The characteristic function $\phi$ of a lattice distribution has the property that there exists a $t_0 \neq 0$ such that $|\phi(t_0)| = 1$.

The most common lattice distributions are on the nonnegative unit lattice $0, 1, 2, \ldots$.

(CHARACTERISTIC FUNCTIONS)

LATTICE SYSTEMS

Lattice systems are a class of random processes indexed by discrete subgroups of $R^*$, such as $Z^*$ (the lattice of $\nu$-tuples of integers). They have their origin in statistical mechanics* and are of special interest as models of critical phenomena*. They are also of some significance as discrete approximations to Euclidean quantum field theories, (see QUANTUM PHYSICS AND FUNCTION INTEGRATION) and also in probability theory: the most elementary multitime Markov chains* are included among the lattice gases.

The simplest and most famous of the lattice systems is the nearest-neighbor Ising model. Let $\nu$ be an integer and $Z^*$ the lattice of $\nu$-tuples of integers. For each $\alpha \in Z^*$, $s_\alpha$ is a random variable taking the values $\pm 1$. To describe the joint probability distributions, we need some auxiliary functions. Given a finite subset $\Lambda$ of $Z^*$, the symbol

$$\sum_{\langle \alpha \gamma \rangle : \alpha, \gamma \in \Lambda}$$

denotes the sum over all those pairs in $\Lambda$ with $|\alpha - \gamma| = 1$ (Euclidean distance). The finite volume Hamiltonian is the function on $\{ -1, 1 \}^\Lambda$:

$$H_\Lambda(s_\alpha) = -J \sum_{\langle \alpha \gamma \rangle : \alpha, \gamma \in \Lambda} s_\alpha s_\gamma - h \sum_{\alpha \in \Lambda} s_\alpha,$$

where $J$ and $h$ are parameters. The finite volume Gibbs measure is the measure, $dm_\Lambda$, on $\{ -1, 1 \}^\Lambda$ giving weight

$$e^{-\beta H_\Lambda(s_\alpha) / Z_\Lambda}$$
to the point $\{ s_\alpha \}$. Here $Z_\Lambda$ is a normalization factor called the partition function and is given by

$$Z_\Lambda = \sum_{x_\alpha = \pm 1 : \alpha \in \Lambda} e^{-\beta H_\Lambda(s_\alpha)}.$$

$\beta$ is another parameter (which is redundant since it can be absorbed into $J$ and $h$). (See GIBBS DISTRIBUTION.)

One useful way of thinking of this setup is as the model of a magnet: each $s_\alpha$ is viewed as a "spin" pointing up (+1) or down (-1). If $J > 0$, the first term in $H_\Lambda$ describes a tendency (lower energy and correspondingly higher weight in $dm_\Lambda$) for neighboring spins to align parallel and the model is that of a ferromagnet. The second term in $H_\Lambda$ can be viewed as the interaction with an externally applied magnetic field: $h$ is then the product of the magnitude of this field and the magnetic moment. If $\beta = 1/kT$ with $k$ Boltzmann's constant and $T$ the temperature, then $dm_\Lambda$ is the measure associated to the canonical ensemble according to the rules of statistical mechanics.

Another interpretation is as a "lattice gas." We change variables to $\rho_\alpha = \frac{1}{2} (1 + s_\alpha)$ and interpret $\rho_\alpha = 1$ ($s_\alpha = 1$) as "occupied" and $\rho_\alpha = 0$ ($s_\alpha = -1$) as "unoccupied." If $z = \exp(-4\nu J + h\beta)$, then

$$w = \tilde{Z}_\Lambda^{-1} \exp \left( -4\nu J \sum_{\langle \alpha \gamma \rangle : \alpha, \gamma \in \Lambda} \rho_\alpha \rho_\gamma \right) \prod_{\alpha \in \Lambda} z^{\nu_\alpha},$$

where $\tilde{Z}_\Lambda^{-1}$ is a normalization factor. [This is not quite true; $\prod_{\alpha \in \Lambda} z^{\nu_\alpha}$ should really be $\prod_{\alpha \in \Lambda} z^{n_\alpha}$ with $z_\alpha = \exp(-2\nu_\alpha J + h\beta)$ and $n_\alpha$ the number of neighbors of $\alpha$ lying in $\Lambda$. Thus $z_\alpha = z$ except for boundary $\alpha$'s; the role of such boundary terms is described below.] In this view, $dm_\Lambda$ is a weight associated to a grand canonical ensemble and $z$ is a fugacity. $J > 0$ now corresponds to an attraction between particles in the gas.

Still a third interpretation of the model is as that of an alloy with one of two allowed species at each site.

Given a configuration $t = (t_\alpha)_{\alpha \in \Lambda} \in \{ -1, 1 \}^{Z^* \Lambda}$, we define the Hamiltonian, partition function, and Gibbs measure with
boundary condition $t_\Lambda$ as

\[ H_A(s | t) = H_A(s) - J \sum_{(\alpha \gamma) : \alpha \in \Lambda, \gamma \not\in \Lambda} s_\alpha t_\gamma \]

\[ Z_A(t) = \sum_{s_\alpha = \pm 1, \alpha \in \Lambda} e^{-\beta H_A(s | t)} \]

\[ \int f(s) \, d\mu_A(s | t) = Z_A(t)^{-1} \sum_{s_\alpha = \pm 1, \alpha \in \Lambda} \times f(s) e^{-\beta H_A(s | t)} \]

Note that all those objects depend only on those $t_\alpha$ with $\text{dist}(\alpha, \Lambda) = 1$. The $\alpha$ are often called the boundary of $\Lambda$ and denoted $\partial \Lambda$.

Most of the interesting and subtle phenomena in the model are associated with the passage to an infinite volume limit, say by taking $\Lambda$ through the sequence of hypercubes $[-n, n]^v$. This limit is usually called the thermodynamic limit. One important quantity is $\lim_{|\Lambda| \to \infty} |\Lambda|^{-1} \ln Z_A(t)$ (with $|\Lambda|$ the number of points in $\Lambda$), which is known to be independent of the boundary condition, $t$. We will denote this quantity by $p(\beta, h)$, where we imagine fixing $J = 1$ and make explicit the dependence on $\beta$ and $h$. In magnetic language, $p$ is a free energy (per unit volume) and in the lattice gas language it is a pressure. $p$ is jointly convex in $\beta$ and $\beta h$ and so jointly continuous. Even though $Z_\Lambda$ is manifestly analytic in $\beta$ and $h$ for each finite $\Lambda$, $p$ may not be smooth. For example, when $v = 2$, $p(\beta, h = 0)$ is explicitly known (the celebrated Onsager solution). There is a critical value, $\beta_c$, of $\beta$ so that $p$ is real analytic in $\beta$ away $\beta_c$, but $\partial^2 p / \partial \beta^2$ diverges logarithmically at $\beta_c$; this is called a second-order phase transition in $\beta$. As $h$ is varied, an even more interesting situation results; while an explicit formula is not known, the following can be proven: $p$ is jointly real analytic in the region $h \neq 0$, $\beta > 0$, and $C^\infty$ in the region $\beta < \beta_c$. For $\beta > \beta_c$, $p$ is not $C^1$ and $\lim_{h \to \beta_c^+} \partial p / \partial h = 0$ [the first equality comes from $\partial p(\beta, -h) = p(\beta, h)$, which follows from $s_\alpha \to -s_\alpha$ symmetry]. A discontinuous first derivative is called a first-order phase transition and is discussed further below. Since $\partial p / \partial h$ is a magnetization (per unit volume) in the magnetic phase, the discontinuity above is the spontaneous magnetization so typical of ferromagnets. In lattice gas language $\partial p / \partial h$ is a density and the discontinuity is that typical of the change of density in passing from a liquid phase to a gaseous phase.

Of special interest is the behavior of $p$ and its derivatives near $\beta_c$. A nonrigorous but extremely stimulating and significant method for studying this behavior in arbitrary dimension $v$ (and other models) is the renormalization group method.

Further insight is obtained by the study of the limits of the Gibbs measures $d\mu_\Lambda$. An equilibrium state (for fixed $\beta$, $j$, $h$) is a measure on $\mathcal{S} = \{-1, 1\}^\Lambda$ obtained by taking arbitrary weak limits of $d\mu_A(\cdot | t)$ and then taking arbitrary weak limits of convex combinations of such limits as the $t_\alpha$'s are varied but $\beta$, $j$, and $h$ held fixed. This procedure yields exactly the measures that obey the so-called DLR (for Dobrushin, Lanford, and Ruelle) equations: for each finite $\Lambda$, the conditional expectations for functions $f$ of the $(s_\alpha)_{\alpha \in \Lambda}$ conditioned on the configuration $(t_\alpha)_{\alpha \in \Lambda}$ outside $\Lambda$ is given by

\[ E(f | t) = \int f(s) \, d\mu_A(s | t) \]

with $d\mu_A(\cdot | t)$ as given before.

For the nearest-neighbor models we have been discussing, the DLR equations say that the conditional expectation of the interior, $E(f | t)$, conditioned on the exterior $(t_\alpha)$ depends only on the boundary (i.e., $t_\alpha$ with $\alpha \in \partial \Lambda$), so their states describe certain multidimensional Markov processes.

The group of translations of $\mathcal{S}^\Lambda$ acts in a natural way on $\mathcal{S}$; of particular interest are those equilibrium states which are translation invariant. Extreme points of the (weakly closed, convex) set of translation-invariant equilibrium states are often called pure phases.

For the two-dimensional model discussed above, the structure of equilibrium states is well understood. For $h \neq 0$ or $\beta < \beta_c$, $h = 0$, there is exactly one equilibrium state but for $\beta > \beta_c$, $h = 0$, there is a one-parameter family with two pure phases. All these states are translation invariant. The structure need not be so simple: in three or more dimensions it
is known that there are nontranslation-invariant equilibrium states for \( h = 0 \) and \( \beta \) sufficiently large. And for the three-dimensional model with \( h = 0 \) and \( \sum_{\langle \alpha \beta \lambda \gamma \rangle} \xi_{\alpha} \xi_{\beta} \), replaced by \( \sum_{\langle \alpha \beta \lambda \gamma \rangle} \xi_{\alpha} \xi_{\beta} \xi_{\lambda} \xi_{\gamma} \) over all sets of four sites forming a planar square, it is known that there are an uncountable infinity of pure phases when \( \beta \) is large!

There is a close connection between multiple phases and first-order phase transitions: indeed, one can construct certain Banach spaces of interactions and for an interaction, \( \Phi \), a pressure \( p(\Phi) \), convex in \( \Phi \), and a notion of equilibrium state for \( \Phi \) so that there is a unique equilibrium state for \( \Phi_0 \) if and only if \( p \) is (Gateaux) differentiable at \( \Phi_0 \). (See Statistical Functionals.) In this theory \([1,2]\), a major role is played by entropy\(^*\) and the Gibbs variational principle.

The distinct phases that occur in the two-dimensional model are easy to describe. Let \( d\mu_{h,\beta} \) be the finite volume state with all \( t_{\alpha} \), \( \alpha \in \Lambda \), equal to \( \pm 1 \). Then one can show that the \( d\mu_{h,\beta} \) have weak limits \( d\mu_\pm \). If \( h = 0 \) and \( \beta > \beta_0 \), \( d\mu_+ \neq d\mu_- \) and these are the two pure phases. Indeed, \( \int s_\alpha d\mu_+ = \lim_{h \to 0} dp/dh \neq 0 \). Thus even though there are only short-range interactions, the setting of all boundary spins has an effect even in the infinite volume limit: this is called long-range order; it is a cooperative phenomenon. Notice that even though the basic Hamiltonian with \( h = 0 \) has the symmetry \( s_\alpha \to -s_\alpha \) (all \( \alpha \)), the states \( d\mu_\pm \) do not have this symmetry (rather this transformation interchanges \( d\mu_+ \) and \( d\mu_- \)). This is called the phenomenon of spontaneously broken symmetry. These notions, more properly the notion of spontaneous broken continuous symmetries, play a major role in modern theories of elementary particles.

Having described the nearest-neighbor Ising model, let us briefly describe the form of some of the other popular models.

**General Ising Models.** Let \( J(A) \) be a translation-invariant function on the finite subsets of \( \mathbb{Z}^* \), let

\[
\sigma^A = \prod_{\alpha \in A} \sigma_\alpha
\]

and take

\[
H_\Lambda = - \sum_{A \subseteq \Lambda} J(A) \sigma^A.
\]

Usually, one requires that

\[
\sum_{0 \in A} |J(A)|/|A| < \infty
\]

to get sensible infinite volume limits, although this condition eliminates the important Coulomb lattice gases where cancellations account for a reasonable thermodynamic limit.

**One-Component Models.** If one relaxes the condition that \( S_\alpha \) take the values \( \pm 1 \) and allow it to be real-valued, one has a larger class of models. To define the model, one needs not only a Hamiltonian, \( H_\Lambda(s_\alpha) \), but also a measure \( dy \) on \( R \), called the a priori measure, to form the Gibbs state

\[
Z_\Lambda^{-1} e^{-\beta H_\Lambda(s) \prod_{\alpha \in \Lambda} dy(s_\alpha)}.
\]

Technically, the theory is very close in spirit to the ordinary Ising model if \( dy \) has compact support; otherwise, one deals with unbounded spins and there are many technical problems; for example, there can be "spurious" solutions of the DLR equation which are very singular at spatial infinity. Popular choices include the spin \( S \) Ising model, where \( s_\alpha \) takes values \( -2S, -2S+2, \ldots, 2S-2, 2S \), and the lattice \( \phi^4 \) theory, where \( dy(x) = \exp(-ax^4 - bx^2)dx \). The latter is connected with discrete approximations of quantum field theories.

**N-Vector Models.** \( \vec{s}_\alpha \) is now a vector-valued random variable, say with values in \( \mathbb{R}^N \). Particularly interesting is the case where the a priori measure is the isotropic one on the unit sphere \( S^{N-1} \), often called the \( N \)-vector model. The case \( N = 3 \) with Hamiltonian \( H_\Lambda = -\sum_{\alpha} \vec{s}_\alpha \cdot \vec{s}_\alpha \) is called the classical Heisenberg model. In many ways it is a better model of a magnet than the Ising model. The \( N \)-vector models with \( N > 2 \) and suitable Hamiltonian have the continuous symmetry group, \( \text{SO}(N) \), of simultaneous rotations of all spins. One interesting aspect of
the theory is that while the discrete $s_\alpha \rightarrow -s_\alpha$ symmetry of the Ising model is broken in $\nu > 2$ dimensions, the continuous symmetry of nearest-neighbor $N$-vector models is broken only in $\nu > 3$ dimensions.

**Spin glasses.** These are a class of Ising models with Hamiltonian $-\sum J_{\alpha \beta} s_\alpha s_\beta$, but now the $J$'s are also random variables.

**Six- and Eight-vertex models.** The random variables are now indexed by bonds in the lattice rather than sites; a bond is a nearest-neighbor pair. The variable takes two values which inform one in which direction to place an arrow on the bond. These are two-dimensional models. In the six-vertex model, only configurations are allowed with exactly two arrows in and two arrows out at each vertex. In the eight-vertex model, one also allows at each vertex the possibility of all arrows in or all arrows out. With various statistical weightings for given vertices these are models of ferroelectrics. One interest of these models is that the pressures have been exactly calculated—first by Lieb for six-vertex models and then by Baxter for eight-vertex models.

**Lattice gauge models.** These are of extreme interest as discrete versions of the (Euclidean region) non-Abelian gauge theories believed to be fundamental to an understanding of elementary particle interactions. Variables are now indexed by directed bonds in $\mathbb{Z}^\nu$ and take values in some Lie group, $G$. There is the restriction that if $a$ and $-a$ are the same bond with opposite directions, then $s_{-a} = s_a^{-1}$. Each planar square of bonds is called a plaquette, $P$, and one defines $s^P = s_\alpha s_\beta s_\gamma s_\delta$, where $a, \beta, \gamma, \delta$ is an ordering of successive sides of $P$ directed so that $\delta$ comes out of the site where $\Delta$ comes in. If $\varphi$ is a real character of $G$, then $\varphi(s^P)$ is independent of which bond among $\alpha, \beta, \gamma, \delta$ is put first. The Hamiltonian is

$$H_\Lambda = c \sum_P \varphi(s^P)$$

the sum being over all plaquettes in $\Lambda$. Notice that if one assigns a group element $h_i$ to each site in the lattice and if we map $s_\alpha$ to $h_\alpha s_\alpha h_\alpha^{-1}$ when $\alpha$ runs from site $i$ to site $j$, then $H$ is left invariant. This is the group of gauge transformations of the model.

**Quantum models.** There is a large class of noncommutative models, where the basic variables $s_\alpha$ are operators in some $C^*$-algebra rather than functions.

**References**


[2] Ruelle, D. (1969). *Statistical Mechanics: Rigorous Results*. W. A. Benjamin, New York. (The classic work on rigorous results. It is not restricted to lattice systems and it is somewhat out of date as regards these systems.)

**Bibliography**

See the following works, as well as the references just given, for more information on the topic of lattice systems.


(QUANTUM PHYSICS AND FUNCTION INTEGRATION) STATISTICAL MECHANICS STATISTICAL PHYSICS)

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