The characteristic function ϕ of a lattice distribution has the property that there exists a $t_0 \neq 0$ such that $|\phi(t_0)| = 1$.

The most common lattice distributions are on the nonnegative *unit lattice* $0, 1, 2, \ldots$.

(CHARACTERISTIC FUNCTIONS)

LATTICE SYSTEMS

Lattice systems are a class of random processes indexed by discrete subgroups of R^ν, such as Z^ν (the lattice of ν-tuples of integers). They have their origin in statistical mechanics* and are of special interest as models of critical phenomena*. They are also of some significance as discrete approximations to Euclidean quantum field theories, (*see* QUANTUM PHYSICS AND FUNCTION INTEGRATION) and also in probability theory: the most elementary multitime Markov chains* are included among the lattice gases.

The simplest and most famous of the lattice systems is the *nearest-neighbor Ising model*. Let ν be an integer and Z^ν the lattice of ν-tuples of integers. For each $\alpha \in Z^\nu$, s_α is a random variable taking the values ± 1. To describe the joint probability distributions, we need some auxiliary functions. Given a finite subset Λ of Z^ν, the symbol

$$\sum_{\langle\alpha\gamma\rangle;\alpha,\gamma\in\Lambda}$$

denotes the sum over all those pairs in Λ with $|\alpha - \gamma| = 1$ (Euclidean distance). The *finite volume Hamiltonian* is the function on $\{-1,1\}^\Lambda$:

$$H_\Lambda(s_\alpha) = -J \sum_{\langle\alpha\gamma\rangle;\alpha,\gamma\in\Lambda} s_\alpha s_\gamma - h \sum_{\alpha\in\Lambda} s_\alpha,$$

where J and h are parameters. The *finite volume Gibbs measure* is the measure, $d\mu_\Lambda$, on $\{-1,1\}^\Lambda$ giving weight

$$e^{-\beta H_\Lambda(s_\alpha)}/Z_\Lambda \equiv w$$

to the point $\{s_\alpha\}$. Here Z_Λ is a normalization factor called the *partition function* and is given by

$$Z_\Lambda = \sum_{s_\alpha = \pm 1;\alpha\in\Lambda} e^{-\beta H_\Lambda(s_\alpha)}.$$

β is another parameter (which is redundant since it can be absorbed into J and h). (*See* GIBBS DISTRIBUTION.)

One useful way of thinking of this setup is as the model of magnet: each s_α is viewed as a "spin" pointing up $(+1)$ or down (-1). If $J > 0$, the first term in H_Λ describes a tendency (lower energy and correspondingly higher weight in $d\mu_\Lambda$) for neighboring spins to align parallel and the model is that of a *ferromagnet*. The second term in H_Λ can be viewed as the interaction with an externally applied magnetic field: h is then the product of the magnitude of this field and the magnetic moment. If $\beta = 1/kT$ with k Boltzmann's constant and T the temperature, then $d\mu_\Lambda$ is the measure associated to the *canonical ensemble* according to the rules of statistical mechanics.

Another interpretation is as a "lattice gas." We change variables to $\rho_\alpha = \frac{1}{2}(1 + s_\alpha)$ and interpret $\rho_\alpha = 1$ $(s_\alpha = 1)$ as "occupied" and $\rho_\alpha = 0$ $(s_\alpha = -1)$ as "unoccupied." If $z = \exp(-4\nu J\beta + h\beta)$, then

$$w \equiv \tilde{Z}_\Lambda^{-1}\exp\Bigl(-4\beta J \sum_{\langle\alpha\gamma\rangle;\alpha,\gamma\in\Lambda} \rho_\alpha\rho_\gamma\Bigr) \prod_{\alpha\in\Lambda} z^{\rho_\alpha},$$

where $\tilde{Z}_\Lambda^{-1}$ is a normalization factor. [This is not quite true; $\prod_{\alpha\in\Lambda} z^{\rho_\alpha}$ should really be $\prod_{\alpha\in\Lambda} z_\alpha^{\rho_\alpha}$ with $z_\alpha = \exp(-2n_\alpha J\beta + h\beta)$ and n_α the number of neighbors of α lying in Λ. Thus $z_\alpha = z$ except for boundary α's; the role of such boundary terms is described below.] In this view, $d\mu_\Lambda$ is a weight associated to a *grand canonical ensemble* and z is a *fugacity*. $J > 0$ now corresponds to an attraction between particles in the gas.

Still a third interpretation of the model is as that of an alloy with one of two allowed species at each site.

Given a configuration $t = \{t_\alpha\}_{\alpha\notin\Lambda} \in \{-1,1\}^{Z^\nu\setminus\Lambda}$, we define the Hamiltonian, partition function, and Gibbs measure with *boundary condition* t_Λ as

$$H_\Lambda(s\,|\,t) = H_\Lambda(s) - J \sum_{\langle\alpha\gamma\rangle;\alpha\in\Lambda,\gamma\notin\Lambda} s_\alpha t_\gamma$$

$$Z_\Lambda(t) = \sum_{s_\alpha=\pm1;\alpha\in\Lambda} e^{-\beta H_\Lambda(s\,|\,t)}$$

$$\int f(s)\,d\mu_\Lambda(s\,|\,t) = Z_\Lambda(t)^{-1} \sum_{s_\alpha=\pm1;\alpha\in\Lambda} \times f(s) e^{-\beta H_\Lambda(s\,|\,t)}.$$

Note that all those objects depend only on those t_α with $\text{dist}(\alpha,\Lambda) = 1$. The α are often called the *boundary* of Λ and denoted $\partial\Lambda$.

Most of the interesting and subtle phenomena in the model are associated with the passage to an infinite volume limit, say by taking Λ through the sequence of hypercubes $[-n,n]^\nu$. This limit is usually called the *thermodynamic limit*. One important quantity is $\lim_{|\Lambda|\to\infty}|\Lambda|^{-1}\ln Z_\Lambda(t)$ (with $|\Lambda|$ the number of points in Λ), which is known to be independent of the boundary condition, t. We will denote this quantity by $p(\beta, h)$, where we imagine fixing $J = 1$ and make explicit the dependence on β and h. In magnetic language, p is a *free energy* (per unit volume) and in the lattice gas language it is a *pressure*. p is jointly convex in β and βh and so jointly continuous. Even though Z_Λ is manifestly analytic in β and h for each finite Λ, p may *not* be smooth. For example, when $\nu = 2$, $p(\beta, h = 0)$ is explicitly known (the celebrated *Onsager solution*). There is a critical value, β_c, of β so that p is real analytic in β away β_c, but $\partial^2 p/\partial\beta^2$ diverges logarithmically at β_c; this is called a *second-order phase transition* in β. As h is varied, an even more interesting situation results; while an explicit formula is not known, the following can be proven: p is jointly real analytic in the region $h \neq 0$, $\beta > 0$, and C^∞ in the region $\beta < \beta_c$. For $\beta > \beta_c$, p is not C^1 and $\lim_{h\downarrow 0}\partial p/\partial h = -\lim_{h\uparrow 0}\partial p/\partial h \neq 0$ [the first equality comes from $p(\beta, -h) = p(\beta, h)$, which follows from $s_\alpha \to -s_\alpha$ symmetry]. A discontinuous first derivative is called a *first-order phase transition* and is discussed further below. Since $\partial p/\partial h$ is a magnetization (per unit volume) in the magnetic phase, the discontinuity above is the *spontaneous magnetization* so typical of ferromagnets. In lattice gas language $\partial p/\partial h$ is a density and the discontinuity is that typical of the change of density in passing from a liquid phase to a gaseous phase.

Of special interest is the behavior of p and its derivatives near β_c. A nonrigorous but extremely stimulating and significant method for studying this behavior in arbitrary dimension ν (and other models) is the *renormalization group method.*

Further insight is obtained by the study of the limits of the Gibbs measures $d\mu_\Lambda$. An *equilibrium state* (for fixed β, j, h) is a measure on $\mathscr{X} \equiv \{-1,1\}^{Z^\nu}$ obtained by taking arbitrary weak limits of $d\mu_\Lambda(\cdot\,|\,t)$ and then taking arbitrary weak limits of convex combinations of such limits as the t's are varied but p, J, and h held fixed. This procedure yields exactly the measures that obey the so-called *DLR* (for Dobrushin, Lanford, and Ruelle) *equations*: for each finite Λ, the conditional expectations for functions f of the $\{s_\alpha\}_{\alpha\in\Lambda}$ conditioned on the configuration $\{t_\alpha\}_{\alpha\in\Lambda}$ outside Λ is given by

$$E(f\,|\,t) = \int f(s)\,d\mu_\Lambda(s\,|\,t)$$

with $d\mu_\Lambda(\cdot\,|\,t)$ as given before.

For the nearest-neighbor models we have been discussing, the DLR equations say that the conditional expectation of the interior, $E(f\,|\,t)$, conditioned on the exterior (t), depends only on the boundary (i.e., $\{t_\alpha\}$ with $\alpha \in \partial\Lambda$), so their states describe certain multidimensional Markov processes.

The group of translations of Z^ν acts in a natural way on $\mathscr{X}$; of particular interest are those equilibrium states which are translation invariant. Extreme points of the (weakly closed, convex) set of translation-invariant equilibrium states are often called *pure phases*.

For the two-dimensional model discussed above, the structure of equilibrium states is well understood. For $h \neq 0$ or $\beta \leqslant \beta_c$, $h = 0$, there is exactly one equilibrium state but for $\beta > \beta_c$, $h = 0$, there is a one-parameter family with two pure phases. All these states are translation invariant. The structure need not be so simple: in three or more dimensions it is known that there are nontranslation-invariant equilibrium states for $h = 0$ and β sufficiently large. And for the three-dimensional model with $h = 0$ and $\sum_{\langle\alpha\gamma\rangle} s_\alpha s_\gamma$ replaced by $\sum_{\langle\alpha\gamma\delta\lambda\rangle} s_\alpha s_\gamma s_\delta s_\lambda$ over all sets of four sites forming a planar square, it is known that there are an *uncountable* infinity of pure phases when β is large!

There is a close connection between multiple phases and first-order phase transitions: indeed, one can construct certain Banach spaces of interactions and for an interaction, Φ, a pressure $p(\Phi)$, convex in Φ, and a notion of equilibrium state for Φ so that there is a unique equilibrium state for Φ_0 if and only if p is (Gateaux) differentiable at Φ_0. (*See* STATISTICAL FUNCTIONALS.) In this theory [1, 2], a major role is played by *entropy** and the *Gibbs variational principle.*

The distinct phases that occur in the two-dimensional model are easy to describe. Let $d\mu_{\Lambda,\pm}$ be the finite volume state with all t_α, $\alpha \notin \Lambda$, equal to ± 1. Then one can show that the $d\mu_{\Lambda,\pm}$ have weak limits $d\mu_\pm$. If $h = 0$ and $\beta > \beta_c$, $d\mu_+ \neq d\mu_-$ and these are the two pure phases. Indeed, $\int s_\alpha \, d\mu_+ = \lim_{h \downarrow 0} dp/dh \neq 0$. Thus even though there are only short-range interactions, the setting of all boundary spins has an effect even in the infinite volume limit: this is called *long-range order*; it is a *cooperative phenomenon.* Notice that even though the basic Hamiltonian with $h = 0$ has the symmetry $s_\alpha \to -s_\alpha$ (all α), the states $d\mu_\pm$ do not have this symmetry (rather this transformation interchanges $d\mu_+$ and $d\mu_-$). This is called the phenomenon of *spontaneously broken symmetry*. These notions, more properly the notion of spontaneous broken continuous symmetries, play a major role in modern theories of elementary particles.

Having described the nearest-neighbor Ising model, let us briefly describe the form of some of the other popular models.

GENERAL ISING MODELS. Let $J(A)$ be a translation-invariant function on the finite subsets of Z^ν, let

$$\sigma^A = \prod_{\alpha \in A} \sigma_\alpha$$

and take

$$H_\Lambda = -\sum_{A \subset \Lambda} J(A)\sigma^A.$$

Usually, one requires that

$$\sum_{0 \in A} |J(A)|/|A| < \infty$$

to get sensible infinite volume limits, although this condition eliminates the important *Coulomb lattice gases* where cancellations account for a reasonable thermodynamic limit.

ONE-COMPONENT MODELS. If one relaxes the condition that S_α take the values ± 1 and allow it to be real-valued, one has a larger class of models. To define the model, one needs not only a Hamiltonian, $H_\Lambda(s_\alpha)$, but also a measure $d\gamma$ on R, called the *a priori measure,* to form the Gibbs state

$$Z_\Lambda^{-1} e^{-\beta H_\Lambda(s)} \prod_{\alpha \in \Lambda} d\gamma(s_\alpha).$$

Technically, the theory is very close in spirit to the ordinary Ising model if $d\gamma$ has compact support; otherwise, one deals with *unbounded spins* and there are many technical problems; for example, there can be "spurious" solutions of the DLR equation which are very singular at spatial infinity. Popular choices include the *spin S Ising* model, where s_α takes values $-2S, -2S+2, \ldots, 2S-2, 2S$, and the *lattice φ^4 theory*, where $d\gamma(x) = \exp(-ax^4 - bx^2)\,dx$. The latter is connected with discrete approximations of quantum field theories.

N-VECTOR MODELS. $\vec{s}_\alpha$ is now a vector-valued random variable, say with values in R^N. Particularly interesting is the case where the a priori measure is the isotropic one on the unit sphere S^{N-1}, often called the N-vector model. The case $N = 3$ with Hamiltonian $H_\Lambda = -\sum_{\alpha\gamma} \vec{s} \cdot \vec{s}_\gamma$ is called the *classical Heisenberg model.* In many ways it is a better model of a magnet than the Ising model. The N-vector models with $N \geqslant 2$ and suitable Hamiltonian have the continuous symmetry group, SO(N), of simultaneous rotations of all spins. One interesting aspect of the theory is that while the discrete $s_\alpha \to -s_\alpha$ symmetry of the Ising model is broken in $\nu \geqslant 2$ dimensions, the continuous symmetry of nearest-neighbor N-vector models is broken only in $\nu \geqslant 3$ dimensions.

SPIN GLASSES. These are a class of Ising models with Hamiltonian $-\sum J_{\alpha-\gamma} s_\alpha s_\gamma$, but now the J's are also random variables.

SIX- AND EIGHT-VERTEX MODELS. The random variables are now indexed by bonds in the lattice rather than sites; a *bond* is a nearest-neighbor pair. The variable takes two values which inform one in which direction to place an arrow on the bond. These are two-dimensional models. In the six-vertex model, only configurations are allowed with exactly two arrows in and two arrows out at each vertex. In the eight-vertex model, one also allows at each vertex the possibility of all arrows in or all arrows out. With various statistical weightings for given vertices these are models of *ferroelectrics*. One interest of these models is that the pressures have been exactly calculated—first by Lieb for six-vertex models and then by Baxter for eight-vertex models.

LATTICE GAUGE MODELS. These are of extreme interest as discrete versions of the (Euclidean region) non-Abelian gauge theories believed to be fundamental to an understanding of elementary particle interactions. Variables are now indexed by *directed* bonds in Z^ν and take values in some Lie group, G. There is the restriction that if α and $-\alpha$ are the same bond with oppostie directions, then $s_{-\alpha} = s_\alpha^{-1}$. Each planar square of bonds is called a *plaquette*, P, and one defines $s^P \equiv s_\alpha s_\beta s_\gamma s_\delta$, where a, β, γ, δ is an ordering of successive sides of P directed so that δ comes out of the site where Δ comes in. If φ is a real character of G, then $\varphi(s^P)$ is independent of which bond among α, β, γ, δ is put first. The Hamiltonian is

$$H_\Lambda = c \sum_P \varphi(s^P)$$

the sum being over all plaquettes in Λ. Notice that if one assigns a group element h_i to each *site* in the lattice and if we map s_α to $h_i s_\alpha h_j^{-1}$ when α runs from site i to site j, then H is left invariant. This is the group of *gauge transformations* of the model.

QUANTUM MODELS. There is a large class of noncommutative models, where the basic variables s_α are operators in some C^*-algebra rather than functions.

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(QUANTUM PHYSICS AND
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