The anharmonic oscillator: a singular perturbation theory*

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Introduction

In these lectures, we will discuss the energy levels of the Hamiltonian $p^2 + x^2 + \beta x^4$ and its variants. This is the simplest example of an operator for which the perturbation series does not converge no matter how small β is. It allows us to ask the questions:

(1) Why doesn't perturbation theory converge? That is, if the energy levels aren't analytic at $\beta = 0$, what is the nature of the singularity?

(2) If the series doen't converge, what does it mean?

(3) Can the levels be recovered from the perturbation series by some method more devious than straightforward summing?

Since the Feynman series of field theory are known to be divergent in some cases (Ref. B), the answers to these questions are of obvious interest, especially since our model is formally very similar to field theories with Lagrangians $\mathscr{L} = \partial^\mu \phi \, \partial_\mu \phi - m^2\phi^2 - \beta\phi^4$.

The material in the last sections of these notes describes work begun at the summer school and continued at Les Houches. I have included it, despite the fact that it wasn't covered in my lectures because it is such a natural continuation of the material covered.

It is a great pleasure to thank Andre Voros for his excellent set of notes of my lectures which formed the core of this discussion.

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* This research partially sponsored by the Air Force Office of Scientific Research under Contract AF 49(638) 1545.

I Regular *vs.* Singular perturbation theory

For comparison purposes, let us first review the analytic theory of perturbations which are not singular. We consider Hamiltonians acting on wave functions ψ that are square integrable on an n-dimensional coordinate space: $\psi \in L^2(R^n)$. The *domain*, $D(H)$, of such an operator, H, is a technical object[1] which one should think of as those vectors $\psi \in L^2(R^n)$ for which $H\psi$ is also in L^2, i.e.

$$\int_{R^n} |(H\psi)(x)|^2 \, dx < \infty .$$

The pertrubed Hamiltonian is allowed to depend on a coupling constant β. Typically $H(\beta) = H_0 + \beta V$.

The basic theorem of regular perturbation theory is[2]:

Theorem (*Kato, Rellich*) If $H(\beta)$ is a family of operator functions of β, β belonging to some complex domain, O, such that

(1) $D(H(\beta))$ is independent of β[3].

(2) For all $\psi \in D(H(\beta))$, $\langle\psi, H(\beta)\psi\rangle$ is an analytic function of β in O,

Then: for any $\beta_0 \in O$ and for any *isolated*, non-degenerate eigenvalue $E(\beta_0)$ of $H(\beta_0)$, there is a neighborhood, V, of β_0, $V \subset O$, such that $H(\beta)$ has only one eigenvalue, $E(\beta)$, near $E(\beta_0)$ for $\beta \in V$. $E(\beta)$ is analytic in V and there is an analytic vector valued function, $\psi(\beta)$, on V such that $H(\beta)\psi(\beta) = E(\beta)\psi(\beta)$.

—if $H = H_0 + \beta V$, the Taylor series of $E(\beta)$ around β_0 is given by the Rayleigh-Schrödinger (R-S) perturbation series.

—if $E(\beta_0)$ has finite multiplicity k, the perturbation splits the level into at most k pieces $E_1(\beta), \ldots, E_m(\beta)$ $(m < k)$, which are the values of one or more multivalued analytic functions near β_0 with algebraic singularities, at worst at $\beta = \beta_0$.

—if $E(\beta_0)$ has finite multiplicity and $H(\beta)$ is self-adjoint when $\mathrm{Re}(\beta - \beta_0) = 0$, then the possible algebraic singularities at $\beta = \beta_0$ do not occur.

Of course, condition (1) is not very direct or transparent but one has:

Kato's criterion If $a > 0$, $b > 0$ so that for all $\psi \in D(H_0)$, $\psi \in D(V)$ and

$$\|V\psi\| \leq a\,\|H_0\psi\| + b\,\|\psi\|,$$

then (1) and (2) of the Kato-Rellich theorem hold for β sufficiently small, with $H(\beta) = H_0 + \beta V$.

Thus, if the perturbation is small in a very simple sense, one has convergence of the Rayleigh-Schrödinger series for $|\beta|$ small. In fact there exist explicit lower bounds to the radius of convergence in terms of the numbers a, b above, the distance of $E(\beta_0)$ to the nearest eigenvalue distinct from it[4].

The *anharmonic oscillator* is the Hamiltonian

$$\boxed{H(\beta) = p^2 + x^2 + \beta x^4} \qquad (\beta > 0).$$

There are various ways of seeing this is not a regular perturbation about $\beta = 0$:

(1) As soon as the perturbation is turned on, the domain changes: $D(H(0)) = D(p^2) \cap D(x^2)$ while $D(H(\beta)) = D(p^2) \cap D(x^4)$.

(2) One can prove rigorous bounds on the coefficients of the perturbation series which prove that the series diverges for all β (see below).

(3) There is a non-rigorous but convincing argument[5]: Power series converge in whole circles but for β negative the potential goes to $-\infty$ at $x = \infty$ and so the character of the bound states changes completely.

The Bender-Wu approximation

In a rigorous study of the analytic structure of the levels of the oscillator (which by Kato's criterion are analytic in a neighborhood of the real axis), one has been guided by some approximate calculations of Bender and Wu. By pasting together approximations in different regions, these authors obtain an approximate wave function which can be used to examine singularities of the levels. Despite the fact that this approximation does not appear to be rigorously justifiable and the fact that "nearby functions" may not have similar singularities[6], many of the gross features of the structure of the Bender-Wu approximate energy have been proven to occur in the actual energy levels. The major properties of this approximate level are (see Figure 1).

—there is a "global" three-sheeted structure around $\beta = 0$. We speak of a "global" branch point because $\beta = 0$ is not an isolated singularity and intend the term to mean that a path going three times around $\beta = 0$ circling conjugate singularities in conjugate ways returns one to the original level.

$-\beta = 0$ is a limit point of square root branch points with asymptotic phase $\pm 3\pi/2$ (which are on the second and third sheet if we cut along the negative axis).

—no singularities occur on the first sheet, $|\arg \beta| < \pi$.

—a detailed level structure is presented[7].

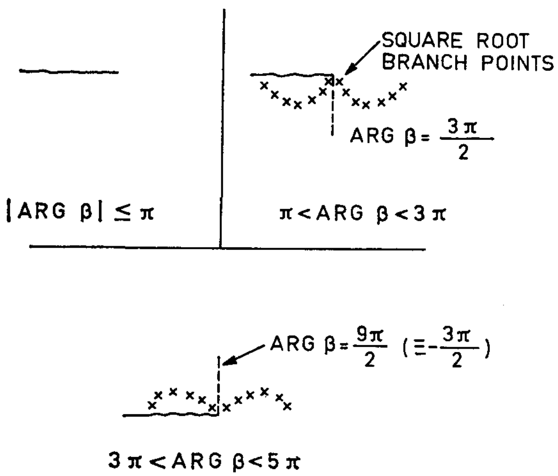

FIGURE 1

In addition, Bender and Wu present two other morsels:

—Rigorous bounds are proven on the coefficients, a_n, of the ground state energy level:

$$Ac^n n^{n/2} < (-1)^{n+1} a_n < B\, d^n n^{5n/2} \qquad (n > 0)$$

for some A, B, c, d. In particular, perturbation theory diverges.

—The first 75 a_n are computed to 12 places and one numerical "finds" the asymptotic behavior

$$a_n \sim (-1)^{n+1} \left(\frac{3}{2}\right)^{n+1/2} \frac{\pi}{4}^{-3/2} \Gamma\left(n + \frac{1}{2}\right)$$

as $n \to \infty$.[8]

III The Symanzik scaling

Symanzik remarked that an elementary covariance of the Hamiltonian $H(\beta)$, has deep consequences. Consider the unitary operator (λ real, >0).

$$(U(\lambda)\psi)(x) = \lambda^{+1/2}\psi(\lambda^{+1}x)$$

so

$$U(\lambda)\, xU(\lambda)^{-1} = \lambda x; \quad U(\lambda)\, pU(\lambda)^{-1} = \lambda^{-1}p.$$

Introduce the general Hamiltonian (α real, $\beta > 0$):

$$H(\alpha, \beta) = p^2 + \alpha x^2 + \beta x^4.$$

Then

$$U(\lambda)\, H(\alpha, \beta)\, U(\lambda)^{-1} = \lambda^{-2} H(\alpha\lambda^4, \beta\lambda^6);$$

since unitarily equivalent operators have identical eigenvalues:

$$E_n(\alpha, \beta) = \lambda^{-2} E_n(a\lambda^{+4}, \beta\lambda^{+6})$$

so, in particular, $\lambda = \beta^{-1/6}$, $\alpha = 1$:

$$E_n(1, \beta) = \beta^{1/3} E_n(\beta^{-2/3}, 1).$$

We are thus able to shift the coupling constant from the x^4 to the x^2 term (in Wightman's terminology, to the subdominant coupling). This may not seem like a special accomplishment but it is one nonetheless. For $p^2 + \alpha x^2 + x^4$ is a "nice" operator for any complex α (essentially by Kato's criterion), while $p^2 + x^2 + \beta x^4$ is misbehaved if $|\arg \beta| = \pi$. This is reflected in the fact: *Any continuation of $E_n(\alpha, 1)$ is an eigenvalue of $p^2 + \alpha x^2 + x^4$, but a continuation of $E_n(1, \beta)$ accross the negative axis is not in general, an eigenvalue of $p^2 + x^2 + \beta x^4$.* (For a proof, see II.3 of my Ann. Phys. article.)

The scaling law has several important consequences:

(1) *The cube root singularity* Suppose we can continue[9] $E_n(\alpha, 1)$ along a curve winding once around $\alpha = \infty$, which is symmetric about the real axis, i.e. a curve γ: $[0, 1] \to C$ with $\gamma(0) = \gamma(1)$ and $\gamma(1 - t) = \overline{\gamma(t)}$. $E_n(\alpha, 1)$ is real everywhere along the real axis (since it is an eigenvalue of $p^2 + \alpha x^2 + x^4$!), so the Schwartz reflection principle implies $E_n(\gamma(t), 1) = \overline{E_n(\gamma(1 - t), 1)}$. But then $E_n(\gamma(1), 1)$

$= \overline{E_n(\gamma(0), 1)} = E_n(\gamma(0), 1)$ since $\gamma(0)$ is real. Thus continuation of $E_n(\alpha, 1)$, once around takes us back to the initial level. By scaling, *$E_n(1, \beta)$ has a "global" cube root singularity at $\beta = 0$* and obeys the symmetry $E_n(1, \beta\, e^{3\pi i}) = -E_n(1, \beta)$. This symmetry also follows by noting $E_n(1, \beta)$ is real for $\arg \beta = 0$ and pure imaginary for $\arg \beta = \pm 3\pi/2$ *so the singularity structure in $0 < \beta < 3\pi/2$ is repeated, four fold in the regions $3\pi/2 < \arg \beta < 3\pi$, etc.*

(2) *The divergence of perturbation theory* We can also use scaling to prove $E_n(1, \beta)$ cannot be analytic about $\beta = 0$; put differently, we can establish that going once about $\beta = 0$ cannot return us to the original function, thus establishing that the cube root singularity is actually present. For suppose $E_n(1, \beta)$ is analytic near $\beta = 0$. Then the function $f(\lambda) = \lambda^2 E_n(\lambda^2, 1) = \lambda^3 E_n(1, \lambda^{-3})$ is analytic near $\lambda = \infty$, with the symmetry $f(\lambda\, e^{2\pi i/3}) = f(\lambda)$. Consider continuing f along the contour in Figure 2, defining f on C_3 by preserving this last symmetry. (The analyticity of f near the real axis follows from Kato's criterion.) Then, when we return to $\lambda = 0$, we must have $f(\lambda) = \lambda^2 E_m(\lambda^2, 1)$ (m may not be n). Thus, the symmetry implies (looking at the lowest terms in the Taylor series) $E_m(0, 1) = e^{4\pi i/3} E_n(0, 1)$. Since $E_m(0, 1) \neq 0 \neq E_n(0, 1)$, we have a contradiction which verifies that the function $E_n(1, \beta)$ cannot be analytic at $\beta = 0$.

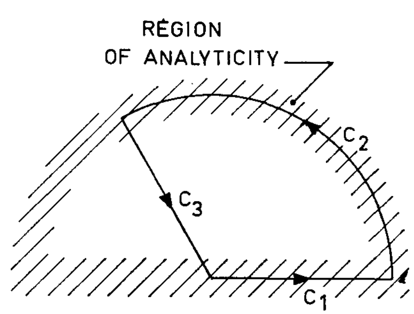

FIGURE 2

(3) *Strong coupling expansion* Since $E_n(\alpha, 1)$ is analytic about $\alpha = 0$, our scaling tells us $E_n(1, \beta)$ has an expansion $\sum_{m=0}^{\infty} a_m \beta^{1-2m/3}$ convergent for $|\beta|$ large. In particular $E_m(1, \beta)/\beta^{1/3} \to E_m(0, 1)$ as $|\beta| \to \infty$.

IV Asymptotic perturbation theory

We have seen that because of the cube root singularity at $\beta = 0$, perturbation theory for the x^4 oscillator does not converge. We are thus immediately faced with the question: what do the series (which are finite term by term) mean? The natural first answer is that the series is asymptotic. We recall the definition:

Definition Let $f(z)$ be a function regular in a region[10]

$$D = \{z \,|\arg z| < \theta; 0 < |z| < B\}.$$

We say $\sum a_n z^n$ is asymptotic (uniformly in the sector) to f if for each N,

$$\left| f(z) - \sum_{n=0}^{N} a_n z^n \right| \Big/ z^N \to 0 \text{ as } |z| \to 0, |\arg z| < \theta .$$

We note first that if $f \sim \sum a_n z^n$ and $f \sim \sum b_n z^n$, then $a_n = b_n$ all n. However, it may happen that $f \sim \sum a_n z^n$ and $g \sim \sum a_n z^n$ without having $f = g$. For example, if $\theta < \pi/2$, and $f - g = M \exp(-z^{-1})$ for any M, then f and g have identical asymptotic series if f has one.

Typically, if $f \sim \sum a_n z^n$ and if z is small[11], the partial sums $\sum_{n=0}^{N} a_n z^n$ will be a fair approximation for small N and then it will become very bad; see, for example, table I which lists the Taylor approximates for the ground state energy of $p^2 + x^2 + .2x^4$.

The earliest results on asymptotic perturbation series are due to Kato and there is a discussion in his book[12]. The first fact one must establish to prove asymptotic series is a result on the norm convergence of resolvants[13]:

Theorem Let $\beta > 0$. Then for any λ, not an eigenvalue of $p^2 + x^2$, one has

$$\|(p^2 + x^2 + \beta x^4 - \lambda)^{-1} - (p^2 + x^2 - \lambda)^{-1}\| \to 0.$$

Proof Using $(A - \mu)^{-1} = (A - \lambda)^{-1} [1 + (\lambda - \mu)(A - \lambda)^{-1}]^{-1}$ it is enough to prove this result for one λ. Now $(p^2 + x^2 + \beta x^4 - \lambda)^{-1} - (p^2 + x^2 - \lambda)^{-1} = \beta[(p^2 + x^2 - \lambda)^{-1} x^2] [x^2(p^2 + x^2 + \beta x^4 - \lambda)^{-1}]$. Because of the $1/x^2$ in $(p^2 + x^2 - \lambda)^{-1}$, each factor in [ ] is bounded as $\beta \to 0$, so the difference converges to zero because of the factor β in front.

Remark The condition $\beta > 0$ does not appear to have been used and indeed the proof can be made as long as $|\arg \beta| < \theta$ with $\theta < \pi$. The proof that $x^2(p^2 + x^2 + \beta x^4 - \lambda)^{-1}$ is bounded breaks down as $\arg \beta$ approaches π.

The basic asymptotic series result is:

Theorem For any n, the Rayleigh-Schrödinger series is asymptotic to $E_n(1, \beta)$ as $\beta \downarrow 0$ with $\arg \beta = 0$. ($\beta > 0$).

Sketch Let

$$P_n(\beta) = -(2\pi i)^{-1} \int_{|z-(2n+1)|<1} dz[p^2 + x^2 + \beta x^4 - z]^{-1}.$$

Then the Cauchy integral theorem implies $P_n(\beta)$ is a projection (!) (think of diagonalizing $p^2 + x^2 + \beta x^4$) and it is in fact the projection onto all eigenvalues with $|E - (2n + 1)| < 1$. For β small then $P_n(\beta)$ is the projection onto the eigenvector with eigenvalue $E_n(1, \beta)$. Thus, if ψ_n is the unperturbed eigenvector: $E_n(1, \beta) = \langle H(\beta)\psi_n, P_n(\beta)\psi_n\rangle / \langle \psi_n, P_n(\beta)\psi_n\rangle$. It is thus enough to establish an asymptotic series for $P_n(\beta)\psi_n$ with remainders bounded in norm. By the formula for $P_n(\beta)$, we only need an asymptotic series for $(p^2 + x^2 + \beta x^4 - z)^{-1}\psi_n$. But one has the geometric series with remainder:

$$\begin{aligned}(p^2 + x^2 + \beta x^4 - z)^{-1} = \sum_{n=0}^{N} &(-\beta)^n (p^2 + x^2 - z)^{-1} \\ &\times [x^4(p^2 + x^2 - z)^{-1}]^n + (-\beta)^{N+1} \\ &\times (p^2 + x^2 + \beta x^4 - z)^{-1} \\ &\times [x^4(p^2 + x^2 - z)^{-1}]^{N+1}.\end{aligned}$$

Since the resolvants converge in norm, the last term is bounded when applied to ψ_n.

Remark We have sloughed over the proof of stability of $E_n(1, \beta)$, i.e. the fact that $H(\beta)$ has one and only one eigenvalue near $E_n(1, 0)$. This follows from norm convergence of resolvents.

We have thus far considered only the limit $\beta > 0$ but our proof extends to $|\arg \beta| < \theta < \pi$. Actually, one can do considerably better. We shall only describe the intuitive idea. Consider $p^2 + \gamma x^2$ with $|\gamma| = 1$, $\gamma \neq -1$. This is a "nice" operator as can be intuitively seen by noting it has eigenvalues $(2n + 1)\gamma^{1/2}$ [with eigenfunctions proportional to $H_n(\gamma^{1/4}x)$ which falls off as $\exp(-\gamma^{1/2}x^2)\, x^n$ as $x \to \infty$ and so is square integrable]. Not surprisingly then as $|\beta| \downarrow 0$, $p^2 + \gamma x^2 + |\beta|\, x^4$ approaches $p^2 + \gamma x^2$; in fact one can prove:

Theorem Let n and $\eta < \pi$ be given. Then there is a B so that $p^2 + \gamma x^2 + |\beta|\, x^4$ has only one eigenvalue near $(2n + 1)\gamma^{1/2}$ if $|\arg \gamma| < \eta$; $|\beta| < B$.

By scaling γ into the x^4 term, one finds:

Theorem Let n and $\theta < 3\pi/2$ be given. Then there is a B so that $E_n(1, \beta)$ is analytic in $\{\beta \mid |\beta| < B,\ |\arg \beta| < \theta\}$. The Rayleigh-Schrödinger series for $E_n(1, \beta)$ is asymptotic uniformly in the sector.

Remarks 1. We thus have some analyticity information on the second sheet.

2. Due to the symmetry properties about $\arg \beta = 0$, $\pm 3\pi/2$ if $\beta = 0$ is a limit of singularities, we see their asymptotic phase must be $\pm 3\pi/2$.

3. For x^{2m} oscillators, one proves there is an $(m + 1)$-sheeted surface and an analog of the above theorem with $\theta = (m + 1)\pi/2$.

V Herglotz functions and singularities of $E(\beta)$

We have seen that the perturbation series is non-convergent because of the cube root singularity at $\beta = 0$, but we do not know yet that the strong coupling expansion is inapplicable to all $\beta \neq 0$, i.e. that $E_n(\alpha, 1)$ is not entire. The Bender-Wu singularities, if they can be proven to occur are precisely able to prevent the convergence of the strong coupling expansion for some $\beta \neq 0$.

We present an argument due to Andre Martin that $E_n(\alpha, 1)$ is not entire and by shaking it we will be able to prove $E_n(\alpha, 1)$ has an infinity of singularities. This argument of Martin has two absolutely characteristic features: First, it is very clever and very simple; secondly, it makes crucial use of the positivity of something.

Definition A function f is called Herglotz in a domain $\mathscr{D}$ of the complex plane if and only if $\operatorname{Im} f$ and $\operatorname{Im} z$ have the same sign[14] in all of $\mathscr{D}$.

Lemma An entire Herglotz function is linear.

Proof If $f(z) = \sum_{0}^{\infty} a_n z^n$, then $\operatorname{Im} f(re^{i\theta}) = \sum a_n r^n (\sin n\theta)$. $\operatorname{Im} f(re^{i\theta})$ has the sign of $\sin \theta$ by assumption, which is also the sign of $m \sin \theta \pm \sin m\theta$ (since $|\sin m\theta / \sin \theta| \leqq m$), so we must have:

$$0 \leqq \int_{-\pi}^{\pi} (m \sin \theta \pm \sin m\theta) \operatorname{Im} f(re \operatorname{in}^\theta) = m a_1 r \pm a_m r^m \qquad (m > 1).$$

If $a_m \neq 0$, we can choose $\pm$ and r so this is violated. Thus

$$a_m = 0, \qquad m > 1; \quad \text{i.e.} \quad f(z) = a_0 + a_1 z.$$

Lemma (Martin) $E_n(\alpha, 1)$ is a Herglotz function on its domain of analyticity.

Proof $E_n(\alpha, 1)$ is always an eigenvalue[15], so $E_n(\alpha, 1) = \langle \psi_n, (p^2 + \alpha x^2 + x^4) \psi_n \rangle$. Thus $\operatorname{Im} E_n(\alpha, 1) = \langle \psi_n, x^2 \psi_n \rangle (\operatorname{Im} \alpha)$. Since $x^2 > 0$, the proof is complete.

Corollary $E_0(\alpha, 1)$ is not entire.

Proof It is not hard to show E_0 cannot be linear.

To improve this result, we must first strengthen the lemma on Herglotz functions:

Lemma If $f(z)$ is a Herglotz function for $|z| > R$, then its Laurent series at $\sum_{n=-\infty}^{\infty} a_n z^n$ has $a_n = 0$ if $n > 1$.

Proof If we mimic the entire function argument, we find

$$a_1 r - a_{-1} r^{-1} \pm (a_m r^m - a^{-m} r^{-m}) > 0$$

for all r large, which implies $a_m = 0$, $m > 1$.

E. Lieb (private communication) has suggested the following more classical proofs of the two Herglotz lemmas:

The lemma on entire Herglotz functions Let $g(z) = f(z) - f(0)$ which is still Herglotz. If $\{z \mid |z| < R\}$ has n zeros, $\arg g(\mathrm{Re}^{i\theta})$ changes by $2\pi n$ as θ runs from 0 to 2π (the argument principle). But in the upper half plane $\arg g$ cannot change by more than π, so $\Delta \arg g(re^{i\theta}) \leqq 2\pi$.

As a result $g(z)$ has no zeros at $z \neq 0$. Thus $h(z) = g(z)/z = e^{H(z)}$ for some entire function H. But $|\operatorname{Im} H| \leqq |\arg(h(z))| \leqq |\arg g| + |\arg z| \leqq 4\pi$. This can only happen if $H =$ constant.

The lemma Herglotz on functions near infinity By a $-z^{-1}$ change of variable, we can suppose f is analytic and Herglotz in a punctured disc. As in the above, f has at most one zero with $z > 0$ and one with $z < 0$, so we may suppose (by shrinking the disc) f is never zero. Let us consider $\Delta \arg f(\mathrm{Re}^{i\theta})$. As above, it is 0, $+2\pi$ or -2π. Since $\Delta \arg f(\mathrm{Re}^{i\theta}) = \dfrac{1}{2\pi i} \displaystyle\int_{|z|=R} \dfrac{f'(z)\,dz}{f(z)}$ it is independent of R. Thus, by letting $g = f$, $z^{-1} f$ or zf we can assure $\Delta \arg g(\mathrm{Re}^{i\theta}) = 0$. Then $g = e^H$ and since H has a bounded imaginary part, H has a removable singularity; thus so does g. Since $f = g$, zg or $z^{-1} g$, f has at worst a pole of order 1.

We can now combine this Herglotz property, scaling, the asymptotic nature of perturbation theory, and the divergence of the perturbation series to prove:

Corollary $\alpha = \infty$ *is not an isolated singularity of* $E_n(\alpha, 1)$.

Proof If it were, $E_n(\alpha, 1)$ would actually be analytic near $\alpha = \infty$ (there couldn't be a branch point by our argument using reality on the real axis), so by the lemma $E_n(\alpha, 1) = \sum_{n=-\infty}^{1} b_m \alpha^n$ for α near ∞. Thus $E_n(1, \lambda^3) = \sum_{m=-1}^{\infty} b_{-m} \lambda^{(1+2m)}$, convergent for $|\lambda|$ small and non-zero. But $E_n(1, \lambda^3) \sim \sum_{m=0}^{\infty} a_m \lambda^3$ as $\lambda \downarrow 0$. This is only possible[17] if $b_{-1} = 0$; $b_0 = a_0$; $a_1 = b_1$, etc. i.e. only if the series are identical. But, then perturbation theory would converge. This contradiction completes the proof.

Thus, $E_n(1\ \beta)$ *must have* 0 *as a limit point of singularities; by our discussion in* IV, *these singularities have asymptotic phase* $\pm 3\pi/2$.

VI The Loeffel-Martin arguments for one-dimensional oscillators

As we shall see, the presence of no singularities on the first sheet in $E_n(1, \beta)$ is critical for reasons that will be shortly apparent. That no singularities occur on the sheet, $|\arg \beta| < \pi$ is a result due to J. J. Loeffel and A. Martin. The proof has two absolutely characteristic features of an argument of Martin: First, it is very clever and very simple; secondly, it makes critical use of the positivity of something.

First suppose one has a path in the α plane beginning and ending at $\alpha_0 > 0$ and suppose the path stays in the region $|\arg\alpha| < 2\pi/3 - \varepsilon$ and that one can continue $E_n(\alpha, 1)$ along this path. We want to show that after continuation, we return to $E_n(\alpha, 1)$.

One employs a solution $\underset{\infty}{\psi}(\alpha, E, z)$ of $\left[-\frac{d^2}{dz^2} + (\alpha z^2 + z^4 - E)\right]\psi = 0$ which has $\underset{\infty}{\psi} \sim e^{-z^3}$ as $z \to +\infty$. Such a solution can be shown to exist (Ref. H) and to be entire in α, E, z.

Loeffel and Martin prove the following facts about zeros of $\underset{\infty}{\psi}$:

—When α, E are real and positive, every zero with $|\arg z| < \pi/3$ is on the real axis.

—When α is on the above curve of continuation and $E =$ the continuation of $E_n(\alpha, 1)$, $\underset{\infty}{\psi}(\alpha, E, z)$ has no zeros in the shaded regions of Figure 3.

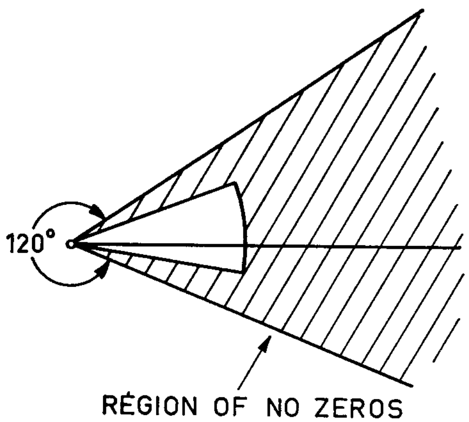

FIGURE 3

Suppose $n = 2m + 1$, so $\underset{\infty}{\psi}(\alpha_0, E_0, 0) = 0$ and $\underset{\infty}{\psi}(\alpha_0, E_0, z)$ has m zeros along the positive real axis. As α varies these zeros may move around but exactly m are "trapped" inside the Leoffel-Martin walls

and no new ones can come in; they cannot "sneak out" at $z = 0$ or collide since double zeros are not allowed ($\underset{\infty}{\psi}$ solves a homogeneous second order equation). Thus when we return to α_0 we have m zeros all caught back on the real axis, i.e. we are at $E_n(\alpha, 1)$.

Note what happens when we let $\arg\alpha > 2\pi/3$ (go onto the second sheet after scaling). $\underset{\infty}{\psi}$ is an entire function and thus it probably has an infinity of zeros[18], so there are presumably infinitely many "barbarian" zeros lurking outside the Loeffel-Martin walls. As $\arg\alpha$ approaches $2\pi/3$, the lateral walls shrink to nothing. If we wander around and then return to the positive real axis, we may have captured or lost a few extra zeros when we rebuild the L.-M. walls; thereby we return to $E_{n+2\Delta n}(\alpha, 1)$.

In the above, we have supposed continuation is always possible; i.e. that no natural boundaries occur. Loeffel and Martin also prove this by showing it is impossible to have $E_n(\alpha, 1) \to \infty$ if we stay in $|\arg\alpha| < 2\pi/3 - \varepsilon$. For they show, if $E \to \infty$, $\underset{\infty}{\psi}$ is well approximated by a simple cosine function which as $E \to \infty$ has infinitely many zeros in the region protected by the L.-M. walls; an application of Rouché's theorem shows $\underset{\infty}{\psi}$ and this cosine have the same number of zeros contradicting the fact that there are only m zeros inside the walls. A Hilbert space argument proves that so long as E stays finite, only isolated algebraic singularities can occur. By keeping track of the zeros within the walls, we see even algebraic singularities cannot occur (the first part of the above argument).

While we have described the L.-M. walls, we have not yet described how they are constructed. Let us attempt to present the general idea. The walls are built up one piece at a time so we only present an overall technique. To understand the exact position, one must see the details. One writes the differential equation for $\underset{\infty}{\psi}(\alpha x + \beta) = \underset{\infty}{\psi}(x)$ [α, β complex] as $-\underset{\infty}{\psi}'' + P_{\alpha,\beta}(x)\underset{\infty}{\psi} = 0$ and integrates it:

$$\underset{\infty}{\psi}'(b)^* \underset{\infty}{\psi}(b) - \underset{\infty}{\psi}'(a)^* \underset{\infty}{\psi}(a) = \int_a^b \left[\left|\underset{\infty}{\psi}'(x)\right|^2 + P_{\alpha,\beta}(x)\left|\underset{\infty}{\psi}(x)\right|^2\right] dx.$$

Then either one picks $\beta = a = 0$ so one can use the boundary condition $\psi_\infty(0) = 0$ or $\psi'_\infty(0) = 0$ or one picks $|\arg \alpha| < \pi/3$, $b = \infty$ so one can use the exponential falloff of ψ_∞ at $z \to \infty (|\arg z| < \pi/3)$. If α, β, a, b are such that $P_{\alpha,\beta}(x)$ has argument in some sector $\gamma < \arg P_{\alpha,\beta} < \delta$ with $\gamma < 0 < \delta \; |\delta - \gamma| < \pi$, then the integral cannot vanish (as a sum of numbers in a sector), thus one concludes $\psi_\infty(a) \neq 0$ (if $b = \infty$) or $\psi_\infty(b) \neq 0$ (if $\beta = a = 0$). In this way the walls are constructed.

Finally, we mention one final conjecture of Martin that would be very useful could it be proven. He conjectures[18a] that as $n \to \infty$, the radius of convergence of $E_n(\alpha, 1)$ about $\alpha = 0$ goes to ∞. This would mean that at any finite α, only finitely many levels can cross each other which implies one need only consider an essentially finite-dimensional problem; this would allow one to prove the non-occurrence of natural boundaries on any sheet.

VII Padé approximants

We have seen that the perturbation series for the energy levels of $p^2 + x^2 + \beta x^4$ diverge but are asymptotic. Since an infinity of functions have the same asymptotic series, the situation is not exactly satisfactory at this point of our development. One might hope that some summability method might allow one to recover the levels from the series. In fact, it was the search for such methods that led Wightman to suggest the analytic study of the levels in the first place[19]. The hope was to find a method which was successful in the oscillator and then attempt to use it on the divergent field theory models. The inspiration came in the opposite direction: the success of Bessis-Pusterla and Copley-Masson in applying Padé approximants to $(\phi^4)_4$ theories suggested one try them for the anharmonic oscillator[20]. What are Padé approximants?

Definition Let $\sum a_n z^n$ be a formal power series. Its $[N, M]$ Padé approximant is the rational function

$$f^{[N,M]}(z) = \frac{P^{[N,M]}(z)}{Q^{[N,M]}(z)}$$

P of degree M, Q of degree N obeying

$$f^{[N,M]}(z) - \sum_{n=0}^{N+M} a_n z^n = 0(z^{N+M+1}).$$

Remarks

1. Let us count the free parameters in P/Q to be certain our asymptotic condition has the correct number of restrictions. P has $N + 1$ free parameters, Q has $M + 1$. Since P/Q does not depend on the individual normalizations of P and Q but only on their ration, P/Q has $N + M + 1$ free coefficients.

2. As Professor Baker will discuss, $f^{[N,M]}$ has a specific elementary formula in terms of determinants.

We make several simple comments about Padé which will no doubt be repeated by several lecturers:

– In many examples, the "diagonal" sequences $f^{[N,N+j]}$, j fixed, converge more rapidly and in larger regions than the Taylor series.

– One possible reason for this is that $f^{[N,N+j]}$ has many zeros and poles as $N \to \infty$ and these can mock up singularities; we will see how this happens for a particular class of functions in the next section.

– The $f^{[N,N]}(z)$ have a covariance property which may help explain why they are so nice. If g is the function obtained from f by the substitution $z \to z' = az + b/cz + a$ then $f^{[N,N]}(z) = g^{[N,N]}(z')$. Thus, if one could prove some subsequence of the $f^{[N,N]}$ converged on every compact of the circle of convergence (the Padé conjecture), it would follow that it converged in every circle of analyticity containing 0 (for one can find a fractional linear transformation taking $0 \to 0$ and the circle into one with 0 as center).

VIII Stieltjes functions and series

Definition A function of Stieltjes is a function of the form:

$$F(z) = \int_0^\infty \frac{d\varrho(x)}{1 + xz} \qquad (z \neq \text{negative real})$$

where ϱ is a positive measure with $\int_0^\infty x^n \, d\varrho(x) < \infty$.

Definition A series of Stieltjes is a series of the form $a_n = (-1)^n \int_0^\infty x^n \, d\varrho(x)$; ϱ as above.

We first note two elementary theorems; one relates Stieltjes functions and series and the other defines Stieltjes functions in terms of their analytic properties.

Theorem (1) Every Stieltjes function has an asymptotic series valid uniformly in every sector $|\arg z| < \theta$ for any $\theta < \pi$. The coefficients of the series are a series of Stieltjes.

(2) For every series of Stieltjes, there is at least one Stieltjes function for which it is the asymptotic series.

Proof (1) If $f = \int_0^\infty d\varrho(x)/1 + xz$, let $a_n = (-1)^n \int_0^\infty x^n \, d\varrho$. Then

$$f(z) - \sum_0^N a_n z^n \equiv R_N(z)$$

obeys

$$R_N(z) = \int_0^\infty d\varrho \left[(1 + xz)^{-1} - \sum_{n=0}^N (-1)^n z^n x^n \right]$$

$$= (-1)^{N+1} \int_0^\infty d\varrho(x) \, (1 + xz^{-1}) \, (xz)^{N+1}$$

so

$$|R_N(z)/z^N| = |z| \int_0^\infty (x^{n+1} \, d\varrho(x)) \, |1 + xz|^{-1} \to 0 \quad \text{as} \quad |z| \to 0$$

uniformly in sectors avoiding the negative axis.

(2) Given $a_n = (-1)^n \int_0^\infty d\varrho(x)$, just let $f(z) = \int_0^\infty d\varrho(x) \, (1 + xz)^{-1}$ and use (1).

Theorem Let f be a function with the following properties:

(1) f is analytic in the cut plane.

(2) $f \to 0$ at ∞ (uniformly if $|\arg z| < \theta < \pi$).

(3) $-f$ is Herglotz.

(4) f has an asymptotic expansion $f \sim \sum_{n=0}^\infty a_n z^n$ valid as $z \downarrow 0$, $z > 0$.

Then f is a function of Stieltjes (and a_n a series of Stieltjes). Conversely, if f is a function of Stieltjes, it obeys (1)–(4).

Sketch That Stieltjes functions obey (1)–(4) is simple. Given (1), (2) write down a dispersion relation for f,

$$f(z) = \frac{1}{\pi} \int_{-\infty}^0 \frac{dz'}{z' - z} \operatorname{Im} f(z' + i\varepsilon) = \int_0^\infty \frac{d\varrho(x)}{1 + xz} \qquad (21)$$

where $d\varrho(x) = -(\pi x)^{-1} \operatorname{Im} f(-x^{-1} + i\varepsilon)$. Since $-f$ is Herglotz, we see f has a Stieltjes form, so we need only show $\int_0^\infty x^n \, d\varrho < \infty$. But $(1 + xz)^{-1} \to 1$ monotonically as $z \uparrow 0$, so $f(z) \downarrow a_0$ implies $a_0 = \int_0^\infty d\varrho(x)$. By repeated application of (4) and the monotone convergence theorem, one proves $\int_0^\infty d\varrho(x) \, x^n < \infty$.

There are two simple questions raised by these two theorems. First, when is there a *unique* Stieltjes functions with perturbation series a_n? The answer is simply answered by the last theorem. If ϱ_1 and ϱ_2 both are positive measures with $a_n = (-1)^n \int x^n d\varrho_i(x)$, $i = 1, 2$ then $f_i = \int_0^\infty d\varrho_i$ $(1 + xz)^{-1}$ are both Stieltjes functions with the same asymptotic series. On the other hand, if f_1 and f_2 are distinct functions with same asymptotic series we must have $\operatorname{Im} f_1(x^{-1} + i\varepsilon) \neq \operatorname{Im} f_2(x^{-1} + i\varepsilon)$ [or else $f_1 - f_2$ is entire with zero asymptotic series!], so $\varrho_1 \neq \varrho_2$. Thus:

Proposition Let a_n be a series of Stieltjes. Then, there is a unique Stieltjes function f with $\sum a_n z^n$ as asymptotic series if and only if there is a unique measure $d\varrho$ with $a_n = (-1)^n \int x^n \, d\varrho$ (in which case we say the *moment problem* for $|a_n|$ has a unique solution).

While there are necessary and sufficient conditions for the moment problem to have a unique solution, they are not simple; there is a simple *sufficient* condition however:

Theorem (Carleman's criterion)[22] If $\sum_{n=1}^{\infty} |a_n|^{-1/2n} = \infty$, and a_n is a series of Stieltjes, then the moment problem for $|a_n|$ has a unique solution.

The second point involves allowing subtractions in our dispersion relations:

Definition If f is analytic in the cut plane with asymptotic series as $z \downarrow 0$ and obeys either

(a) $|f(z)| < A|z| + B$ and $-f$ is Herglotz

or (b) $|f(z)| < A|z|^n + B$ and $-\mathrm{Im}\, f(z + i\varepsilon)$ exists as a positive measure for $z \varepsilon (-\infty, 0)$ then we say f is a Stieltjes function in the extended sense[23].

Using subtracted dispersion relations implies:

Proposition If $f(z)$ is a Stieltjes function of extended type, then we have

$$f(z) = \sum_{n=0}^{m} b_n z^n + z^{m+1} \int_0^{\infty} \frac{d\varrho(z)}{1 + xz}$$

for a measure $d\varrho$ with all moments finite.

Finally, we come to the question of the relation between Stieltjes functions and Padé approximates. The basic theorem was proven by Stieltjes before 1900 and is one of the most impressive pieces of classical analysis; it remains the only really basic result in Padé theory:

Theorem (Stieltjes) Let a_n be a series of Stieltjes. Then

(1) The diagonal approximants $f^{[N,N+J]}$ converge as $N \to \infty$ (j fixed) uniformly on compact subsets of the cut plane to a Stieltjes function f_j with a_n as asymptotic series.

(2) If j is odd and $x > 0$, the $f^{[N,N+J]}$ increase monotonically in N; if j is even and $x > 0$, the $f^{[N,N+J]}$ decrease monotonically in N.

(3) The moment problem for $|a_n|$ has a unique solution if and only if $f_1 = f_0$ and in that case all the f_j are equal to that unique Stieltjes function with asymptotic series $|a_n|$.

Before making some remarks about the proof, we note:

Corollary Let f be a Stieltjes function of extended type. Then all the diagonal Padé sequences, $f^{[N,N+J]}(j > m - 1)\, a_n$, the asymptotic series for f, converge uniformly compacts of the cut plane and if the moment problem for $|a_n|^{(n>j)}$ has a unique solution, then they converge to f.

Sketch of Proof Write $f = P + z^m g$ with P a polynomial and g Stieltjes. It is not hard to show the Padé's for f and g are simply related.

We have no intention of proving Stieltjes' theorem completely but we will make a series of comments which almost comprise a proof of sorts.

1) The theorem is actually in a weaker form than appears in say Baker's review article. If $(-1)^n a_n = \int_0^{\infty} x^n \, d\varrho(x)$, one concludes $\sum_{n,m=1}^{k} \lambda^n \lambda^m |a_{n+m+l}| > 0$ all λ, k_j; $\lambda = 0, 1$. One remarkable discovery of Stieltjes is that these conditions are also sufficient for there to be a solution of the moment problem for $|a_n|$. His proof involves showing that when the positivity conditions hold the Padé's converge to a Stieltjes function with a_n as aymptotic series. If one wishes to prove the theorem with the apriori weaker positivity conditions (instead of the assumption that a_n is a series of Stieltjes) one must use several obscure theorems on determinants; alternately the Hahn-Banach theorem implies their conditions are sufficient for the moment problem to have a solution.

2) The Pade's $f^{[N,N]}$ have all their poles and zeros on the negative axis (thus mocking up the cut). There is a very beautiful way of seeing this due to Basdevant, Bessis and Zinn-Justin[24]. Let $Q_N(z) = z^N Q^{[N,N]}\left(-\frac{1}{z}\right)$ is a polynomial of degree, N. There is a measure $\tilde{\varrho}$ on $[0, \infty]$, related to ϱ with $\int_0^{\infty} Q_N(z)\, Q_M(z)\, d\tilde{\varrho} = 0$ if $N \neq M$. This implies Q_N has N zeros in $[0, \infty]$, i.e. the poles of P/Q lie on the negative axis. Since $-1/f$ is an extended function of Stieltjes, $P^{[N,N]}$ has its zeros on the negative axis. One also sees that the zeros of $Q^{[N,N]}$ interlace as $N \to \infty$ and the zeros of $P^{[N,N]}$ and $Q^{[N,N]}$ interlace so all the residues of the poles of $f^{[N,N]}$ are positive.

3) Tiktopoulos and Treiman (unpublished) have a nice way of looking at the monotonicity properties of the Padé's. Suppose $\{a_n\}_{n=0,\ldots,2N}$ are given such that

$$a_n = (-1)^n \int_0^\infty x^n \, d\varrho$$

has a solution. The non-empty set of solutions S is convex and in a suitable topology it is compact. In this topology the mapping $\varrho \to \int_0^\infty d\varrho(1 + xz)^{-1} = F_z(\varrho)$ is continuous for any z real. Thus the largest value of $F_z(\varrho)$ is actually realized for a $\varrho \in S$ which is an extreme point of S. But the extreme points of S are sums of at most $(2N + 2)$ point-measures. So the maximal value is realized for a function $f(w) = \sum_0^{2N+1} \frac{a_n}{1 + b_n w}$; $b_n, a_n > 0$. By a variation of parameters argument, Tiktopoulos and Treiman show this is precisely the $[N, N]$ approximant. Thus $f[N, N] > f[N-1, N-1] > \cdots$.

4) Once one knows the $f^{[N,N]}(z)$ are monotone for $z > 0$ and are of the form

$$f^{[N,N]}(z) = a_0 - \sum_{n=1}^{N} \frac{b_n z}{1 + c_n z}; \quad b_n, c_n > 0$$

the convergence of the approximants uniformly on compacts is simple. For $f^{[N,N]\prime}(0) - a_1 = \sum_{n=1}^{N} b_n$. Thus $|f^{[N,N]}(z)| < a_0 + a_1 M(z)$ where $M(z) = \max_{0<\lambda<\infty} |z/1 + \lambda z|$ is finite in the cut plane and bounded on compact sets. Thus the $f^{[N,N]}(z)$ converge on the real axis [they decrease and are bounded below by $f^{[1,2]}(z)$], and are uniformly bounded on compacts so they converge uniformly on compacts to an analytic function f_0 by a theorem of Vitali[25]. It is clear that $-f$ is Herglotz as the limit of Herglotz functions; and it is not hard to show it has $\sum a_n z^n$ as asymptotic series.

5) Since each limiting function is Stieltjes, it is clear that they are all equal if there is a unique Stieltjes function with asymptotic series $\sum a_n z^n$, i.e. if the moment problem for $|a_n|$ has a unique solution. On the other hand, if f is a Stieltjes function with series $\sum a_n z^n$, then $f^{[N,N]}(x) > f(x) > f^{[N,N+1]}(x)$ for all N; $x > 0$ so $f_0(x) > f(x) > f_1(x)$. If $f_0 = f_1$, then only one such f exists, i.e. the moment problem has a unique solution.

IX Padé approximants for the anharmonic oscillator

We have seen for the levels of the anharmonic oscillator that:

- $E_n(1, \beta)$ has an asymptotic series around $\beta = 0$.
- $|E_n(1, B)| < A + B|\beta|^{1/3}$ in the cut plane.
- $E_n(1, \beta)$ is analytic in the cut plane.
- $E_n(1, \beta)$ is a Herglotz function.

Thus: $-E_n(1, \beta)$ *is a Stieltjes function in the extended sense* and so *the Padé approximants* $f^{[N,N+J]}$ *formed from the R.S. series converge.* In order to be certain that they converge to the "right answer", we must know the moment problem for the Rayleigh-Schrödinger series has a unique solution. To apply Carelman's criterion one needs an upper bound on the coefficients[26], in fact one can prove:

Theorem Let a_n be the Rayleigh-Schrödinger coefficients of some energy level of a $p^2 + x^2 + \beta x^{2m}$ oscillator. Then, for some C, D:

$$|a_n| < CD^n n^{(m-1)n}.$$

Thus, for the x^4 (or x^6) oscillator $\sum |a_n|^{-1/2n} = \infty$, so:

For any level of the one-dimensional x^4 oscillator, and any j, the $f^{[N,N+J]}$ Padé approximates converge uniformly on compacts of the cut plane to the energy level. If j is even, they are monotonically increasing for $\beta > 0$ and if j is odd they are monotonically decreasing.

Of course there is no reason that the convergence should be very rapid. But numerically[27] it is quite rapid for β not too large. For $\beta = 0.1$, $f^{[5,5]}$ is accurate to one part in 10^5 and $f^{[12,12]}$ to one part in 10^{11}. Even when $\beta = 1$, $f^{[5,5]}$ is good to 0.1%. For β very large, the convergence must be bad for $f^{[N,N]}(\beta) \to$ constant at ∞ while $E(1, \beta)/\beta^{1/3} \to$ constant at ∞. For $\beta = 15$, $f^{[5,5]}$ is still accurate to 30%. For a dramatic demonstration of the divergence of perturbation theory and convergence of the Padé's, see table 1:

TABLE I

N	$\sum_{n=0}^{N} a_n\beta^n$	$f^{[1/2N,1/2N]}$
1	1.150000	
2	1.097500	1.111111
3	1.153750	
4	1.105372	1.117541
5	1.176999	
6	1.049024	1.118183
7	1.314970	
8	0.686006	1.118273
9	2.353090	
10	−2.442698	1.118288
11	13.253968	
12	−42.333586	1.118289
13	168.895730	
14	−796.466406	1.118289
15	3005.179546	

$\beta = 0.2$; $E(\beta) = 1.1182892\ldots$

X Extensions of the Padé method

One can ask to what extent the method generalizes. Since scaling was used so crucially it seems unlikely we can say much about arbitrary singular potentials. There are thus two natural directions to attempt to generalize in: x^{2m} perturbations and to more than one dimensional oscillators; perhaps even to infinitely many degrees of freedom!

The situation is described by the table[28]:

TABLE 2

n	m	Results of § I–V	Bender-Wu	Loeffel-Martin	Carleman Cond.	Padé
1	2	A.S.	A.S.	A.S.	A.S.	Yes
1	$m > 2$	Yes ($\beta^{1/3} \to \beta^{1/m+1}$)	Yes	Yes	No if $m > 3$	U. if $m > 3$
$n > 1$	$m = 2$	Yes	Probably not	U.	Yes	U.
$n > 1$	$m > 2$	Yes	Probably not	U.	No if $m > 3$	U.

Thus, for $p^2 + x^2 + \beta x^{2m}$ Hamiltonians, the Padé's converge but since the moment problem is not known to have a unique solution, they may not converge to the "right answer" or to an answer independent of j. And for multi-dimensional problems, the Loeffel-Martin argument which depended so crucially on keeping track of nodes fail so we no longer have cut plane analyticity.

For infinitely many degrees of freedom, another blow awaits us. Glimm and Jaffe have shown how to construct some well-defined systems with infinitely many degrees of freedom. Let ϕ a free field of mass m in 2-dimension space-time. Let $V_g = \int dx\, g(x){:}\,\phi^{2m}(x){:}$ where $g \varepsilon L^1 \cap L^2$; $g > 0$[29]. For such theories, Simon and Höegh-Krohn have proven:

Theorem Let $H(\beta) = H_0 + \beta V_g$, and let $E(\beta)$ be the ground state energy. Then even if $E(\beta)$ is analytic in a cut plane, $E(\beta) + C\beta$ is not Herglotz for any C.

In some sense this is due to the infinite subtraction involved in : :.

These authors conjecture that one might have $a_n = (-1)\int_0^\infty x^n d\varrho\,(n > 2)$ with $\int_0^\infty x\, d\varrho = \infty$.

XI Carleman's condition revisited

In a sense, the primary question we are asking is whether we can establish some additional property of the energy levels which together with the asymptotic series determines the levels uniquely. Thus, we have seen that once one knows the moment problem for the $|a_n|$ has a unique solution and that $E_0(1,\beta)$ is Stieltjes, $E_0(1,\beta)$ is uniquely determined. The fact that one can compute E_0 easily is an added bonus but in some sense the critical question involves the unique determination from the perturbation series. It is thus worthwhile delving into the reason Carleman's criterion works. It is actually based on a more fundamental theorem of Carleman:

Carleman's theorem Let $f(z)$ be analytic in a region $D = \{z \mid |z| < B; |\arg z| < k\pi/2\}$ and suppose

(1) $|f(z)| < b_n |z|^n$ in D;

(2) $\sum_{n=1}^{\infty} b_n^{-1/kn} = \infty$.

Then $f = 0$.

Corollary Let f and g be two functions analytic in D as above. Let a_n be a sequence so both f and g are have $\sum a_n z^n$ as asymptotic series *in the strong sense*, i.e. for some A, σ, and all $z \varepsilon D$, N:

$$\left| f(z) - \sum_{n=0}^{N} a_n z^n \right| < A\sigma^{N+1}[k(N+1)]!\, |z|^{N+1}$$

$$\left| g(z) - \sum_{n=0}^{N} a_n z^n \right| < A\sigma^{N+1}[k(N+1)]!\, |z|^{N+1}.$$

Then $f = g$.

Proof Let $b_{N+1} = 2A\sigma^{N+1}[k(N+1)]!$. Then $f - g$ obeys the conditions of Carleman's theorem and

$$\sum b_n^{-1/kn} > C \sum n^{-1} = \infty.$$

Remark If such a strong asymptotic condition holds then $|a_n| < A\sigma^n(kn)!$ Given a_n, this suggests what value of n to take. In fact one has:

Theorem Fix m and ϱ. For any $\theta < (m+1)\pi/2$, one can find σ, B and A so that

$$\left| E_\varrho^{(m)}(\beta) - \sum_{n=0} a_{\varrho,n}^{(m)} \beta^n \right| < A\sigma^{N+1}[(m-1)(N+1)]!\, |\beta|^{N+1}$$

if $0 < |\beta| < B$; $|\arg \beta| < \theta$. Here $E_\varrho^{(m)}(\beta)$ represents the ϱth level of an $p^2 + x^2 + \beta x^{2m}$ oscillator.

Sketch By scaling, one needs to obtain bounds as $|\beta| \to 0$ on the ϱth level of $p^2 + \gamma x^2 + |\beta|\, x^{2m}$ uniform in N and $|\gamma| = 1$, $|\arg \gamma| < \left(\frac{2}{m+1}\right)\theta$. It is not hard to show that ratios of functions with strong asymptotic conditions obey a strong asymptotic condition, so as in § 4, by writing $E_\varrho^{(m)}(\gamma, |\beta|) = \langle H(\gamma, |\beta|)\, \psi_\varrho,\ P_\varrho(|\beta|)\, \psi_\varrho \rangle / \langle \psi_\varrho, P_\varrho(|\beta|)\, \psi_\varrho \rangle$ and $P_\varrho(|\beta|)$ as an integral of the resolvant, we need only control the error in the geometric series expansion for the resolvant; it is sufficient to prove:

$$\| [x^{2m}(p^2 + \gamma x^2 - \lambda)^{-1}]^N \psi_\varrho(\gamma) \| < A\sigma^N [(m-1)(N+1)]!$$

with $\psi_\varrho(\gamma)$ the ϱth level of $p^2 + \gamma x^2$ and with λ arbitrary obeying $|\lambda - (2n+1)\gamma^{1/2}| = \frac{1}{2}$. Introducing scaled creation and annhilation operators $a(\gamma)$, $b(\gamma)$ [since $b(\gamma) = a^+(\bar{\gamma})$, we call it b!] and *unnormalized* states: $\psi_\varrho(\gamma) = b(\gamma)^\varrho (\varrho!)^{-1/2} \psi_0(\gamma)$; $\psi_0(\gamma) = \pi^{-1/2} \exp(-\gamma x^2)$. One can show $\|\psi_\varrho(\gamma)\| < CD^\varrho$ where C, D can be choosen independent of γ for $|\arg \gamma| < \left(\frac{2}{m+1}\right)\theta$; $|\gamma = 1|$. Now one just expands x^{2m} in creation and annhilation operators and obtain the required bound on $\| [x^{2m}(\varrho^2 + \gamma x^2 - \lambda)^{-1}]^N \psi_\varrho \|$.

Remarks 1. *This argument goes through without any change to N-dimensional oscillators.*

2. Thus, *for any x^{2m} oscillator in any finite number of dimensions the perturbation series with the strong asymptotic series determines the energy levels.*

3. For additional details of the vague sketch above, see the paper of Graffi, Grecchi and Simon (F 5).

Since the above use of the Carleman theorem only requires analyticity near $\beta = 0$ which may be approached by semi-perturbative methods, there is some hope for extension to infinitely many degrees of freedom. In fact, one can prove:

Theorem Let $g > 0$; $g \varepsilon L^1 \cap L^2$. Let H_0 be the free Hamiltonian for a neutral scalar find $\phi(x)$ in *two-dimensional space time* and let $V = \int dx g(x) : \phi^4(x) :$ Let a_n be the coefficients of the Rayleigh-Schrödinger series for the ground state energy, $E(\beta)$ for $H_0 + \beta V$. Then

(a) For any $\theta < \pi$, there is a B with $E(\beta)$ analytic in $\{\beta | 0 < |\beta| < B; |\arg \beta| < \theta\} = D_\theta$.

(b) For any $\theta < \pi$, there is a σ and an A with $|E(\beta) - \sum_{n=0}^{n} a_n \beta^n| < A\sigma^{N+1}(N+1)!\, |\beta|^{N+1}$.

In particular, the perturbation series determines the ground state (vacuum) energy uniquely.

Remarks 1. The Rayleigh-Schrödinger series for $E(\beta)$ is well known to agree with the Feynman series obtained by summing all connected vacuum diagrams $\left[\text{with } H_I = \int g(x) : \phi^4(x) : dx\right]$.

2. This theorem fails for ϕ^{2m} in two critical ways. First the analyticity result uses the fact that $|V|^{1/2}$ is a Kato small perturbation of H_0[30]. Secondly the $n!$ growth is peculiar to ϕ^4; for ϕ^{2m}, one expects $[n(m-1)]!$ growth and would require analyticity in some higher sheets.

XII Borel summability

While we have seen that the perturbation series and the strong asymptotic estimate determine the eigenvalues, we do not, as yet, have any method for constructing the eigenvalues from the perturbation series. Using the fact that we have a little more analyticity than the bare minimum needed to apply Carleman's theorem, we will find the method of Borel (or its obvious extension due to Leroy) applicable.

The Borel method is based on the observation $n! = \int_0^\infty e^{-x}x^n\, dx$ so that formally:

$$\sum a_n z^n = \sum \frac{a_n}{n!} \int_0^\infty e^{-x}(xz)^n\, dx = \int_0^\infty e^{-x} g(xz)\, dx.$$

It is clear that if $|a_n| < A\sigma^n n!$ then $g(z) = \sum \frac{a_n}{n!} z^n$ has a finite radius of convergence. Of course, we need more that $g(z)$ in a finite circle to try to recover $f(z) = \sum a_n z^n$ from its *Borel transform* $g(z) = \sum \frac{a_n}{n!} z^n$.

That one has more in certain cases is a theorem of Watson:

Theorem (Watson) Suppose $f(z)$ is analytic in a sector $D = \{z | 0 < |z| < B;$ $|\arg z| < (\pi/2) + \theta\}$ and obeys a strong asymptotic condition:

$$\left| f(z) - \sum_{n=0}^{N} a_n z^n \right| < A\sigma^{N+1}(N+1)!\, |z|^{N+1}$$

in D. Then, the Borel transform $g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ which is apriori analytic in a circle of radius $r = \sigma^{-1}$ about $z = 0$, has an analytic continuation into the sector $\{z \mid |\arg z| < \theta\}$ and for any $z\varepsilon\hat{D}$ with $\hat{D} = \{z | 0 < |z| < B; |\arg z| < \theta\}$ one has

$$f(z) = \int_0^\infty e^{-x} g(xz)\, dx.$$

Remarks 1. By taking $z \to w = z^{1/m}$, we can make a trivial extension to Watson's theorem: Replace D with $\{z | 0 < z < B; |\arg z| < k(\pi/2) + \theta\}$ (f may be multivalued, i.e. many sheeted), the strong asymptotic condition with $\left| f(z) - \sum_{n=0}^{N} a_n z^n \right| < A\sigma^{N+1}[k(N+1)]!\, |z|^{N+1}$, the Borel transform with Leroy transform $g(z) = \sum_{n=0}^{\infty} \frac{a_n}{(kn)!} z^n$ and the inversion formula with $f(z) = \int_0^\infty e^{-x} g(zx^k)\, dx$, one has an identical formula.

2. When the above holds we say f is *Borel summable of order k* and call g the Borel transform of order k.

We thus have:

Corollary Let f be the ϱth energy level of an n-dimensional oscillator $H = H_0 + \beta V$ where V is a homogeneous polynomial of degree $2m$, which is strictly positive on $\mathbb{R}^n - \{0\}$. Then, f is Borel summable of order $m - 1$ from the Rayleigh-Schrödinger perturbation series. Its Borel transform is analytic in the cut plane, cut in $(-\infty, -\sigma)$ with $\sigma > 0$.

Corollary Let f be the ground state energy of a two-dimensional spatially cutoff $:\phi^4:$ field theory: $H = H_0 + \beta V$ where $V = \int g(x) : \phi^4(x) : dx$ $g\varepsilon L^1 \cap L^2$; $g > 0$. Then f is Borel summable from the Feynman perturbation series. Its Borel transform is analytic in the right half plane.

We note that while it is much easier to prove Borel summability than Padé summability the Padé method is far superior from a computational point of view. First the Padé approximants have simple explicit formulae and in the Stieltjes' case provide upper and lower bounds (and thus built in error estimates). Most critically, the Borel method is implicit rather than explicit in that it involves an analytic continuation at a critical point. One way of attempting to do this continuation (of the Borel transform) is by forming Padé approximants for the series $\frac{a_n}{n!}$ and then taking the inverse Borel transform. While this procedure is unjustified[31] the numerical results are even better than the ordinary Padé's(!) and give much better approximations at intermediate β.

XIII Bibliography

A. Basic analyticity references

The discussion in sections § 3–§ 5 is based upon:

A.1 B. Simon, *Ann. Phys.*, **58**, 76 (1970).

The Bender-Wu approximation (§ 2) was introduced and developed in

A.2 C. Bender and T. T. Wu, *Phys. Rev. Lett.*, **21**, 406 (1968).
A.3 C. Bender and T. T. Nu, *Phys. Rev.*, **185**, 1231 (1969).
A.4 C. Bender, *J. Math. Phys.*, **11**, 769 (1970).

The Loeffel-Martin results (§ 6) are proven in:

A.5 J. J. Loeffel and A. Martin, Proc. May, 1970 Programme 25, Conference (Strassbourg), reprinted below

For additional discussion of analyticity, see:

A.6 B. Simon, Proc. 1969 Eastern Theoretical Physics Conf. (Syracuse).
A.7 A. S. Wightman, Proc. 1969 C.N.R.S. Conf. on Systems with an Infinite Number of Degrees of Freedom (Gif-sur-Yvette).

B. Divergence of perturbation theory for fields

The classic divergence references are:

B.1 C. A. Hurst, *Proc. Camb. Phil. Soc.*, **48**, 625 (1952).
B.2 W. Thirring, *Helv. Phys. Acta*, **26**, 33 (1953).
B.3 A. Peterman, *Helv. Phys. Acta*, **26**, 291 (1953).

For a careful treatment of $(\phi^4)_2$, see

B.4 A. Jaffe, *Commun. Math. Phys.*, **1**, 127 (1966).

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B.5 F. Dyson, *Phys. Rev.*, **85**, 631 (1952).

On the other hand, convergence of a *highly cutoff* Feynman Q.E.D. series is established in:

B.6 E. R. Caianiello, *Nuovo Cimento*, **9**, 9 (1958), and extended by various authors.

For a summary of the relevant literature, see the introduction of:

B.7 B. Simon, *Nuovo Cimento*, **59 A**, 199 (1969).

C. Regular perturbation theory

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C.1 T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, 1966.

For additional textbook treatment, see:

C.2 F. Rellich, *Perturbation Theory of Eigenvalue Problems*, Gordon and Breach, 1969.
C.3 K. O. Friedrichs, *Perturbation of Spectra in Hilbert Space*, A.M.S. publ., 1965.

And the original research article by Kato is quite readable:

C.4 T. Kato, *Prog. Theor. Phys.*, **4**, 514 (1949); **5**, 95; 207.

D. Asymptotic perturbation theory

In addition to Kato's book (C.1) and article (C.4) and my *Ann. Phys.* article (A.1); see:

D.1 W. Greenlee, *Arch. Rat. Mech. Anal.*, **34**, 143 (1969).
D.2 T. Kato, *J. Fac. Sci. Univ. Tokyo* (Sec. I), **6**, 145 (1951).
D.3 E. Titchmarsh, *Proc. Roy. Soc.*, **A 200**, 34 (1949); **A 201**, 473 (1950).

E. Padé approximants

For the general theory see:

E.1 G. Baker, *Adv. Theo. Phys.*, **1**, 1 (1966).
E.2 J. Basedevant, D. Bessis, and J. Zinn-Justin, *Nuovo Cimento*, **60 A**, 185, (1969). App.
E.3 F. Wall, *Analytical Theory of Continued Fractions*, Chelsea (1949).

The proof of convergence of the Padé approximants for x^4 and x^6 oscillators is due to:

E.4 J. J. Loeffel, A. Martin, B. Simon, and A. S. Wightman, *Phys. Lett.*, **30 B**, 656 (1969).

F. Carleman's criterion and Borel summability

For a discussion of Carleman's criterion, see:

F.1 Carleman, *Les Fonction Quasi-Analytique*, Gauthier-Villars (1926).

F.2 Rudin, *Real and Complex Analysis*, McGraw-Hill (1967).

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G. Two-dimensional boson field theories

For the general theory, see:

G.1 J. Glimm and A. Jaffe, *Comm. Pure and App. Math.*, **22**, 401 (1969).

G.2 J. Glimm and A. Jaffe, *Phys. Rev.*, **176**, 1945 (1968).

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G.4 J. Glimm and A. Jaffe, *Acta Math.* **125**, 203 (1970).

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G.6 E. Nelson, In Proc. 1966 M.I.T. Conference on Mathematical Physics.

G.7 L. Rosen, *Comm. Math. Phys.*, **16**, 157 (1970).

G.8 L. Rosen, *Comm. Pure Appl. Math.* **24**, 417 (1971).

G.9 I. Segal, *Bull. A.M.S.*, **75**, 1390 (1969).

G.10 I. Segal, *Ann. Math.* **92**, 462 (1970).

G.11 B. Simon and R. Höegh-Krohn, J. Func. Anal.

The first results on asymptotic series and $(\phi^{2m})_2$ theories were obtained by:

G.12 B. Simon and R. Höegh-Krohn, Princeton preprint, May, 1970.

The strong asymptotic condition and Borel summability of the ground state energy in $(\phi^4)_2$ theories is due to:

G.13 B. Simon, **25**, 1583 (1970).

H. Solutions of ordinary differential equations

In addition to the appendix by A. Dicke in Al, see:

H.1 G. Hsieh and Y. Sibuya, *J. Math. Anal. Appl.*, **16**, 84 (1966).

References

1. We shall be vague about various subtleties of a mathematical nature throughout these lectures; for mathematical rigor, the reader is referred to the literature quoted above. In particular, I have tried to cross every *t* in my Ann. Phys. paper which is the basis of much of the lectures.

2. This theorem deals with perturbations of so-called type (A). There are several other results which are sometimes useful. See Ref. C.1.
3. Technically, $H(\beta)$ must be closed on this domain and $H(\beta)$ must have a non-empty resolvant.
4. This simple criterion holds in the self-adjoint case. When $H(\beta)$ is not self-adjoint, the distance to the nearest eigenvalue must be replace by norms of $(H(\beta) - \lambda)^{-1}$ for various λ. See Kato's book (C.1) for details.
5. One must exercise some care in arguing about the meaning of perturbation series where they converge; see my Ann. Phys. article, Sec. II.3.
6. Consider $\exp(-\frac{1}{2}z^{-1})$ and $[\exp(-z^{-1}) + \varepsilon]^{1/2}$ near $z = 0$.
7. No features of the structure have been proven to occur. One of the most intriguing of these features is that by analytic continuation one can go from any level to any other of the same parity; i.e. there is only one "coupling constant trajectory" of each parity.
8. Recently, Bender and Wu have given a series of reasonableness arguments which predict this asymptotic behavior.
9. While one expects $E_n(\alpha, 1)$ to have only isolated signularities, it has not been proven that natural boundaries and other hairy beasts don't prevent continuation; we ignore this subtlety. For some (unfortunately feeble) discussion see my *Ann. Phys.* paper. For some real results and a beautiful conjecture of Martin, see § 6.
10. θ larger than π is possible for a function with a many sheeted structure; "regular" means continuous on D, analytic in the interior.
11. What is small is dependent on f; there is no reason why 1/137 is small and 14 is not!
12. The approach we use is less general but easier than that of Kato. For still a third approach, see appendix II of my *Ann. Phys.* paper or the paper of Greenlee (D.1).
13. It is our feeling that one should distinguish regular and asymptotic perturbation theory by saying that the former is associated with resolvant convergence $(H(\beta) - \lambda)^{-1} \to (H(0) - \lambda)^{-1}$ *in norm* as $\beta \to 0$ in a circle while the latter involves convergence *in norm* but in a sector. This is in direct conflict with the Kato notion of comparing convergence in *norm* and convergence in the *strong* operator topology. We remark that if one has norm convergence in a sector, the *stability* criterion needed by Kato is true easily.
14. This definition differs from another sometimes used which only requires $\mathrm{Im} f > 0$ if $\mathrm{Im}\, z > 0$ but with $\mathscr{D}$ the entire upper half plans.
15. This distinguishing characteristic of $p^2 + \alpha x^2 + x^4$ from $p^2 + x^2 + \beta x^4$ enters again!
16. The strange mixture of analytic function theoretic arguments with Hilbert space theory gives the subject an eerie quality.
17. This depends critically on the fact that the $\sum b_{-n}\lambda^{2n+1}$ series has only a finite number of negative term e.g. $\sum_{n=-1}^{\infty} b_n\lambda^{2n+1} \to \infty$ as $\lambda \downarrow 0$ unless $b_{-1} = 0$ which lets us conclude $b_{-1} = 0$, etc.
18. A famous theorem of complex analysis says every entire function which is not a polynomial takes every value with the possible exception of one an infinity of times.

18a. For $p^2 + x^2 + x^{2m}$; $m > 2$, the Martin conjecture has be proven.
19. It was Wightman who first proposed the problem to me and who provided inspiration both directly and through his getting Loeffel, Martin, and Symanzik interested in the problem.
20. I first learned about Padé in a letter Wightman sent me from France (where he spent 1968/69) which began: "The spectre of Padé is haunting Europe. *S*-matricists of the world unite! You have nothing to lose but your Chew."
21. It is a theorem about Herglotz functions that $\mathrm{Im} f(z' + i\varepsilon)\, dz'$ always exists as a measure in the limit, and that (2) justifies an unsubtracted dispersion relation.
22. We discuss this further in § 11.
23. This is not quite standard terminology.
24. See their paper for details.
25. The Vitali convergence theorem says any sequence of analytic functions uniformly bounded on compacts, converging pointwise at a set of points z_n with a limit point, converge uniformly on compacts. Proof: By the Cauchy integral theorem, uniform boundedness implies the f_m are equicontinuous, so any subsequence has a subsubsequence converging uniformly on compacts. All the limits are analytic and agree at the z_n so they are all the same.
26. Note the Bender-Wu rigorous bound $|a_n| < CD^n n^{5n/2}$ does not imply $\sum |a_n|^{-1/2n} = \infty$, but the "conjectured" asymptotic behavior $|a_n| \sim CD^n n^n$ does have $\sum |a_n|^{-1/2n} = \infty$.
27. For numerical tables, see my *Ann. Phys.* paper, the Loeffel et al. Letter or Reid's paper.
28. $n \equiv$ dimension of space; $2m =$ degree of perturbation; a.s. = already seen; $u =$ unknown.
29. We emphasize that these cutoff theories are only the beginning of the Glimm-Jaffe program.
30. For: ϕ^{2m}: Hoegh-Krohn and Simon have proven a similar analyticity result but with $\theta < \pi/2$.
31. In particular, the series is definitely *not* a series of Steiltjes.

Dr. A. Martin has given two series of lectures: one about the pion-pion interaction is almost a word by word repetition of the lectures given at the IX Universität für Kernphysik Schladming 1970 (P. Urban editor).

The other series is about the treatment of the anharmonic oscillator. We reproduce here a lecture in French, given at the University of Strasbourg, which covers the subject fairly well.