COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
27. RANDOM FIELDS, ESZTERGOM (HUNGARY), 1979.

IDENTIFYING THE CLASSICAL LIMIT OF A QUANTUM SPIN SYSTEM

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ABSTRACT

We consider quantum spin systems where each site supports a representation $U^{(l)}$ of a semisimple Lie algebra, g. By combining Lieb's method (for the case $g = \mathrm{so}(3)$) and Weyl's weight theory, we identify the proper classical phase space, Γ (S^2 in Lieb's case), for a classical limit as $l \to \infty$. Γ turns out to be a coadjoint orbit; for example, if $U^{(l)}$ is the degree l spinor representation of $\mathrm{so}(2n)$, then Γ is the manifold of $n \times n$ orthogonal and antisymmetric matrices. The possible relevance to a proof of the Lee – Yang theorem for the S^n model is discussed.

1. INTRODUCTION

In this note, we want to describe some of the results in a fuller paper to be published elsewhere [11]. The starting point of our analysis is a beautiful paper of Lieb [8]. Consider a finite array $\{\alpha\}_{\alpha \in \Lambda}$ of sites and for each α consider an independent set of spin l quantum ($\mathrm{so}(3)$)

*Supported by USNSF grant MCS-78-01885.

spins, $L_{j,\alpha}^{(l)}$ (independent means we operate on $\bigotimes_{\alpha \in \Lambda} C^{2l+1}$) and let $H(x_\alpha)$ be a multiaffine function on $|\Lambda|$-three vectors $x_\alpha = (x_{1,\alpha}, x_{2,\alpha}, x_{3,\alpha})$ i.e. a sum of monomials $c x_{j_1,\alpha_1} \ldots x_{j_k,\alpha_k}$. Since, spins at different sites commute, one can give an unambiguous meaning to the operator $H(L_\alpha^{(l)})$. Define

$$Z_Q^l(\gamma) = (2l+1)^{-|\Lambda|} \operatorname{Tr}(\exp(-H(\gamma l^{-1} L_\alpha^{(l)}))).$$

Let Γ denote the two sphere, S^2, and let

$$Z_{cl}(\gamma) = (4\pi)^{-|\Lambda|} \int_{\Gamma^{|\Lambda|}} \exp(-H(\gamma x_\alpha)) \prod_\alpha d\Omega(x_\alpha)$$

where $d\Omega(x_\alpha)$ is the usual (unnormalized) measure on Γ. Then Lieb proves that for all l, γ.

Theorem ([8]).

(1.1) $$Z_{cl}(\gamma) \leq Z_Q^l(\gamma) \leq Z_{cl}((1 + l^{-1})\gamma).$$

The point of this result is that the estimates are good enough to prove convergence of the spin l pressure to the classical pressure as $l \to \infty$. We are mainly interested in the consequence:

(1.2) $$\lim_{l \to \infty} Z_Q^l(\gamma) = Z_{cl}(\gamma).$$

Our goal here is to examine what happens when so (3) is replaced by a more general Lie algebra. Since we want groups with lots of finite dimensional representations and since Abelian pieces are "uninteresting", we may as well consider semisimple Lie algebra in compact form i.e. so (n), su (n), usp (n), the five exceptional algebras and direct sums of such. In fact, in this note, we will restrict ourselves to so (n). We note that one other case was worked out by Fuller and Lenard [3], namely the spherical harmonic representations of so (n). They found that Γ is a Grassman manifold of oriented two planes in n space, but the exact choice of Γ was found by adhoc means. Our goal here is to illustrate what Γ is in general. In Section 4, we recover the Fuller – Lenard result.

The problem we consider is of some purely mathematical interest but our interest is motivated in part by an idea of Dunlop and Newman [2]. These authors prove a Lee – Yang theorem for S^2 classical spins by using the limit result (1.2) and the known result for quantum spins. As Newman has emphasized [9] this is a rather indirect way of obtaining information on classical spins but it is the only proof we know for the S^2 classical Lee – Yang theorem! We regard the extension of Lieb's result as a possible first step in extending the Dunlop – Newman proof to S^n-spins. One's first hope that S^n will arise as a Γ for so $(n+1)$ is a false one – Γ always supports an invariant symplectic form (not an unreasonable feature of a classical mechanical system!) and only S^2 has such a form. However, as we explain Section 5, it is still possible to reduce the Lee – Yang theorem for S^m to an quantum Lee – Yang theorem which at the present moment remains unproved.

It is a pleasure to thank B. Kostant, A. Lenard and S. Levin for valuable discussions.

2. FUNDAMENTAL WEIGHTS: THE MAIN THEOREM.

In describing the results, we will suppose that the reader has some previous exposure to the Weyl theory of representations of the compact Lie groups. For readable accounts see Samelson [10], which emphasizes the algebraic aspects or Adams [1], which emphasizes the geometric content; there is also a brief description in [11].

The Lie algebra, g, supports a natural action of its Lie group G; i.e. for any $x \in G$, there is a linear map $A(x)$ on g, for example if $g = \text{so}(n)$, if $L \in g$ generates rotations about an axis e, then $A(x)L$ generates rotations about xe, the image of e under the rotation x. A is called the adjoint action and by duality it induces a natural action A^* on g^* the dual of g, i.e. $A^*(x) = [A(x^{-1})]^*$. A^* is called the coadjoint action and special attention should be paid to the orbits in g^* under the action, i.e. $\{A^*(x)l \mid x \in G;\ l \text{ fixed}\}$. These *coadjoint orbits* play a role in the Kostant – Souriau [5], [12] method of geometric quantization and from their point of view our result that they are the classical limit manifolds is most natural. Given a coadjoint orbit, Γ, and $l_0 \in \Gamma$, one defines a bilinear functional ω_{l_0} on the cotangent space $T^*_{l_0}(\Gamma)$ as follows: elements of $T^*_{l_0}(\Gamma)$ are naturally thought of as elements of $T^*_{l_0}(g^*) = g$, i.e. as elements of the Lie algebra. Given $L, M \in g$, we define $\omega_{l_0}(L, M) = l_0([L, M])$ where $[\cdot, \cdot]$ is Lie bracket. ω is easily sent to be non-degenerate and continuous in l_0 making Γ into a symplectic manifold, proving, by the way that $\dim \Gamma$ is even.

The semi-simple compact Lie algebras support a natural positive definite inner product (negative of the Killing form), e.g. for so (n), $(L, M) = \mathrm{Tr}\,(L^*M)$. Thus the adjoint action is naturally equivalent to the coadjoint action but it is wortwhile distinguishing them for clarity.

In g, one chooses a maximal Abelian subalgebra h. For example in so $(2n)$ or so $(2n+1)$, one conventionally chooses $L_{12}, L_{34}, \ldots, L_{2n-1,2n}$ where L_{ij} generates rotations in the ij plane; i.e. if g is viewed as real antisymmetric matrices, then L_{ij} has 1 in the ij position, -1 in the ji position and 0 elsewhere. Given a representation U of g, one simultaneously diagonalizes the $L \in h$. The common eigenvalues define linear functionals λ on h called *weights*. We will extend λ to a $\tilde{\lambda}$ on all of g by setting it equal to zero on $h^\perp$, the orthogonal complement of h in the natural inner product. With this definition, if

(2.1) $$U(L)w = \lambda(L)w \qquad (L \in h)$$

then

(2.2) $$(w, U(L)w) = \tilde{\lambda}(L) \quad \text{all} \quad L \in g.$$

The familiar integrality conditions in so (3), (i.e. that L_z has eigenvalues in $\left(0, \pm\frac{1}{2}, \pm 1, \ldots\right)$) have an analog in a general algebra. The weights λ must lie in a discrete lattice, W, of dimension $r = \dim(h)$ ($= n$ for so $(2n)$ or so $(2n+1)$). For example in so (m), the condition is $\lambda(L_{2i-1,2i}) \in \frac{1}{2}Z$; $\lambda(L_{2i-1,2i} - L_{2j-1,2j}) \in Z$; Z = integers.

A choice leads to a natural basis $\lambda_1, \ldots, \lambda_r$ for W called *the fundamental weights*. For example, in the case of so $(2n)$ or so $(2n+1)$

$$\lambda_1 = w_1, \lambda_2 = w_1 + w_2, \ldots, \lambda_{r-2} = w_1 + \ldots + w_{r-2}$$

and for so $(2n)$

$$\lambda_{r-1} = \frac{1}{2}(w_1 + \ldots + w_{r-1} - w_r);$$

$$\lambda_r = \frac{1}{2}(w_1 + \ldots + w_{r-1} + w_r)$$

and for so $(2n+1)$

$$\lambda_{r-1} = w_1 + \ldots + w_{r-1}; \quad \lambda_r = \frac{1}{2}(w_1 + \ldots + w_r)$$

where w_j is the linear functional with

$$w_j(L_{2i-1,2i}) = \delta_{ij}.$$

Moreover, a special role is played by the weight

(2.3) $$\delta = \lambda_1 + \ldots + \lambda_r.$$

Any weight $\lambda \in W$ can be uniquely written $\lambda = n_1\lambda_1 + \ldots + n_r\lambda_r$ with $n_i \in Z$. We order W by $\sum n_i\lambda_i \geqslant \sum m_i\lambda_i$ if and only if $n_i \geqslant m_i$. Given any irreducible representation, U, of g, there is among its weights a unique one maximal in this order. This weight lies in

$$W^+ = \{\sum n_i\lambda_i \mid n_i \geqslant 0\}.$$

Moreover, two irreducible representations are equivalent if and only if their maximal weights agree and any $\lambda \in W^+$ is the maximal weight of an irreducible representation; that is there is a one-one correspondence between irreducible representations, $U^{(\lambda)}$, of g and weights $\lambda \in W^+$.

For example, if $g = \mathrm{so}\,(2n)$, $U^{(\lambda_1)}, \ldots, U^{(\lambda_{r-2})}$ are the rank $1, 2, \ldots, r-2$ totally antisymmetric tensors; $U^{(\lambda_{r-1})}$ and $U^{(\lambda_r)}$ are the spinor representations of which we will have more to say in Section 4. For $j \leqslant r-2$ and $l \in Z$, $U^{(l\lambda_j)}$ are the representations with $j \times l$ Young tableaux, i.e. the l fold symmetric tensor product of $U^{(\lambda_j)}$ with various partial traces set to zero. In particular, $U^{(l\lambda_1)}$ is just the degree l sperical harmonics.

We can now state our main theorem.

Theorem. *Let* g *be a compact semi-simple Lie algebra of dimension* q *and let* $r = \dim(h)$. *Fix one of the fundamental weights* $\lambda_1, \ldots, \lambda_r$ *(call it* λ*) and fix a Hamiltonian* H *depending in a multiaffine way on* $|\Lambda|$ *q-vectors,* x_α. *Let* $L_\alpha^{(l)}$ *denote independent copies of the generators of* g *in the representations* $U^{(l\lambda)}$ *and let*

$$Z_Q^l(\gamma) = (d_l)^{-|\Lambda|} \operatorname{Tr}(\exp(-H(\gamma l^{-1} L_\alpha^{(l)}))$$

where $d_l = \dim(U^{(l\lambda)})$. *Let* Γ *be the coadjoint orbit containing* λ *and let* $d\mu$ *be the natural invariant normalized measure on* Γ. *Define*

$$Z_{cl}(\gamma) = \int_{\Gamma^{|\Lambda|}} \exp(-H(\gamma x_\alpha)) \prod_\alpha d\mu(x_\alpha).$$

Then

$$(2.4) \qquad Z_{cl}(\gamma) \leq Z_Q^l(\gamma) \leq Z_{cl}((1 + l^{-1} a)\gamma)$$

where

$$a = 2\frac{\langle \lambda, \delta \rangle}{\langle \lambda, \lambda \rangle}$$

with δ *the weight* (2.3) *and* $\langle \cdot, \cdot \rangle$ *the natural inner product on* g^*.

Remark 1. This theorem is probably true for any $\lambda \in W^+$, not just $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_r$ but at this point, in one annoying technical place (see Section 3), the restriction to the fundamental weights enters in the upper bound (the lower bound is always true).

Remark 2. The weight δ enters magically (it often does!) through the fact that the Casimir operator ($\sum L_{2i-1,2i}^2$ for so (m)) has eigenvalue $\langle \lambda, \lambda \rangle + 2\langle \lambda, \delta \rangle$ in the representation $U^{(\lambda)}$.

3. LIEB'S METHOD AND COHERENT VECTORS

Lieb's method [8] depends on three basic ideas:

(1) Use coherent vectors;

(2) Get lower bounds by using Jensen's inequality;

(3) Get upper bounds by using a Golden – Thompson inequality.

The usefulness of these ideas is illustrated by noting that they provide a rather quick proof of the classical limit of the partition function also in non-relativistic systems with a finite number of degrees of freedom [13], [11] and also control on the classical limit of the pressure in realistic models [11].

Steps (2) and (3) are summarized in the following result:

Theorem. *Let* $(X, d\mu)$ *be a probability measure space. Let* $\mathscr{H}$ *be a Hilbert space of dimension* $d < \infty$. *Let* $x \mapsto P(x)$ *be a measurable map of* X *into the one dimensional projections and suppose that*

$$(3.1) \qquad \int P(x)\, d\mu(x) = d^{-1} 1.$$

Then for any self-adjoint A:

$$(3.2) \qquad \frac{1}{d} \operatorname{Tr}(e^{-A}) \geq \int d\mu(x) \exp(-\operatorname{Tr}(AP(x))).$$

Moreover, if

$$(3.3) \qquad A = d \int f(x) P(x)\, d\mu(x)$$

for some $f \in L^\infty$, *then*

$$(3.4) \qquad \frac{1}{d} \operatorname{Tr}(e^{-A}) \leq \int d\mu(x) \exp(-f(x)).$$

For a proof, see [11]. (3.2) is a simple application of Jensen's inequality. (3.4) is more subtle; in [11] it is proven using the Golden – Thompson type inequality $\operatorname{Tr}((CD)^n) \leq \operatorname{Tr}(C^n D^n)$. Berezin [15] also found these estimates at the same time as [8].

To apply this result to get the main theorem of Section 2, we take $X = \Gamma^{|\Lambda|}$ and $P(x)$ a projection onto a coherent vector. Let us describe what happens at a single site, i.e. $|\Lambda| = 1$.

Let $U^{(f)}$ be the irreducible representation associated to the maximal weight, f. Let $P(\tilde{f})$ be the projection onto the eigenspace

$$\{w \mid U^{(f)}(L)w = f(L)w\}.$$

By general principles, $P(\tilde{f})$ is rank 1 and w is the unique vector (up to phase) with

$$(3.5) \qquad (w, U^{(f)}(L)w) = \tilde{f}(L).$$

$\tilde{f}$ the extension of f to g described in Section 2. Let Γ be the coadjoint orbit with $\tilde{f} \in \Gamma$. Given $x \in \Gamma$, pick $y \in G$ with $A^*(y)\tilde{f} = x$ and define

$$P(x) = U^{(f)}(y)P(\tilde{f})U^{(f)}(y)^{-1}.$$

Then the fact that (3.5) determines w uniquely implies that $P(x)$ is independent of which y is choosen with $A^*(y)\tilde{f} = x$. Moreover

$$(3.6) \qquad \mathrm{Tr}\,(P(x)U^{(x)}(L)) = x(L).$$

The irreducibility of $U^{(f)}$ implies that

$$(3.7) \qquad \int P(x)\,d\mu(x) = \dim\,(U^{(f)})^{-1}\,1.$$

(3.2), (3.6) and (3.7) lead to the lower bound $Z_{cl}(\gamma) \leq Z_Q^l(\gamma)$.

To get the upper bound we need the formula

$$(3.8) \qquad U^{(f)}(L) = c \int x(L)P(x)\,d\mu(x)$$

for a constant c. It is not hard to see that both sides of (3.8) are functions $F(L)$ with

$$U^{(f)}(y)F(L)U^{(f)}(y)^{-1} = F(Ad(y)L)$$

for all $L \in g$, $y \in G$. Thus (3.8) holds automatically if $U^{(f)} \otimes \bar{U}^{(f)}$ contains the adjoint representation only once (by the Wigner – Eckart theorem). This is true if and only if f is a multiple of a fundamental weight [6].

We conjecture (3.8) is generally true. A general proof of (3.8) would provide a general proof of the theorem of Section 2 without the restriction to fundamental wrights. The evaluation of the constant in (3.8) is now fairly easy [8], [3], [11] ($d = \dim\,(U^{(f)})$):

$$(3.9) \qquad c = d[\langle f,f\rangle^{-1}][\langle f,f\rangle + 2\langle f,\delta\rangle].$$

(3.4), (3.7), (3.8) and (3.9) lead to the upper bound in (2.4).

4. TWO EXAMPLES

Spherical harmonies. We want to recover the result of Fuller and Lenard [3]. $U^{(l)}$ is the natural representation of so (n) on the polynomials in n variables which are harmonic and homogeneous of degree l. This representation is irreducible [14] and it is easy to see that in terms of the basis of weights described in Section 2, a maximal weight is $l\lambda_1$ with weight vector $(x_1 + ix_2)^l$. The Lie algebra so (n) is antisymmetric real matrices and the inner product $(L,M) = \mathrm{Tr}\,(L^*M)$ associates so $(n)^*$ and so (n). In this realization, λ_1 corresponds to the matrix m_1 with

$$(m_1)_{ij} = \begin{cases} 1 & i = 1,\ j = 2 \\ -1 & j = 1,\ i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

m_1 is a rank 2 matrix with $\mathrm{Tr}\,(m_1^* m_1) = 2$ and it is easy to see that the image of m_1 under SO (n) is precisely all such matrices (except for $n = 2$ which we ignore as Abelian). Thus, $\Gamma = \{m \in \mathrm{so}\,(n) \mid \mathrm{rank}\ m = 2;\ \mathrm{Tr}\,(m^*m) = 2\}$ which is naturally isomorphic to the set of normalized decomposable two forms i.e. the Grassman manifold of oriented two planes in $O(n)$. This yields the result of Fuller – Lenard once one notes that

$$\frac{\langle w_1, \delta\rangle}{\langle w_1, w_1\rangle} = \left(\frac{1}{2}(n-2)\right).$$

(This is either a direct calculation using the exact form of δ or else it follows from the formula, $l(l + (n-2))$, for the value of the Casimir operator in $U^{(l\lambda_1)}$).

Spinors. We consider so $(2n)$. The Clifford algebra is the irreducible realization of the relations among $2n$ matrices, $\sigma_1, \ldots, \sigma_{2n}$, $\{\sigma_\alpha, \sigma_\beta\} = 2\delta_{\alpha\beta}$ on C^m with $m = 2^n$ (i.e. familiar as Pauli matrices for $n = 1$, Dirac matrices for $n = 2$ and CAR for general n). The realization

$$U_1(L_{ij}) = \frac{1}{2}\sigma_i\sigma_j$$

yields a representation of so $(2n)$ which is not irreducible but rather $U_1 = U^{(\lambda n)} \oplus U^{(\lambda n-1)}$ with λ_n, λ_{n-1} the weights given in Section 2. (To see this note that the 2^n vectors with $U_1(L_{2i-1,2i}) = \pm 1$ (n choices) form a basis for C^m and read off the weights). The "spin $\frac{1}{2}l$ spinors" are just

$$U_l = U^{(l\lambda_n)} \oplus U^{(l\lambda_{n-1})}.$$

(To have a group representation we need to use Spin $(2n)$, the double covering of SO $(2n)$ and to get irreducibility we take pin $(2n)$, the double covering of O $(2n)$. U_l is an irreducible representation of pin $(2n)$, i.e. to accomodate parity we need both λ_n and λ_{n-1}. Without parity, we have non-irreducibility since $\sigma_{2n+1} = \sigma_1 \ldots \sigma_{2n}$ commutes with each $\sigma_i\sigma_j$. So long as we use pin $(2n)$, the general theory applies). Under the natural association of so $(2n)^*$ and so $(2n)$, λ_n corresponds to a matrix m_n:

$$(m_n)_{ij} = \begin{cases} \frac{1}{2} & i = 2k-1,\ j = 2k \\ -\frac{1}{2} & i = 2k,\ j = 2k-1 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $2m_n$ is unitary and antisymmetric and any such matrix is of the form $2xm_nx^{-1}$ for $x \in \mathrm{O}(n)$, i.e.

$$\Gamma = \Big\{\frac{1}{2}U \mid U \in U(2n) \cap \mathrm{so}(2n), \text{ i.e. unitary and antisymmetric}\Big\}.$$

Since $\delta = (n-1)w_1 + \ldots + w_{n-1}$ and $\lambda_n = \frac{1}{2}(w_1 + \ldots + w_n)$ we see that

$$\frac{\langle\delta, \lambda_n\rangle}{\langle\lambda_n, \lambda_n\rangle} = \frac{\frac{1}{2}[(n-1) + (n-2) + \ldots + 0]}{\frac{1}{4}n} = (n-1).$$

It is convenient to set

$$\tilde{\Gamma} = \{U \mid \tfrac{1}{2}U \in \Gamma\}$$

and define $Z_{c\lambda}$ in terms of $\tilde{\Gamma}$. If, we do that and define Z_{Ql} for the spin $\frac{1}{2}l$ spinors normalized by $\frac{U^{(l)}(L)}{l}$:

$$Z_{cl}\Big(\frac{\gamma}{2}\Big) \le Z_{Ql}(\gamma) \le Z_{cl}\Big(\frac{1}{2}\gamma[1 + 2(n-1)l^{-1}]\Big).$$

5. TOWARDS A LEE – YANG THEOREM FOR S^d

In this section, we want to reduce the proof of a Lee – Yang theorem for S^d, $d = 2n - 2$ to a conjecture about spinors. As noted in the introduction, for $d \ne 2$, S^d is *not* the classical limit Γ for any sequence of representations of so $(d + 1)$. However:

Theorem. *Fix $m = 2n$. Let $\tilde{\Gamma}$ be the classical limit space for the spinor representations. Map $\tilde{\Gamma}$ to R^{m-1} by*

$$\tau(U) = (U_{12}, U_{13}, \ldots, U_{1m}).$$

Then ran τ *is S^{m-2}, the sphere in R^{m-1} and the measure ν on S^{m-2} defined by $\nu(A) = \mu(\tau^{-1}[A])$, μ the natural measure on $\tilde{\Gamma}$, is just the usual measure on S^{m-2}.*

Proof. Clearly, $\tau(U) \in S^{m-2}$ since U is unitary. If $x \in \mathrm{SO}(2n)$ leaves $(1, 0, \ldots, 0)$ fixed, then $\tau(xUx^{-1})$ is the natural rotation on $\tau(U)$. Thus, Ran τ is rotation invariant and ν is rotation invariant. ∎

As a result, if for each site $\alpha \in A$, we have a copy $\tilde{\Gamma}_\alpha$ of $\tilde{\Gamma}$ and the Hamiltonian is independent of $\{U_{jk}^{(\alpha)}\}_{j \geq 2, k \geq 2, \alpha \in \Lambda}$, by "integrating out" these variables we get an S^{m-2} classical model. Thus,

Theorem. *Let $\mathscr{H} = \bigotimes_{\alpha \in \Lambda} \mathscr{H}_\alpha$, each $\mathscr{H}_\alpha = C^{2k}$, $k = 2^n$. Let L_{ij}^α be basic spinor actions on $\mathscr{H}_\alpha$ and let*

$$(5.1) \qquad -H = \sum_{\alpha,\beta} \left(J_{\alpha\beta} \sum_{i,j} L_{ij}^{\alpha} L_{ij}^{\beta} + K_{\alpha\beta} \sum_{j=2}^{n} L_{1j}^{\alpha} L_{1j}^{\beta} \right) + \sum_{\alpha} h_{\alpha} L_{12}^{\alpha}.$$

Suppose that for all $J_{\alpha\beta}, K_{\alpha\beta} \geq 0$, h_α complex with $\mathrm{Re} h_\alpha > 0$, we have that $\mathrm{Tr}(e^{-H}) \neq 0$. Then,

$$(5.2) \qquad \int \sum_{\alpha} d\Omega(s_\alpha) \exp \left(\sum K_{\alpha\beta} s_\alpha s_\beta + \sum h_\alpha s_{\alpha_1} \right) \neq 0$$

for $K_{\alpha\beta} \geq 0$, h_α complex with $\mathrm{Re} h_\alpha > 0$ and Ω the measure on S^{2n-2}. If $\mathrm{Tr}(e^{-H}) \neq 0$ when an additional term

$$\sum_{\alpha,\beta} M_{\alpha\beta} \sum_{j=2}^{n-1} L_{1j}^{\alpha} L_{1j}^{\beta}$$

is added to $H (M_{\alpha\beta} \geq 0)$, then (5.2) holds for Ω the measure on S^{2n-3}.

Proof. Given $\mathrm{Tr}(e^{-H})$ for basic spinors, we can use the Griffiths trick [4] and couple l basic spinors together with infinitely strong coupling and get $\mathrm{Tr}(e^{-H}) \neq 0$ for spin $\frac{1}{2} l$ spinors. This yields a Lee – Yang theorem for classical spins by our limit theorem in Sections 2-4. If we specialize to $J = 0$ in this classical result ($J \neq 0$ is needed to get the classical result from basic spinors) and integrate out the uncoupled components we get the desired result. If we have an M term and add

$$\sum_{\alpha} M \left(\sum_{j=2}^{n-1} (U_{1j}^{\alpha})^2 \right)$$

in the classical limit and take $M \to \infty$, we get the S^{2n-3} result.∎

Various tests of the conjecture that $\mathrm{Tr}(e^{-H}) \neq 0$ on a small number of sites are positive [7], so we expect that the conjecture is true. However, Asano contraction methods appear not to work [7], so a new understanding of the Lee – Yang theorem for the quantum Heisenberg model, preferably on a more group theoretic level, appears necessary to prove that $\mathrm{Tr}(e^{-H}) \neq 0$ under the situation needed above.

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