spins σ_i, $i > 0$ let θF be the same function of the spins σ_{-i}, i.e. if $F = f(\sigma_{1/2}, \sigma_{3/2} \ldots)$, then $\theta F = f(\sigma_{-1/2}, \sigma_{-3/2}, \ldots)$. OS positivity for the state $\langle\cdot\rangle_0$ is the inequality for real F's that

$$\langle(\theta F)F\rangle_0 \geqslant 0$$

which follows by noting that

$$\langle(\theta F)F\rangle_0 = \langle\theta F\rangle_0 \langle F\rangle_0 = \langle F\rangle_0^2 \geqslant 0 \,. \tag{1}$$

Given a Hamiltonian $H(\sigma)$, we define as usual

$$\langle\cdot\rangle = \langle\cdot e^{-H}\rangle_0 \,/\, \langle e^{-H}\rangle_0 \,.$$

Notice that the nearest neighbor ferromagnet H, viz:

$$-H = \beta \sum_{i=-n+1/2}^{n-3/2} \sigma_i \sigma_{i+1} + \beta\, \sigma_{-n+1/2}\, \sigma_{n-1/2} \qquad \text{can be written}$$

$$-H = A + \theta A + C_1 \theta C_1 + C_2 \theta C_2 \quad \text{with} \quad A = \beta \sum_{i=1/2}^{n-3/2} \ldots,\ C_1 = \beta^{1/2}\, \sigma_{1/2}\ ;\ C_2 = \beta^{1/2}\, \sigma_{n-1/2} \,.$$

Therefore

$$\langle(\theta F)F\rangle \geqslant 0 \tag{2}$$

since $\langle e^{-H}\rangle_0 \langle(\theta F)F\rangle = \langle\theta(Fe^A)[Fe^A](e^{C_1\theta C_1})(e^{C_2\theta C_2})\rangle_0$ is seen to be positive upon expanding $e^{C_i\theta C_i}$ and using (1). Equation (2) is <u>OS positivity</u> for the interacting model.

Notice that since we chose periodic boundary conditions, one could obtain OS positivity with a variety of θ's involving reflecting about various ways of cutting the circle in half. Moreover, (2) implies a Schwarz inequality which we can iterate by using these different θ's. For example, consider a chain of 4 spins: $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ coupled by a Hamiltonian $-H = \beta(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_1)$. Let $f_1, \ldots, f_4$ by 4 real valued functions on ± 1. Then

$$<f_1(\sigma_1)f_2(\sigma_2)f_3(\sigma_3)f_4(\sigma_4)> \leq <f_1(\sigma_1)f_1(\sigma_4)f_2(\sigma_2)f_2(\sigma_3)>^{\frac{1}{2}}<f_4(\sigma_1)f_4(\sigma_4)f_3(\sigma_2)f_3(\sigma_3)>^{\frac{1}{2}}$$

$$\leq \prod_{i=1}^{4} <f_i(\sigma_1)\ldots f_i(\sigma_4)>^{\frac{1}{4}}$$

where in the first step we use the Schwarz inequality for the breakup (12)(34) (i.e. $F(\sigma_1,\sigma_2) = f_1(\sigma_1)f_2(\sigma_2)$; $G(\sigma_1,\sigma_2) = f_4(\sigma_1)f_3(\sigma_2)$; $\theta(\sigma_1) = \sigma_4$, $\theta(\sigma_2) = \sigma_3$ and use $<F\theta G> \leq <F\theta F>^{\frac{1}{2}} <G\theta G>^{\frac{1}{2}}$) and in the second step the breakup (41)(23). Clearly by repeating this argument and using it in different directions, we will obtain:

<u>Theorem 1</u> (Chessboard Estimates) Let Λ be a hypercube with sides $2^{n_1}\times\ldots\times 2^{n_\nu}$ in the lattice $\mathbb{Z}^\nu$ and for each α in Λ, let σ_α be an N-component spin. Let $d\mu$ be some measure on $\mathbb{R}^N$ (e.g. $\delta(|\sigma|-1)d^N\sigma$ for the N-vector model), let $H = \frac{1}{2}\beta \sum_{|\alpha-\gamma|=1,\mathrm{mod}\,\Lambda} (\sigma_\alpha-\sigma_\gamma)^2$ (mod Λ means periodic B.C.), and let $<\cdot> = Z^{-1}\int \cdot\, e^{-H} \prod_{\alpha\in\Lambda} d\mu(\sigma_\alpha)$. Then, for any real valued functions $\{f_\alpha\}_{\alpha\in\Lambda}$ on $\mathbb{R}^N$:

$$< \prod_{\alpha\in\Lambda} f_\alpha(\sigma_\alpha)> \leq \prod_{\alpha\in\Lambda} < \prod_{\beta\in\Lambda} f_\alpha(\sigma_\beta)>^{1/|\Lambda|}$$

<u>Remark</u> By more effort,[2] one can allow any hypercube with $2\ell_1\times\ldots\times 2\ell_\nu$ sides.

<u>Example 1</u>[2] Take $\nu = 2$, $N = 3$; $d\mu(\sigma_\alpha) = \delta(|\sigma|-1)d^3\sigma$ and $-H = \beta[\varepsilon \sum_{|\alpha-\gamma|=1} \sigma_\alpha\cdot\sigma_\gamma + (1-\varepsilon)\sum_{|\alpha-\gamma|=1} \sigma_\alpha^{(3)}\sigma_\gamma^{(3)}]$, i.e. the anisotropic classical Heisenberg model. By scaling $\sigma_\alpha^{(3)}$, this can be brought into the form of Theorem 1. Let P_α^+ (resp P_α^-) be the projection onto those states with $\sigma_\alpha^{(3)} > 0$ (resp $\sigma_\alpha^{(3)} < 0$) and let Q_α^δ be the projection onto those states with $|\sigma_\alpha^{(3)}| < (1-\delta)$. By the simple checkerboard estimate at inverse temperature, β:

$$\langle Q_0^\delta\rangle_\beta \leqslant \langle \prod_{\alpha\in\Lambda} Q_\alpha^\delta\rangle_\beta^{1/|\Lambda|}$$

$$\leqslant \tau_\delta e^{2\beta a(\delta)} \Big/ 1/2\ (1-\tau_\lambda)e^{2\beta b(\lambda)}$$

where τ_δ is the volume of the spherical strip $\{\sigma|\,|\sigma^{(3)}| < 1-\delta\}$, $a(\delta)$ is the maximum value of $\varepsilon\sigma_1\cdot\sigma_2 + (1-\varepsilon)\sigma_1^{(3)}\cdot\sigma_2^{(3)}$, when $|\sigma_i^{(3)}| < 1-\delta$ and $b(\lambda)$ is the minimum of the same function when $\sigma_i^{(3)} > 1-\lambda$. (The 1/2 comes from the fact that only one "cap" of the sphere is taken and the 2 from the fact that each pair is counted only once so there are $2|\Lambda|$ pairs.) Clearly, by choosing λ very small, we can arrange that $b(\lambda) > a(\delta)$, so $\lim_{\beta\to\infty} \langle Q_0^\delta\rangle = 0$, i.e. at low temperature each spin likes to point mainly up or mainly down.

Next let $\alpha_1,\ldots,\alpha_n$ be any n distinct spins and let $\beta_1,\ldots\beta_n$ be their nearest neighbor directly to the right. Then

$$\langle \prod_{i=1}^{n} P_{\alpha_i}^{+}\ P_{\beta_i}^{-}\rangle \leqslant \langle\text{conf}\rangle^{n/|\Lambda|} \equiv C_\Lambda(\beta)^{n/|\Lambda|} \tag{3}$$

where (conf) is the projection onto all states with $\sigma_\alpha^{(3)} > 0$ (resp. $\sigma_\alpha^{(3)} < 0$) for $\alpha_1 = 0,1,4,5,8,9$, etc.; (resp. $\alpha_1 = 2,3,6,7\ldots$) . Equation (3) is obtained by using the reflection idea but with a two-site block rather than a single-site block. Thus, + − reflects to − + + − and the next time to − + + − − + + − , leading to the condition given. By considerations similar to the above $C_\Lambda(\beta)^{1/|\Lambda|} \to 0$ as $\beta\to\infty$ (uniformly in $|\Lambda|$) by crude estimates of free energies and entropies.

(3) is ideal for plugging into a Peierls argument. Explicitly, we will show that $\langle P_\alpha^{+}\ P_\gamma^{-}\rangle \to 0$ as $\beta\to\infty$ uniformly in $|\alpha-\gamma|$ and Λ . Given that $\langle Q_0^\delta\rangle \to 0$, this will imply the existence of long-range order for β large . In the usual way, one surrounds α with contours where all spins are up inside the contour and all spins immediately outside are down. If the contour has length ℓ , it must have at least $\ell/4$ pairs of the kind considered in (3) (or of

one of the other three $-+$, $\pm$, $\mp$ possibilities) so $\langle P_\alpha^+ P_\gamma^- \rangle \leq \sum_{\ell=4}^{\infty}$ (# of contours of length ℓ) $C(\beta)^{\ell/4} \to 0$ as $\beta \to \infty$. This proves that long-range order occurs.

The moral of the above is that estimates of contour can be reduced to estimates of "thermodynamic quantities" via chessboard estimates.

3. INFRARED BOUNDS

We want to illustrate the new method of Ref. 3 by proving a phase transition in the classical isotropic Heisenberg model in $\nu = 3$ dimensions; we use ideas from Refs. 4 and 5 as well as Ref 3. For a fixed model of this type (see Theorem 1) let $\langle\cdot\rangle_\Lambda$ denote a finite volume expectation. Griffiths[30] has proven that (for models even under $\sigma_\alpha \to -\sigma_\alpha$)

$$m^2 \geq \lim_{\Lambda\to\infty} \langle(\frac{1}{|\Lambda|} \sum_{\alpha\in\Lambda} \vec{\sigma}_\alpha)^2\rangle_\Lambda \qquad (4)$$

where m denotes the infinite volume spontaneous magnetization so that $m \neq 0$ implies long-range order and multiple phases: (4) is quite easy to prove (see also Ref. 4) and has obvious interpretations in terms of bulk fluctuations.

Let Λ^* be the dual volume to Λ , i.e. if Λ is a cube with $\ell_1 \times \ell_2 \times \ell_3$ sites then Λ^* is the set of p's with $p_i = 2\pi n_i/\ell_i$; $n_i = 0,1,\ldots,\ell_i-1$. Introduce the Fourier spins: $\hat{\sigma}_p = \sum_{\alpha\in\Lambda} |\Lambda|^{-\frac{1}{2}} e^{-ip\cdot\alpha} \sigma_\alpha$. Then H becomes $H_\Lambda = \beta \sum_{p\in\Lambda^*} E_p \hat{\sigma}_p \cdot \hat{\sigma}_{-p}$ where E_p is the spin wave energy $E_p = 3-\cos p_1 - \cos p_2 - \cos p_3$. Since $\{|\Lambda|^{-\frac{1}{2}} e^{-ip\cdot\alpha} | p\in\Lambda^*\}$ is an orthonormal basis for $\mathbb{C}^{|\Lambda|}$ we have the Plancherel relation:

$$\frac{1}{|\Lambda|} \sum_{p\in\Lambda^*} \langle\hat{\sigma}_p \hat{\sigma}_{-p}\rangle_\Lambda = 1 \qquad (5)$$

since $\sigma_\alpha^2 = 1$. Equations (4) and (5) present a simple "spin wave" picture of phase transitions: Since $|\Lambda^*| = |\Lambda|$, (5) says that under "normal" circumstances, each $\langle\hat{\sigma}_p\hat{\sigma}_{-p}\rangle_\Lambda$ is to be expected to be $O(1)$ as $\Lambda \to \infty$. Equation (4) says that for a phase transition, we need $\langle\hat{\sigma}_{p=0}^2\rangle_\Lambda$ to be $O(|\Lambda|)$ as $\Lambda \to \infty$.

The basic bound we require is

$$\langle\hat{\sigma}_p\hat{\sigma}_{-p}\rangle_\Lambda \leqslant 3/(2\beta E_p) \qquad (6)$$

From this bound we see that

$$\lim_{|\Lambda|\to\infty}\Big[\frac{1}{|\Lambda|}\sum_{p\neq 0}\langle\hat{\sigma}_p\hat{\sigma}_{-p}\rangle_\Lambda\Big] \leqslant 3/2\beta \int_{|p_i|\leqslant\pi} E_p^{-1}\, d^3p/(2\pi)^3 \equiv 3/2\beta\, G_3(0)$$

since the Fourier sum approaches a Fourier integral. This says that if $\beta > 3/2\, G_3(0)$, then $\lim |\Lambda|^{-1} \langle\hat{\sigma}^2_{p=0}\rangle \neq 0$ so there is long-range order. To summarize the bound (6) and the sum rule (5) force macroscopic occupation of the $p = 0$ mode and thereby long-range order. $\nu \geqslant 3$ is essential for this argument (as it must be since[31] there is no spontaneous magnetization if $\nu = 2$) because the analog of $\int E_p^{-1}\, d^3p$ diverges if $\nu = 2$ since $E_p \sim 1/2\, p^2$ for p small.

Before proving (6) we make two remarks about it: looking at the formula for H and rewriting (6) as $E_p\langle\hat{\sigma}_p\hat{\sigma}_{-p}\rangle_\Lambda \leqslant 3/2\, kT$, we see that (6) is a kind of "equipartition inequality"(the 3 comes from the 3 degrees of freedom, σ^i_α, $i=1,2,3$); since the modes are far from uncoupled there is no reason for an equipartition theorem but an inequality holds nonetheless. Secondly, the field theoretic analog of (6) is trivial which is what suggested it in the first place (in the field theory case,[3] the analog of the sum rule (5) is non-trivial): it turns out to say that $\hat{F}(k)$, the Fourier transform of the two-point function, obeys $\hat{F}(k) \leqslant k^{-2}$; but $\hat{F}(k)$ has a Källén-Lehmann representation $\hat{F}(k) = \int d\rho(m^2)\, (k^2+m^2)^{-1}$ with $\int d\rho(m^2) = 1$ (for canonical theories) so that $(k^2+m^2)^{-1} \leqslant k^{-2}$ yields $\hat{F}(k) \leqslant k^{-2}$.

<u>Theorem 2</u> (Infrared Bounds) Under the hypotheses of Theorem 1:

$$\langle\hat{\sigma}_p\hat{\sigma}_{-p}\rangle \leqslant N/2\beta E_p$$

<u>Proof</u> Define for $(h_\alpha) \in (\mathbb{R}^N)^{|\Lambda|}$:

$$Z(h_\alpha) = \int e^{-H} \prod_{\alpha\in\Lambda} d\mu(\sigma_\alpha - h_\alpha)$$

We first claim that

$$Z(h_\alpha) \leqslant Z(0) \qquad (7)$$

It suffices to prove (7) in case $d\mu(\sigma) = F(\sigma)d^N\sigma$ with F everywhere non-zero and then use a limiting argument. In that case, let $G_\alpha(\sigma) = F(\sigma-h_\alpha)/F(\sigma)$ so that

$$Z(0)^{-1}Z(h_\alpha) \equiv \langle \prod_{\alpha\in\Lambda} G_\alpha(\sigma_\alpha)\rangle \leqslant \prod_{\alpha\in\Lambda} \langle \prod_{\beta\in\Lambda} G_\alpha(\sigma_\beta)\rangle^{1/|\Lambda|} = 1$$

where the inequality follows from the chessboard estimates and the last step follows by noting that

$$\langle \prod_{\beta\in\Lambda} G_\alpha(\sigma_\beta)\rangle = \int e^{-H} \pi d\mu_\beta(\sigma_\beta-h_\alpha)/\int e^{-H}d\mu_\beta(\sigma_\beta) = 1$$

since H is only a function of $\sigma_\beta - \sigma_\gamma = (\sigma_\beta-h_\alpha) - (\sigma_\gamma-h_\alpha)$. This proves (7).

Now we rewrite (7) by making the change of variables, $\sigma_\alpha - h_\alpha \to \sigma_\alpha$ so $H(\sigma_\alpha) \to H(\sigma_\alpha+h_\alpha) = H(\sigma_\alpha) + \beta \sum_{|\alpha-\gamma|=1} (h_\alpha-h_\gamma)(\sigma_\alpha-\sigma_\gamma) + \beta/4 \sum_{|\alpha-\gamma|=1} (h_\alpha-h_\gamma)^2$. Thus (7) becomes:

$$\langle\exp[-\beta \sum_{|\alpha-\gamma|=1} (h_\alpha-h_\gamma)\cdot(\sigma_\alpha-\sigma_\gamma)]\rangle \leqslant \exp(\beta/4 \sum_{|\alpha-\gamma|=1} (h_\alpha-h_\gamma)^2) \tag{8}$$

from which we conclude that

$$\beta^2 \langle(\Sigma(h_\alpha-h_\gamma)\cdot(\sigma_\alpha-\sigma_\gamma))^2\rangle \leqslant \beta/2\, \Sigma(h_\alpha-h_\gamma)^2 \tag{9}$$

by taking $h_\alpha \to \lambda h_\alpha$ and expanding to second order about $\lambda = 0$. (9) has only been proven for h_α real but if $(\ )^2$ is interpreted as $|\cdot|^2$ it extends immediately to complex h_α. Take $h_\alpha = e^{ip\cdot\alpha}(1,0,\ldots)$ in (9) and one obtains

$$\langle\hat\sigma_p^{(1)}\hat\sigma_{-p}^{(1)}\rangle \leqslant 1/2\beta E_p$$

so the theorem follows by summing over components of σ. Q.E.D.

4. ADDENDA

Finally we make a series of remarks about extensions and problems:

1. The natural setting for proving theorems 1 and 2 is the following:[5] Let $\mathfrak{A}$ be an algebra (of observables), and $\mathfrak{A}_+$, $\mathfrak{A}_-$ two subalgebras with an automorphism $\theta : \mathfrak{A}_+ \to \mathfrak{A}_-$. An expectation $\langle\cdot\rangle_0$ on $\mathfrak{A}$ is said to obey

Generalized OS Positivity (GOS) (resp. Generalized Schwarz Inequality (GSI)) if and only if for all $A_1,\dots,A_n \in \mathfrak{A}_+$ and $B_1,\dots,B_n \in \mathfrak{A}_+$, we have $\langle A_1(\theta A_1)\dots A_n(\theta A_n)\rangle \geq 0$ (resp. $\langle A_1(\theta B_1)\dots A_n(\theta B_n)\rangle \leq \langle A_1(\theta A_1)\dots A_n(\theta A_n)\rangle^{\frac{1}{2}} \langle B_1(\theta B_1)\dots B_n(\theta B_n)\rangle^{\frac{1}{2}}$) . If $\langle\cdot\rangle$ obeys GOS, and $-H = A + \theta A + \Sigma\, B_i\theta B_i$, then by the argument in Theorem 1 and the Trotter product formula $\langle\cdot\rangle = Z^{-1}\langle\cdot e^{-H}\rangle_0$ is OS positive in the sense that $\langle A\theta A\rangle \geq 0$ so that with translation invariance, one can obtain Chessboard type estimates. If $\langle\cdot\rangle_0$ obeys GSI, then

$$\langle\exp(A+\theta B+\Sigma\, C_i\theta D_i)\rangle^2 \leq \langle\exp(A+\theta A+\Sigma\, C_i\theta C_i)\rangle \langle\exp(B+\theta B+\Sigma\, D_i\theta D_i)\rangle$$

by using the Trotter product formula, expanding the exponential in $\exp((\Sigma\, C_i\theta D_i)_n)$ and using GSI. This bound and translation invariance imply analogs of Equation (8).

2. When can one prove GOS and GSI? Clearly, if $\mathfrak{A}_+$ and $\mathfrak{A}_-$ commute with each other and if $\langle\cdot\rangle_0$ is OS positive, we have $\langle A_1\theta A_1\dots A_n\theta A_n\rangle = \langle A_1\dots A_n\theta(A_1\dots A_n)\rangle \geq 0$ and similarly GSI. Three cases of interest occur: (i) (Classical systems; reflections about planes containing no bounds[3]) This is the case discussed in Sections 2 and 3; $\mathfrak{A} = L^\infty(M\times M)$; $\mathfrak{A}_+ = L^\infty(M)$ as $F(m,m') = F(m)$; $\mathfrak{A}_- = L^\infty(M)$ as $F(m,m') = F(m')$ and $(\theta F)(m,m') = f(m')$ if $F(m,m') = f(m)$. $\langle\cdot\rangle_0 = \int \cdot\, d\mu(m)d\mu(m')$; $\mathfrak{A}_+$ commutes with $\mathfrak{A}_-$ and $\langle f(\theta f)\rangle = \int F(m)F(m')d\mu(m)\, d\mu(m') = (\int F(m)d\mu(m))^2 \geq 0$. (ii) (Classical systems; reflections about planes containing bounds[5]) This allows one to deal with lattices like the face centered cubic where the perpendicular bisectors of bounds contain lattice sites $\mathfrak{A} = L^\infty(M\times N\times M)$; $\mathfrak{A}_+$ = functions of m and n , $\mathfrak{A}_-$ = functions of n and m' ; $\theta: F(m,n) \mapsto F(m',n)$; $\langle\cdot\rangle_0 = \int \cdot\, d\mu(m)d\nu(n)d\mu(m')$ and $\langle f(\theta f)\rangle = \int d\nu(n) (\int f(m,n)d\mu(m))^2 \geq 0$. (iii) ("Real" Quantum Systems; reflections about planes containing no sites.) Here $\mathfrak{A}$ is an algebra of matrices on $\mathcal{H}\otimes\mathcal{H}$ which are all simultaneous <u>real</u> , $\mathfrak{A}_+$ (resp. $\mathfrak{A}_-$) is matrices of the form $A\otimes 1$ (resp. $1\otimes A$) and $\theta(A\otimes 1) = 1\otimes A$, $\langle\cdot\rangle_0 = \mathrm{Tr}(\cdot)$. Then $\mathfrak{A}_+$ commutes with $\mathfrak{A}_-$ and $\langle B\theta B\rangle = \mathrm{Tr}(B)^2 > 0$ <u>since Tr(B) is real</u> . This allows one to treat models like

the x,y model and also, in effect, the antiferromagnet (where the Hamiltonian becomes, after suitable rotations, $-H = \sigma_x\sigma'_x + \sigma_z\sigma'_z + (i\sigma_y)(i\sigma'_y)$ and $\sigma_x, \sigma_z, i\sigma_y$ are simultaneously real) but <u>not</u> the ferromagnet.

3. There are two cases of interest in which GOS fails: (i) ("Real" quantum systems, reflection about planes with sites). Here $\mathfrak{A}$ is an algebra of real matrices on $\mathcal{K}\otimes\mathcal{K}\otimes\mathcal{H}$ with $\mathfrak{A}_+$ = matrices of $A\otimes B\otimes 1$ and $\theta(A\otimes B\otimes 1) = 1\otimes B\otimes A$. Here $\mathfrak{A}_+$ and $\mathfrak{A}_-$ do not commute. If $\langle\cdot\rangle_0 = \mathrm{Tr}$ it always happens that $\langle\cdot\rangle_0$ is OS positive but[5] GOS may fail. (ii) (General Quantum Systems) Note that if $\mathfrak{A}_+$ commutes with $\mathfrak{A}_-$ and $\langle\cdot\rangle_0$ is OS positive, then $\langle(A_1\theta A_1+\ldots+A_n\theta A_n)^k\rangle \geqslant 0$. For two quantum Pauli spins $\mathrm{Tr}((\vec{\sigma}_1\cdot\vec{\sigma}_2)^3) < 0$ so OS positivity fails; in fact $\langle(\sigma_x\sigma_y\sigma_z)(\sigma'_x\sigma'_y\sigma'_z)\rangle_0 = -1$. We should emphasize that we do not know that OS positivity restricted to functions of σ_z fails for the interacting expectation in either case (this is all that is missing to extend Example 1 to quantum ferromagnets); all we know is that the proofs thus far used, all of which expand an exponential, do fail.

4. One can ask when $\sum_{n\neq m} J(|n-m|)s_{n+\frac{1}{2}}\, s_{m+\frac{1}{2}}$ is of the form $A+\theta A+\Sigma\, C_i\theta C_i$ in terms of the θ of Section 2. For finite range J's, only for $J(n) = \delta_{n1}$ but infinite range σ's of the form $|n|^{-\alpha}$ are also OK.[5] In particular, one obtains infrared bounds in that case. If one notes that $\int dp\, E_p^{-1} < \infty$ only for $\alpha < 2$, one has a simple proof[5] of Dyson's result [18] (see Ref. 33).

5. There is one major step in pushing the argument of Section 3 through in the quantum case (like the xy model) that we haven't yet mentioned. The sum rule (5) (with suitable replacements for "1") is true for the thermal expectations. But the infrared bound, (6), holds for a different kind of "two point function", namely $(\hat{\sigma}_p, \hat{\sigma}_{-p})$ where (A,B) is the "DuHamel two-point function":

$$(A,B) = \mathrm{Tr}(e^{-H})^{-1}\int_0^1 \mathrm{Tr}(e^{-xH}A\, e^{-(1-x)H}B)\, dx$$

This enters since going from (7) to (9) involves an expansion of $Z(\lambda h)$ to second order and

$$\mathrm{Tr}(e^{-H+\lambda A}) = \mathrm{Tr}(e^{-H})\ [1+\lambda\ \langle A\rangle + 1/2\ \lambda^2(A,A) + 0(\lambda^3)]$$

To complete the proof of phase transitions one needs an upper bound on $g(A) = 1/2\ \langle A^*A+AA^*\rangle$ in terms of $b(A) = (A^*,A)$. The following bound of Bruch and Falk[32] (rediscovered in Ref. 4) does the trick: $b(A) \geq g(A)f(c(A)/4g(A))$ where $c(A) = \langle [A^*,[H,A]]\rangle$ and f is the function $f(x \tanh x) = x^{-1} \tanh x$.

6. Among the important open questions are the following: (i) Do infrared bounds hold for the nearest neighbor isotropic quantum Heisenberg ferromagnet? (ii) Is the nearest neighbor anisotropic Heisenberg ferromagnet OS positive for functions of the σ_z's alone? (iii) For which interactions $J(n)$ do the infrared bounds hold in the sense $\langle \hat{\sigma}_p \hat{\sigma}_{-p}\rangle \leq N/2\beta E_p$ with $E_p = 1/2 \sum_n J(n)(1-\cos np)$ with $(n \in Z^\nu)$?

FOOTNOTES

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