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Large Orders and Summability of Eigenvalue Perturbation Theory: A Mathematical Overview

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Abstract

The study of large orders of perturbation theory in various problems is reviewed: the anharmonic oscillator, the Zeeman and Stark problems, double wells, and the like. Padé and Borel summability and path integral ideas are discussed. The rigorous results on the subject are summarized.

1. Introduction

Perturbation theory is a theoretical physicist's most powerful tool. In much of nonrelativistic quantum mechanics, the relevant series are the Reyleigh–Schrödinger (RS) series and in quantum field theory, one has the Feynman series. These are closely related; indeed, for the anharmonic oscillator, the RS series can be written in terms of Feynman diagrams (see e.g., Bender and Wu [1] or Simon [2]†).

The convergence of the RS series was initially studied by Rellich in the 1930s with important later contributions by Kato and Nagy. These things are well described in the encyclopedic book of Kato [3] with shorter presentations in Friedrichs [4], Reed and Simon [5], and Rellich [6]. In studying eigenvalues $E(\beta)$ or $H_0 + \beta V = H(\beta)$, a key role is played by the existence of an estimate of the form

$$\|V\varphi\| \leq a\|H_0\varphi\| + b\|\varphi\| \tag{1.1}$$

for some a, b and "all" φ. Indeed, if Eq. (1.1) holds, if E_0 an isolated simple eigenvalue of H_0, then for $|\beta|$ small, $H(\beta)$ has a unique eigenvalue $E(\beta)$ near E_0 and $E(\beta)$ is analytic in β about $\beta = 0$. Condition (1.1) holds automatically for finite matrices, H_0, V, and also for some cases of physical interest; e.g., the $1/Z$ expansion of atomic physics

$$H_0 = \sum_{i=1}^{N} -\Delta_i - |r_i|^{-1}, \quad V = \sum_{i<j} |r_i - r_j|^{-1}. \tag{1.2}$$

(If $\beta = 1/Z$, then $H_0 + \beta V$ is up to rescaling of space and energy, the Hamiltonian of N electrons moving in the field of a charge Z, infinite mass nucleus.)

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† See Section 20 of Ref. 2.

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However, Eq. (1.1) fails in a number of important cases, indeed in the three most standard textbook examples

anharmonic oscillator: $H_0 = p^2 + x^2, \quad V = x^4;$ (1.3)

Stark problem: $H_0 = -\Delta - r^{-1}, \quad V = \hat{e}\cdot\mathbf{x};$ (1.4)

Zeeman problem: $H_0 = -\Delta - r^{-1}, \quad V = x^2 + y^2.$ (1.5)

Equation (1.3) is considered with one degree of freedom; in Eq. (1.4), $\hat{e}$ is a fixed unit vector in the direction of an applied electric field which is proportional to β; in Eq. (1.5), x and y are the components of r orthogonal to an applied magnetic field B; β is proportional to B^2, and the Hamiltonian $H(\beta)$ differs from the true Hamiltonian in magnetic field by a term $B\cdot L_z$. Since $[L_z, H(\beta)] = 0$, we can diagonalize L_z and each $H(\beta)$. We see that the effect of this extra term is to add a B term to the eigenvalues.

Equation (1.1) fails in these examples for a good reason; the eigenvalue perturbations series almost surely have zero radius of convergence in all three cases: for Eqs. (1.3) and (1.4), it is a rigorous result that the series diverge [see Ref. 7 for Eq. (1.3); Ref. 8 for Eq. (1.4)]; in Eq. (1.5) there is no rigorous theorem, but there is tremendous evidence both numerically [9] and theoretically [10] that the radius of convergence is zero.

The amazing fact is that one can recover the eigenvalues of objects like $p^2 + x^2 + \beta x^4$ from the perturbation series in spite of the fact that these series diverge; indeed one has obtained the lowest eigenvalues of $p^2 + x^2 + x^4$ (as an example) to better than 20 places [11,12]!

The situation vis à vis convergence in quantum field theory is similar: There is a very special model, the :cosϕ: model in two space time dimensions, for which the Schwinger functions have convergent Feynman series [13]; in all other renormalized field theories (except for the linear [14] and quadratic Boson theories [15] which are physically trivial), the Feynman series are generally believed to diverge (although there are no rigorous theorems, except for the very simplest models: see Ref. 16 and references therein). We remark that for certain fermion theories *with cutoff* it is known that there is a nonzero radius of convergence [17,18], but there is good reason to believe that as cutoffs are removed, this radius of convergence shrinks to zero.

While we will later discuss the anharmonic oscillator in more detail, we pause now to explain why it should have an RS series $\Sigma a_n\beta^n$ with zero radius of convergence. There are two reasons: One goes back to a celebrated paper of Dyson [19] who argues that quantum electrodynamic should have a zero radius of convergences: For if $e^2 < 0$, like charges attract and the vacuum is unstable under decay into a large number of pairs; since there is no reasonable meaning for the theory when $e^2 < 0$ and series converge in circles, the series should not converge. A similar argument applies to $p^2 + x^2 + \beta x^4$; if $\beta < 0$, the potential goes to $-\infty$ to $\pm\infty$ and there should be no eigenvalues. While this intuition is very useful, we warn the reader that it can be misleading; for example, the operator [20]

$$H(\beta) = p^2 + x^2 + 2\beta(x^3 - x) + \beta^2 x^4$$

has a ground state energy $E(\beta)$, with an RS series which converges for all β (albeit to the wrong answer!) despite the applicability of a Dyson-type argument. Another example [21] is

$$H_0 = -\Delta - r^{-1}, \quad V = r^{-1},$$

where the series $E(\beta) = -\frac{1}{4}(\beta - 1)^2$ is an entire function of β even though $H_0 + \beta V$ has no eigenvalue if $\beta > 1$.

The second reason involves the structure of the RS series. a_n is built out of n matrix elements of V and $(n-1)$ energy denominators. Since x^4 couples a given eigenstate of H_0 to at most five others, the number of terms is bounded by 5^n and the maximum individual term is roughly $n^{2n}/n^{(n-1)}$, so a_n should diverge like $n!$ This argument will yield a rigorous upper bound

$$|a_n| \le C^{n+1}n!$$

(see, e.g., Ref. 21), but it seems difficult to get lower bounds in this way because of possible cancellations in the RS coefficients. One can get lower bounds from the Feynman diagram formulas since they all have the same sign; see, e.g., Ref. 2, Section 20 or Ref. 22. (The original argument is in Ref. 7.)

If $\Sigma a_n\beta^n$ has a finite radius of convergence B, then B determines the leading asymptotics of the a_n, e.g., $B^{-1} = \overline{\lim}_{n\to\infty}|a_n|^{1/n}$, and if a_n is regular, then $a_n \approx (B^{-1})^n \cdots$. If $B = 0$, then even the leading asymptotics is not *a priori* clear. For the anharmonic oscillator, normalized by $H_0 = p^2 + \frac{1}{4}x^2$, $V = \frac{1}{4}x^4$, Bender and Wu [1] computed the first 75 a_n's and, on the basis of these numbers, conjectured the asymptotic formula

$$a_n \approx (\sqrt{6}/\pi^{3/2})(-1)^{n+1}(3)^n\Gamma(n + \tfrac{1}{2}). \tag{1.6}$$

Formulas like Eq. (1.6) are one-half the basic theme to be developed here. At first sight they appear to be intriguing but somewhat only of academic interest. That this is not true is connected with the second half of the basic theme.

What is the meaning of perturbation theory when it does not converge? The answer is somewhat reminiscent of the final chapters of Agatha Christie's *Murder on the Orient Express*, which are entitled something like "The Answer that Doesn't Satisfy" and "The Answer that Satisfies." The traditional answer is that $\Sigma a_n\beta^n$ is an *asymptotic series* for $E(\beta)$, i.e., for each fixed N:

$$\lim_{\beta\downarrow 0}\beta^{-N}\left(E(\beta) - \sum_{n=0}^{N} a_n\beta^n\right) = 0. \tag{1.7}$$

Equation (1.7) says that $E(\beta)$ uniquely determines the a_n; indeed

$$a_N = \lim_{\beta\downarrow 0}\beta^{-N}\left(E(\beta) - \sum_{0}^{N-1} a_n\beta^n\right). \tag{1.8}$$

However just knowing the a_n does not determine E; indeed if $f(\beta)$ has a given asymptotic series a_n, so will $g(\beta) = f(\beta) + 10^{1000}\exp(-1/10^{1000}\beta^2)$ even though for any reasonable value of β, f and g will differ by enormous amounts. Of course, Eq. (1.7) is not totally unsatisfactory; if β is small and E is not too unreasonable, then one can hope to approximate E by taking the first few terms. And physically, β is often very small: $\beta \approx 1/137$ in quantum electrodynamics; the natural unit ($\beta \approx 1$) in the Zeeman problem is about 10^9 G and in the Stark problem is about 10^9 V/cm so that laboratory fields are small. Results on when series are asymptotic were obtained first by Titchmarsh and are described, for example, in Refs. 3 and 5.

From a purely mathematical point of view, one would like a condition weaker than convergence of the series and stronger than Eq. (1.7), so that at most one function could be associated to the series. Calculationally, one would like an explicit procedure for recovering this uniquely determined function from the series. Together, success in these searches comprise "the answer that satisfies." As for the first question, one has the following theorem of Carleman [23]:

Theorem 1.1 (Carleman's theorem): Let $F(z)$ be a function analytic in $D \equiv \{z \mid |z| < R, |\arg z| < \tfrac{1}{2}\pi\}$, continuous in $\overline{D}$. Suppose that for some fixed $A > 0$,

$$|F(z)| \leq A^{n+1} n! |z|^n \tag{1.9}$$

for all n. Then $F = 0$.

With this in mind, we say that $\Sigma a_n \beta^n$ is a *strong asymptotic series* (SAS) for a function $f(\beta)$ if f is analytic in a region of the form D and

$$\left| f(\beta) - \sum_{n=0}^{N} a_n \beta^n \right| \leq A^{N+1} N! |\beta|^{N+1} \tag{1.10}$$

for all N. Clearly, at most one function f has a given series as SAS; since if g also obeys Eq. (1.10), $F(z) \equiv f(z) - g(z)$ will obey Eq. (1.9) (with a changed value of A). Moreover, if f has $\Sigma a_n \beta^n$ as SAS, and g has $\Sigma b_n \beta^n$ as SAS, then (i) $f + g$ has $\Sigma(a_n + b_n)\beta^n$ as SAS; (ii) fg has $\Sigma c_n \beta^n$ with $c_n = \Sigma_{m=0}^{n} a_m b_{n-m}$ as SAS; (iii) if $b_0 \neq 0, f/g$ has $\Sigma d_n \beta^n$ as SAS, where d_n is defined inductively by $d_n = -b_0^{-1}[(\Sigma_{m=0}^{n-1} d_m b_{n-m}) - a_n]$; (iv) if f has $\Sigma a_n \beta^n$ as SAS, $df/d\beta$ has $\Sigma n a_n \beta^{n-1}$ as SAS. We will see shortly that SAS are closely related to Borel summability.

The final element of the answer that satisfies is how to beat a divergent series into submission and force it to yield an answer. Hardy's whole book *Divergent Series* [24] is on this subject. Unfortunately, more of the book deals with only marginally divergent series like $1 - 1 + 1 - 1 + 1 - 1 + \cdots$, which "clearly" sums to $\tfrac{1}{2}$ [e.g., $\tfrac{1}{2} = \lim_{t \uparrow 1} 1 - t + t^2 \cdots$ (Abel sum) and if $s_n = \Sigma_{j=1}^{n} (-1)^{j+1}$, then $\tfrac{1}{2} = \lim_{n \to \infty} (1/n) \Sigma_1^n s_j$ (Cesaro sum)]. There are at least two summability methods which have been successfully applied to some of the badly divergent series encountered above: the *Padé method* and the *Borel method.*

Given a formal power series $\Sigma a_n z^n$, and two non-negative integers N,M, we define the $[N,M]$-*Padé* approximant, to be the unique function

$$f^{[N,M]}(z) = P^{[N,M]}(z)/Q^{[N,M]}(z),$$

with P a polynomial of degree M, Q a polynomial of degree N and with

$$\left| f^{[N,M]}(z) - \sum_{n=0}^{N+M} a_n z^n \right| = O(z^{N+M+1}). \tag{1.11}$$

[The reader should check that Eq. (1.11) places the right number of conditions to uniquely determine f.] Padé approximants are further discussed in Refs. 25 and 26.

As for convergence theorems, one has the following remarkable theorem of Stieltjes:

Theorem 1.2 (the Stieltjes theorem): Suppose that there is a positive measure $d\rho$ on $[0, \infty)$ so that (*series of Stieltjes*)

$$a_n = (-1)^n \int_0^\infty x^n \, d\rho(x). \tag{1.12}$$

Then, for each integer j and all $z \in \mathbb{C}\backslash(-\infty, 0)$, the limit

$$f_j(z) = \lim_{N \to \infty} f^{[N,N+j]}(z)$$

exists and has the form

$$f_j(z) = \int_0^\infty (1 + xz)^{-1} \, d\rho_j(x) \tag{1.13}$$

where $d\rho_j$ solves Eq. (1.12). In particular, if there is only one measure $d\rho$, solving Eq. (1.12) all the f_j are equal. Moreover, if j is even (respectively, odd), then for $z > 0$, $f^{[N,N+j]}(z)$ is monotone decreasing (respectively, increasing) as N increases.

Much more is known, e.g., Eq. (1.12) has a unique solution if and only if $f_1 = f_0$. Note that this theorem provides a constructive solution of the moment problem, i.e., given a_n which are the moments of a measure, one determines $d\rho$. Obviously, it is very useful to have conditions for there to be at most one $d\rho$ solving Eq. (1.12) for then all $f^{[N,N+j]}$ converge to the same function which is given by Eq. (1.13). The following is useful (see, e.g., Ref. 27, p. 343).

Theorem 1.3: If $|a_n| \leq C^{n+1}(2n)!$, then there is at most one $d\rho$ solving Eq. (1.12).

As we explain in Section 2, the RS coefficients for any eigenvalue of $p^2 + x^2 + \beta x^{2m}$ essentially (after changing "sign" and dropping a_0) obey Eq. (1.12). For $m = 2,3$, the coefficients obey the hypothesis of the last theorem and one knows that the Padé approximants converge to the eigenvalue. For $m \geq 4$, there is more than one solution of Eq. (1.12) and the Padé's do not converge to the eigenvalue [28]. Indeed, using the Bender–Wu formula (only rigorously proven [8] subsequent to Ref. 28) and Lemma 3.1 in Ref. 28, this is easy.

Padé approximants have the following advantages: (i) They are very easy to compute. There are explicit determinant formulas for $P^{[N,M]}$ and $Q^{[N,M]}$ in terms of z and a_n. (ii) For series of Stieltjes, they provide rigorous upper and lower bounds. (iii) Since f has lots of poles, one can hope to be approximating more singularities than just the leading singularity one sees in the power series. (iv) The diagonal Padé $f^{[N,N]}$ has a kind of invariance under fractional linear transformations. (v) If the formal series $\Sigma a_n z^n$ obeys

$$(\textstyle\sum a_n z^n)^* (\textstyle\sum a_n z^n) = 1, \quad z \text{ real},$$

in the sense of formal power series, then $f^{[N,N]}$ will obey $|f^{[N,N]}(z)|^2 = 1$, z real. This is relevant when summing up Feynman series.

Some disadvantages relative to the Borel method we will describe next are (i) the method is not known to be *regular*; i.e., even if $\Sigma a_n z^n$ converges for small z, we do not know if $f^{[N,N]}$ converge; (ii) the only *pointwise* convergence result (there are weaker convergence results in general situations) is Theorem 1.2 which requires *global* information on $f(z)$; thus we know Borel summability for n-coupled x^4 oscillators but not Padé; (iii) there is no simple extension to treat a_n growing rapidly, e.g., the x^8 oscillator; (iv) most importantly, there seems no efficient way of using information on the large n behavior of a_n to improve convergence of the approximation.

The other summability method is known as Borel summation. This is based on the formula

$$\int_0^\infty x^n e^{-x/t}\, dx = t^{n+1} n!. \tag{1.14}$$

Given a series a_n obeying

$$|a_n| \le C^{n+1} n!, \tag{1.15}$$

one can define the *Borel transform* $B(z)$ for $|z| < C^{-1}$ by

$$B(z) = \sum_{n=0}^\infty a_n z^n/n!. \tag{1.16}$$

Suppose that $B(z)$ has an analytic continuation in a neighborhood of $[0, \infty)$ and that for $x > 0$, we have

$$|B(x)| \le e^{Dx}.$$

Then, for t real and positive with $t < D^{-1}$, we can form

$$f(t) = t^{-1} \int_0^\infty B(x) e^{-x/t}\, dx \tag{1.17}$$

called the *Borel sum* of $\Sigma a_n t^n$. If we formally interchange sum and integral and use Eq. (1.14), we see why $f(t)$ is formally $\Sigma a_n t^n$. If the series does converge, it is fairly easy to justify this interchange and see that f really is just the sum, i.e., this method is regular.

Here is a divergent series which is Borel summable; let

$$a_n = (-1)^n n!$$

Then $B(z) = (1 + z)^{-1}$ and

$$f(t) = \int_0^\infty (1 + yt)^{-1} e^{-y}\, dy.$$

Note that this a_n is also a series of Stieltjes which satisfies Theorem 1.3, so the Padé method applies here also.

There is a beautiful theorem of Watson, as extended by Nevanlinna (see Sokal [29] for discussion) which justifies Borel summability under suitable hypothesis. Let

$$D(d) \equiv \{z \mid \mathrm{Re}(z^{-1}) > d^{-1}\},$$

which is a circle of diameter d tangent to the imaginary axis at the point 0. Let

$$D(d, \epsilon) = \bigcup_{|\theta|<\epsilon, \theta \text{ real}} [e^{i\theta} D(d)].$$

Let

$$R(b) = \{z \mid |z| < b\} \cup \{z \mid \mathrm{Re}z > 0, |\mathrm{Im}z| \le b\},$$

which is a half-strip capped by a semicircle. Let

$$R(b, \epsilon) = \bigcup_{|\theta|<\epsilon, \theta \text{ real}} [e^{i\theta} R(b)].$$

If $\epsilon = 0$, set $D(d, \epsilon) = D(d)$; $R(b, \epsilon) = R(b)$. The first time through, the reader should think primarily about this case.

Theorem 1.4 (Watson–Nevanlinna theorem): Let $\Sigma a_n z^n$ be a power series and let f be a function analytic in $D(d, \epsilon)$ for some $0 \le \epsilon \le \pi$, and suppose that in that region

$$\left| f(z) - \sum_{n=0}^N a_n z^n \right| \le AB^{N+1}(N + 1)!|z|^{N+1}. \tag{1.18}$$

Then the Borel transform B is analytic in $R(b, \epsilon)$ for any $b < B^{-1}$ and Eq. (1.17) holds for all t (even nonreal t) in $D(d)$. More generally, if $|\theta| < \epsilon$ and $z \in e^{i\theta}D(d)$, then

$$f(z) = e^{i\theta} z^{-1} \int_0^\infty B(e^{i\theta}x) \exp\left(\frac{-e^{i\theta}x}{z}\right) dx. \tag{1.19}$$

Remarks:

(a) Under the above Eq. (1.17) converges absolutely. Since $|\exp(-x/z)| = \exp[-\mathrm{Re}(z^{-1})x]$, we see that $D(d)$ is the natural region for the absolute convergence of integrals like Eq. (1.17). Analyticity in D is *necessary* if Eq. (1.17) converges absolutely.

(b) If a_n is a series of Stieltjes, with $|a_n| \le C^{n+1} n!$, then Eq. (1.18) holds with d arbitrary for any $\epsilon < \frac{1}{2}\pi$. The functions are both Borel and Padé summable.

(c) If $\epsilon > \frac{1}{2}\pi$, the region $D(d, \epsilon)$ is interpreted as multisheeted region.

(d) One can form modified Borel transforms by replacing $n!$ by $(n + b)!$ $[x! \equiv \Gamma(x + 1)]$ for any b and adding an extra x^b in the inverse transform. If one knows the leading asymptotics of a_n, a clever choice of b can make the modified Borel transform have a singularity of prescribed type as its nearest singularity.

(e) If $|a_n| \le C^{n+1}(nk)!$ but f is analytic in a region of opening angle $\frac{1}{2}k\pi$ with suitable estimates, then one can use a modified Borel transform with $n!$ replaced by $(kn)!$ and $B(x)$ replaced by $B(x^k)$.

The advantages of Borel summability are that only local properties are needed and that remarks (d) and (e) give it considerable flexibility. For example, for $p^2 + x^2 + \beta x^{2m}$, one only has Padé summability to the eigenvalue if $m = 2,3$; modified Borel summability with $k = (m - 1)$ works for any integer m. The main advantage is that in numerical analysis, one can feed the asymptotics of a_n into the calculation and improve accuracy; e.g., by clever conformal mapping to remove the nearest singularity. Ironically, even though summability for the x^4 oscillator and asymptotics of a_n for large n had been studied for about five years (by Simon and co-workers and by Bender and Wu, respectively), it was only with the work of Lipatov and of the Saclay group that this connection was appreciated.

There is a *very* unfortunate tendency in the literature to use the phrase "$\Sigma a_n z^n$ is Borel summable" to mean that $a_n \approx (-1)^n a^n (n!) C n^b$ for some a, b, C. We emphasize that first not every such series is Borel summable, e.g., if $\epsilon = \exp(-10^{1000})$, then

$$a_n = n![(-1)^n + \epsilon^{n+1}] \tag{1.20}$$

is not Borel summable. Moreover, for any $\theta \in (0, \pi]$, the sequence

$$a_n = n! \cos(n\theta)$$

is Borel summable even though its coefficients do not alternate sign. The example (1.20) is most upsetting. It really says that one cannot hope to *prove* a series is Borel summable just by looking at it. In fact it appears one needs to be able to construct $f(z)$ *first* and then prove it is the Borel sum of a series by proving Eq. (1.18). For eigenvalue problems, this is easy but it means in field theory that summability cannot at present be used as a mathematical constructive tool.

In Section 2, we discuss Eq. (1.3); in Section 3 we discuss Eq. (1.5); and in Section 4 we discuss Eq. (1.4). We concentrate on an overview of ideas and on the WKB approach to the asymptotics of a_n. In Section 5, we sketch the Lipatov idea in the simplest case. In Section 6, we say something about double wells, the main examples being

$$p^2 + x^2 + 2\beta x^3 + \beta^2 x^4 \tag{1.21}$$

and

$$-\Delta + |x|^{-1} + |x - \beta^{-1}\hat{e}|^{-1} \tag{1.22}$$

(i.e., H_2^+ in the large R limit). In Section 7, we summarize the rigorous results and in Section 8, we tell some personal details of the history of the basic discoveries for the x^4 oscillator.

Nothing will be said here about detailed numbers. To some that will be like the simpleton who tells the whole long buildup to the joke and then forgets the punch line! The reader should not mistake this attitude. The numbers are very important and should certainly be urged to consult the other papers here for them, but the ideas are even more important here. It is impressive that one can compute x^4 oscillator eigenvalues to better than 20 places with the perturbation series. But it seems even more impressive that one can in principle determine the eigenvalues at all from a divergent series.

I should like to close this introduction by expressing my thanks to J. Čížek for the initial suggestion of this workshop, to P.-O. Löwdin for embracing this idea as warmly as he did, and to both of them for their work in implementing the idea.

2. Anharmonic Oscillator

Let $E(\alpha, \beta)$ be the ground state energy of $p^2 + \alpha x^2 + \beta x^4 \equiv H(\alpha, \beta)$, α, β real, $\beta \geq 0$. The reason for adding the extra parameter is made clear by a clever scaling argument of Symanzik [30]. The map $x \to \lambda x$, $p \to \lambda^{-1} p$ is unitarily implementable for λ and so

$$E(\alpha, \beta) = \lambda^{-2} E(\lambda^4 \alpha, \lambda^6 \beta).$$

In particular, for $\beta > 0$,

$$E(1, \beta) = \beta^{1/3} E(\beta^{-2/3}, 1). \tag{2.1}$$

From Eq. (2.1), we see that $E(\beta) \equiv E(1, \beta)$ has a kind of three sheeted structure. Moreover, since $H_0 = p^2 + x^4$, $V = x^2$ does obey Eq. (1.1), $E(\alpha, 1)$ is analytic about $\alpha = 0$ so that $E(1, \beta)$ has an expansion

$$E(\beta) = a_0 \beta^{1/3} + a_1 \beta^{-1/3} + a_2 \beta^{-1} + a_3 \beta^{-5/3} + \cdots \tag{2.2}$$

converging for $|\beta|$ large.

Two papers appearing in 1969–1970 considerably expanded our knowledge of the analytic structure of $E(\beta)$ and of its RS series. The first two by Bender and Wu [1, 7] did many things: (a) computed the first 75 coefficients of the RS series for $E(\beta)$ about $\beta = 0$. If one thinks of the formulas for the RS coefficients or of Feynman diagrams, this seems very forbidding but by using the fact that V only links an unperturbed state to five others, one can get recursion relations. As discussed by Čížek at this conference, this is a theme which in a sense can be repeated in Eqs. (1.4) and (1.5); (b) numerically analyzed the asymptotics, obtaining the empirical ***Bender–Wu*** formula (1.6); (c) studied the analytic structure of $E(\beta)$ in a modified WKB approximation. While this was not rigorous, all the qualitative features are consistent with the rigorous results obtained subsequently. The major features they found are (i) there are no singularities in $|\arg\beta| < \pi$, i.e., in a plane cut along the negative axis, there is an analytic continuation of $E(\beta)$ from $\beta \in [0, \infty)$; (ii) on the natural three sheeted surface given by Symanzik scaling, parametrized by $-3\pi \leq \beta \leq 3\pi$, there are lots of singularities; indeed, an infinite number of ***Bender–Wu*** singularities which accumulate at $\beta = 0$; (iii) these singularities have asymptotic phase $\pm\frac{3}{2}\pi$. Since $E(\beta)$ is real for β real and has a definite phase for $\arg\beta = \frac{3}{2}\pi$, the singularity structure in $0 \leq \arg\beta \leq \frac{3}{2}\pi$ is repeated fourfold, so this asymptotic phase says that the singularities do their utmost to stay out of the way; (iv) the singularities are all square root branch points; (v) by starting at $\beta > 0$, analytically continuing in β then returning to $\beta > 0$, one always obtains an even parity eigenvalue. Moreover, *all* even parity eigenvalues are obtained in this way. Unfortunately, this later fact has never been proven but an analogous fact is known for one-dimensional energy bands as functions of quasimomentun, see Kohn [31] (and Avron and Simon [32]) and the analog is "generically" true for finite matrices; see Refs. 32 and 33. Feature (iv) also is generically true [32].

In a long paper motivated in part by Bender and Wu [7], Simon [21] studied many of these questions rigorously:

(i) He proved feature (ii) in the sense that 0 cannot be an isolated singularity of $E(\beta)$ on a three sheeted surface.

(ii) He proved feature (iii) in the sense that for any $\epsilon > 0$, there is a $B_\epsilon > 0$ with $E(\beta)$ as analytic in $\{\beta | 0 < |\beta| < B_\epsilon, |\arg\beta| \leq \frac{3}{2}\pi - \epsilon\}$.

(iii) He proved the first part of feature (v).

(iv) He noted that if feature (i) holds, then one can write a once subtracted dispersion relation for $E(\beta)$ and read off the formula for the RS coefficients a_n, $n \geq 1$:

$$a_n = (-1)^{n+1} \int_0^\infty x^n \, d\rho(x), \tag{2.3}$$

where

$$d\rho(x) = (\pi x)^{-1}\,\mathrm{Im}E(-x^{-1} + i0). \tag{2.4}$$

(v) If $(p^2 + x^2 + \beta x^4)\psi = E\psi$, then

$$\mathrm{Im}E = \mathrm{Im}\beta \int x^4|\psi(x)|^2dx\left(\int|\psi(x)|^2\,dx\right)^{-1},$$

valid if $|\arg\beta| < \pi$. Thus $\mathrm{Im}\beta > 0$, implies $\mathrm{Im}E > 0$ (on the first sheet), so $d\rho$ in Eq. (2.4) is positive.

(vi) Not knowing, initially that feature (i) held, he numerically computed Padé approximants (using the computed coefficients of Ref. 7) finding rapid convergence and the monotonicity properties that would hold by the Stieltjes theorem if feature (i) were true. (Earlier, similar calculations were done in Refs. 34 and 35).

(vii) He noted that *formally* the asymptotic formula (1.6) is equivalent via Eqs. (2.3) and (2.4) to

$$\mathrm{Im}E(-\beta + i0) = (2/\pi)^{1/2}\exp(-1/3\beta)[1 + O(1)]. \tag{2.5}$$

The main moral to be drawn from the above is that the analytic properties can be quite complicated and that summability can work in spite of the Bender–Wu singularities. Those singularities will not enter in our discussion again although they do enter in some elements of the Stark problem as discussed by Benassi and Grecchi [36].

With this background we turn to the main themes of summability and large orders.

Subsequent to the work of Bender and Wu and Simon, Loeffel, and Martin [37] were able to prove feature (i) by exploiting some very clever ODE arguments which do not extend to multidimensional oscillators. Loeffel et al. [31] then combined these results to announce Padé–Stieltjes summability of x^4 and x^6 oscillators and Graffi et al. proved Borel summability of $E(\beta)$ including the result that the Borel transform $B(x)$ is analytic in a plane cut only in $(-\infty, -B)$ for some B [38].

As for deriving Eq. (1.6), Bender and Wu have published three different demonstrations (in the sense of convincing a reasonable but not a stubborn man). Their first [39] analyzed the recurrence relation and I must confess to have never been convinced that this was much more than the numerical analysis already given. The third demonstration [40,41] is an intriguing analysis of the size of a typical Feynman diagram; in my opinion, this approach deserves much more attention and development than it has received.

The middle demonstration of Bender and Wu is one of the definitive approaches to the large order problem which we will call (along with others) the *Bender–Wu method*. It has three steps [39,42].

Step (1): Obtain, as Simon [21] did, the RS coefficients via a dispersion relation and thereby reduce the problem to obtaining the asymptotics of $\mathrm{Im}E(-\beta + i0)$ as $\beta \downarrow 0$. Much later, Herbst and Simon [43] noted that this step does not even require global analyticity. Suppose $E(\beta)$ is a function obeying (a) E analytic in $\{\beta | 0 < |\beta| < R;\ |\arg\beta| < \pi\}$, continuous in the closure of the this region; (b) $\mathrm{Im}E(-\beta + i0) \geq 0$ for $\beta > 0$; (c) $E(\beta)$ has $\Sigma_{n=0}^{\infty} a_n\beta^n$ as asymptotic series for $\beta \downarrow 0$, β real. Then write $E(\beta)$ for $\beta > 0$ using the Cauchy integral formula and a contour starting at $-R - i0$, running around the circle of radius R to $-R + i0$, then just above the negative axis, around 0 and finally just below the axis to $-R - i0$. One finds [using (b) for technical reasons], that

$$a_n = (-1)^{n+1}\int_0^R x^n\,d\rho(x) + E_n,$$

with $d\rho$ still given by Eq. (2.3) and with

$$|E_n| \leq CR^{-n}.$$

Thus again, the large n behavior of a_n can be obtained by finding the small β behavior of $\mathrm{Im}E(-\beta + i0)$.

Step (2): Interpret $\mathrm{Im}E(-\beta + i0)$ as the lifetime of a decaying state. In the anharmonic oscillator case, when β is very small, $p^2 + x^2 - \beta x^4$ is the formal Hamiltonian in a potential which has high bumps before going to $-\infty$ at $x = \pm\infty$. Thus the eigenvalues of $p^2 + x^2$ should decay by tunneling.

Step (3): Compute the width $\mathrm{Im}E(-\beta + i0)$ for β small by using WKB ideas. Indeed by doing a detailed calculation, Bender and Wu [42] obtained not only Eq. (1.6), but the leading errors in the format of multiplying the right-hand side by $[1 + a/n + O(1/n^2)]$ for explicit a.

These ideas were further developed and extended to other types of oscillators in papers by Bender, Wu, and Banks [44–46]. The paper of Bender and Wu [42], while not rigorous, is sufficiently careful that the rigorous proof of Harrell and Simon [8] could be based on Ref. 42 not only in strategy, but at some points in tactics also. The main problems in the proof are that $p^2 + x^2 - \beta x^4$ is not a reasonable operator (indeed it is not essentially self-adjoint) and so Step (2) must be properly interpreted and that one must do a lot of slugging to get rigorous error estimates in WKB. One exploits a variation of parameters approach to WKB developed especially by Froman and Froman and co-workers [47–49].

3. Zeeman Problem

By the (hydrogen) Zeeman Hamiltonian, we mean the Hamiltonian of a "hydrogen" atom in a constant magnetic field B. If the field is in the z direction it is convenient to use the azimuthal gauge

$$\mathbf{a}(x, y, z) = (-\tfrac{1}{2}By, \tfrac{1}{2}Bx, 0),$$

so that (in at. u., $m = e = \hbar = 1$)

$$H(B) = \tfrac{1}{2}(i\nabla - \mathbf{a})^2 + |r|^{-1} \tag{3.1}$$

commutes with $L_z = -i(x\,\partial/\partial y - y\,\partial/\partial x)$, indeed,

$$H(B) \equiv -\tfrac{1}{2}\Delta - |r|^{-1} - BL_z + \tfrac{1}{8}B^2\rho^2, \tag{3.2}$$

where

$$\rho^2 = x^2 + y^2.$$

Because L_z and $H(B)$ commute, L_z can be diagonalized and so the BL_z term acts like a constant; i.e., in studying Eq. (3.2), we may as well drop the BL_z and think of the operator

$$-\tfrac{1}{2}\Delta - |r|^{-1} + \alpha p^2. \tag{3.3}$$

Analytic properties of the eigenvalues in α were studied by Avron, Herbst, and Simon [9,50,51] (who did not restrict themselves to hydrogen). The region of analyticity includes a cut circle so they were able to obtain Borel summability for hydrogen and for all other atomic systems.

Mathematically, there is a subtle problem Avron et al. had to solve which does not enter in the oscillator case. An important step in controlling complex coupling constant is *stability of eigenvalues*, i.e., there are eigenvalues of $H_0 + \alpha V$ (α complex) close to those of H_0. In the oscillator case this follows from *norm* convergence of the resolvent of $p^2 + x^2 + \beta x^4$ to that of $p^2 + x^2$ as $|\beta| \downarrow 0$ as long as $|\arg\beta|$ stays in $(-\pi + \epsilon, \pi - \epsilon)$. This cannot hold for Eq. (3.3) because for $\alpha \neq 0$, $|\arg\alpha| < \pi$, Eq. (3.3) cutoff in z, has a compact resolvent but $-\tfrac{1}{2}\Delta - |r|^{-1}$ does not. An interesting alternative to the approach Avron et al. used for this stability question has been presented by Hunziker and Vock [52].

The above concludes the first step in the Bender–Wu scheme as described in Section 2. The other steps in the scheme were implemented by Avron [10] (see his contribution to this issue) who found $E(B) \sim \Sigma_{n=0}^{\infty} a_{2n}B^{2n} + a_1B$ with

$$a_{2n} \approx (-1)^{2n+1}(4/\pi)^{5/2}\pi^{-4n}(4n + \tfrac{1}{2})! \tag{3.4}$$

in agreement with numerical calculations [9]. The difficulty is that even after taking azimuthal symmetry into account, the problem is intrinsically two dimensional so that WKB is conceptually and especially mathematically in much worse shape than in one dimension. For this reason, Eq. (3.4) has alluded mathematical proof thus far.

Another aspect of this problem is that the path integral formalism does not appear to be applicable; see Section 5.

Finally, we note that the units here are such that $B = 1$ corresponds to a field of about 10^9 G. The laboratory fields are extremely small but in astrophysical applications, one would like to be able to compute $E(B)$ fairly accurately for $B \approx 5$ to 1000. In the analogous regions for anharmonic oscillators, one can compute E to 20 places, but so far accurate computations in the hydrogen Zeeman problem have not been made. (At this conference, Zinn-Justin described preliminary calculations which seem to be very accurate.) One of the difficulties is that the large B asymptotics is slow and complicated containing $(\ln B)^{-1}$ and $\ln(\ln B)/\ln B$ terms [53,54].

See Ref. 55 for additional information including degenerate levels.

4. The Stark Problem

By this, we mean the family of Hamiltonians

$$H(F) = -\Delta - r^{-1} + Fz \equiv H_0 + FW, \tag{4.1}$$

describing hydrogen in a constant electric field. At first sight, it appears that summability methods cannot possibly be applicable here, because the eigenvalues of H_0

dissolve into continuous spectrum for $F \neq 0$ (and real). In some sense, what occurs is a resonance and so it should have an imaginary part. But direct summation of a real series (and the RS series is real) should be real. In addition, one finds that for the ground states, the series have the form $\Sigma a_{2n}F^{2n}$ and calculation of the first few a_{2n} show they do not seem to alternate sign but to all be negative.

The beautiful discovery of the applicability of summability ideas to the Stark problem in hydrogen by Graffi and Grecchi [56] was a considerable surprise. Their work is based on an old idea of Schrödinger [57] and Epstein [58] that if $H(F)\psi = \lambda\psi$ is multiplied by r, the system separates in "squared parabolic coordinates" into anharmonic oscillators; explicitly, if $\lambda(F)$ is an eigenvalue of angular momentum l of Eq. (4.1), and if $E(\alpha, \beta)$ are the eigenvalues of $-d^2/dx^2 + \alpha x^2 + \beta x^4 + (l^2 - \tfrac{1}{4})x^{-2}$,

$$E(-\lambda, \tfrac{1}{2}F) + E(-\lambda, -\tfrac{1}{2}F) = 2. \tag{4.2}$$

{Subsequently, Benassi et al. [59] realized that Eq. (4.2) implies that a formula of Banks et al. [46] for $\mathrm{Im}E(1, -\beta + i0)$ is "equivalent" to "Oppenheimer's formula" for $\mathrm{Im}\lambda(F)$.}

The Graffi–Grecchi work was considerably illuminated by simultaneous work by Herbst [60] on extension of complex scaling ideas [see the review issue of Int. J. Quantum Chem. **14** (1978)] to Eq. (4.1). The two were then synthesized by Herbst and Simon who were able to prove summability in general complex atoms [20,43].

What arises is the following picture:

(a) If $\lambda(F)$ is defined to be the resonance of $H(F)$ near $E_0 = -\tfrac{1}{4}$ with $\mathrm{Im}\lambda(F) > 0$, then $\lambda(F)$ has an analytic continuation to a region $\{F \mid |F| < F_0; -\delta \leq \arg F \leq \pi + \delta\}$ (actually δ can be taken to $\tfrac{1}{2}\pi$).

(b) $\lambda(F)$ has an asymptotic series in the region above of the form $\Sigma a_{2n}F^{2n}$.

(c) By writing λ as a function of F^2 and using the ideas in Step (1) of Section 2 or equivalently using a Cauchy formula with a contour consisting of a semicircle and a piece above the axis, one reads off the Herbst–Simon formula

$$a_{2n} = 2\pi \int_0^R F^{-2n-1}\,\mathrm{Im}E(F)\,dF + O(R^{-2n}). \tag{4.3}$$

(d) The function $g(x) \equiv \lambda(ix)$ is Borel summable from the series $(-1)^n a_{2n}$. $\lambda(F)$ for real F cannot be recovered directly from Eq. (1.17), but it can be recovered from Eq. (1.19) by choosing θ suitably.

If one plugs the asymptotic formula for $\mathrm{Im}\lambda(f)$ [61],

$$\mathrm{Im}\lambda(F) \approx -\tfrac{1}{4}F^{-1}\exp(-\tfrac{1}{6}F) \tag{4.4}$$

into Eq. (4.3), one finds

$$a_{2n} = -6^{2n+1}(2\pi)^{-1}(2n)![1 + O(1)] \tag{4.5}$$

in agreement with numerical calculations by Silverstone [62] and by the Waterloo group [63]. For further details including excited, especially degenerate states, see Ref. 63. Privman [64] has found that with proper normalization the a_n are all integers and computed the first 30 a_n exactly, e.g., a_{30} is given as a 130 digit number!

5. Path Integrals

There is a very elegant and powerful approach to the large orders problem invented by Lipatov [65] and extended and developed especially by the Saclay group in a series of papers including [66–69]. To describe the idea, let $H(\beta) = \frac{1}{2}(p^2 + x^2 - 1) + \beta x^4$, and for T fixed, look at the perturbation series

$$\sum b_n(T)\beta^n \tag{5.1}$$

for

$$\mathrm{Tr}\{\exp[-TH(\beta)]\}. \tag{5.2}$$

The theory of path integrals (see, e.g., Ref. 2) yields an expression for Eq. (5.2), viz.,

$$\int \exp\left(-\beta \int_0^T q^4(s)\, ds\right) d\nu(q), \tag{5.3}$$

where $d\nu(q)$ is a measure on continuous functions $q:[0, T] \to (-\infty, \infty)$ [with boundary condition $q(T) = q(0)$] which is formally

$$N^{-1} \exp\left(-\frac{1}{2}\int_0^T \dot{q}^2(s)\, ds - \frac{1}{2}\int_0^T q^2(s)\, ds\right) d^\infty q. \tag{5.4}$$

Here N is a formally infinite normalization constant arranged so that $\int d\nu = 1$ and $d^\infty q$ is formally $\pi\, dq(s)$. Reading $b_n(T)$ from Eq. (5.3), we see that

$$b_n(T) = \frac{(-1)^n}{n!} \int \exp\left[n \ln\left(\int_0^T q^4(s)\, ds\right)\right] d\nu(q) \tag{5.5}$$

Changing variables from q to $\sqrt{n}\, q$, we get a formal expression

$$b_n(T) = (-1)^n \frac{n^{2n}}{n!} \int \frac{\exp[-nF(q)]\, d^\infty q}{N'}, \tag{5.6}$$

with

$$F(q) = \frac{1}{2}\int_0^T \dot{q}^2(s)\, ds + \frac{1}{2}\int_0^T q^2(s) - \ln \int_0^T q^4(s)\, ds.$$

If $f_0(T)$ is the minimum of F over all q obeying the $q(T) = q(0)$ boundary condition, this suggests that the leading asymptotics for b_n should be $(-1)^n[n^{2n}/n!]\exp[-nf_0(T)]$. Using the formula

$$E(\beta) = -\lim_{T\to\infty} -\frac{1}{T}\ln \mathrm{Tr}\{\exp[-TH(\beta)]\}$$

and interchanging the T and n limits, one finds $E(\beta) \approx \Sigma a_n\beta^n$ with

$$a_n \approx (-1)^{n+1}[n^{2n}/n!]\exp(-cn),$$

where

$$c = \min\left(\frac{1}{2}\int_{-\infty}^{\infty} \dot{q}^2(s)\, ds + \frac{1}{2}\int_{-\infty}^{\infty} q^2(s)\, ds - \ln \int_{-\infty}^{\infty} q^4(s)\, ds\right).$$

This, in fact, yields the leading [i.e., the $n!$ and $(3)^n$ in Eq. (1.6)] behavior in the Bender–Wu formula. By making a Gaussian approximation about the minimizing point, one finds that one can also compute the constant and a systematic $1/n$ series.

This "Lipatov method" has the advantage of being formally extendable to field theories and other situations.

Interestingly enough, the asymptotics of integrals like

$$\int \exp[-nG(x/\sqrt{n})]\, d\nu(q)$$

using these ideas was studied in the mid-sixties by two students Pincus [70] and Schilder [71] of Donsker although the context was rather different and their G not so singular. For more recent developments of these ideas, see Donsker and Varadhan [72], Simon [2], Ellis and Rosen [73–75], and Davies and Truman [76]. Spencer [77] found an elementary way of rigorously proving the very leading [$n!$ and $(3)^n$] behavior in the Bender–Wu formula with these ideas. Breen [78] has with more work obtained for the full formula.

Avron et al. [50] have noted that these ideas do not seem to be applicable to the Zeeman problem. The corresponding $\exp[-TH(\beta)]$ does not have a finite trace. The natural thing to look at is

$$(\psi_0, \exp[-TH(\beta)]\psi_0)$$

with ψ_0 the unperturbed ground state. But the corresponding $b_n(T)$ are bounded for T fixed by $c(T)^n(n/2)!$ while the coefficients of $E(\beta)$ go like $n!$ so that one cannot interchange the T and n limits.

As a final remark, we note that the Bender–Wu method is applicable to excited states while the Lipatov method seems to be limited to the ground state.

6. Double Wells

A very interesting class of examples is illustrated by the double-welled anharmonic oscillator:

$$H(\beta) = -\frac{d^2}{dx^2} + x^2 + 2\beta x^3 + \beta^2 x^4. \tag{6.1}$$

Since $V(x) = x^2 + 2\beta x^3 + \beta^2 x^4 = x^2(1 + x\beta)^2$ is even about the point $x = -1/2\beta$ V has identical minimum about the points $\lambda = 0$ and $\lambda = -\beta$ separated by a high wall of height $\frac{1}{16}\beta^2$ for β small. Thus for small β, we expect two eigenvalues near each eigenvalue of $-d^2/dx^2 + x^2$. The following are the main features:

(1) The pair of levels is separated by an amount going to zero exponentially in $1/\beta$ as $\beta \downarrow 0$, with asymptotics given by WKB [5,68,79–81].

(2) The perturbation coefficients a_{2n} of the eigenvalue defined by $E(\beta) \sim \Sigma a_{2n}\beta^{2n}$ diverge like $DC^n n!$ (to be compared with $(-1)^{n+1}DC^n n!$ in the single-welled case) [68].

(3) There is a *formal*, only *partly correct* formula analogous to (2.3), (2.4):

$$a_n \approx \int_0^{\beta_0} \beta^{-n}\, d\rho(\beta), \tag{6.2}$$

$$d\rho(\beta) = \beta^{-1}[\tfrac{1}{2}\Delta E(\beta)]^2\, d\beta, \tag{6.3}$$

where $\Delta E(\beta)$ is the splitting of the corresponding eigenvalue [82,83]. Equations (6.2) and (6.3) are semiempirical, based in part on a Lipatov intuition. They seem to give the right answer for D and C but not for the $1/n$ corrections.

An interesting application of these ideas has been to the $1/R$ expansion in H_2^+, i.e., to $E(R)$, the eigenvalues of

$$-\Delta - |\mathbf{x}|^{-1} - |\mathbf{x} - R\hat{e}|^{-1},$$

with $\hat{e}$ a fixed unit vector. Since this looks like a double well [79,80], Morgan and Simon [84] expected the coefficients a_n in an asymptotic series to grow like $DC^n n!$ The first 11 a_n which had been previously calculated did not seem to have $n!$ growth but almost magically at $n = 12$, the ratio a_{n+1}/a_n locked into a proper ratio. They numerically computed C and D to two figures. Brezin and Zinn-Justin [83] tried formulas (6.2) and (6.3) and from the known [85,86] asymptotics for $(\Delta E)(R)$, they determined C and D analytically. See Ref. 87 for more accurate formulas for a_n and numerical asymptotics.

At this conference both Damburg and Silverstone (the latter describing joint work with Graffi and Harrell) indicated an approach for computing asymptotics of double well a_n which might lead to a rigorous proof. The idea is that the models are summable along $\beta = i$ and can be continued back to real β where they have a nonzero imaginary part. (In some double wells, this has been done by Caliceti, Graffi, and Maioli [88]). On the real axis, the value is the eigenvalue of the initial differential equation but with a non-self-adjoint boundary condition. If WKB methods can compute the asymptotics of the imaginary part of this eigenvalue, then a dispersion relation will yield the asymptotics of a_n.

7. A Summary of Rigorous Results

Here we want to give references for which proofs are given for things discussed here are proven rigorously and to state the more important open questions from a mathematical point of view. We emphasize we use "Borel summable" to mean more than just $(-1)^n CD^n n!$ asymptotics.

A. Anharmonic Oscillator

The series are asymptotic (see Refs. 3 and 5 and references therein), Borel summable for any finite number of degrees of freedom [38] and Stieltjes summable for one degree of freedom [89]. The Borel transform is analytic in a plane cut in $(-\infty, -A)$. The Bender–Wu formula has been proven by a method following the Bender–Wu method [8] and by a method following the Lipatov method [78].

B. Field Theories

The earliest Borel summability results for cutoff ϕ_2^4 theories are in Refs. [90,91]. The Feynman series for the Schwinger functions for ϕ_2^4 [92], Y_2 [93,94] and ϕ_3^4 are Borel summables [95]. (For earlier results on asymptotic series see Ref. 96 for ϕ_2^4, Refs. 97 and 98 for Y_2, and Refs. 99 and 100 for ϕ_3^4.) Many other series in the ϕ_2^4 and ϕ_3^4 theories are also Borel summable [101,102].

C. Zeeman Effect

The RS series are asymptotic and Borel summable [54]. Enough analyticity is known to reduce the large n behavior of a_n to small β behavior of $\operatorname{Im}E(-\beta + i0)$.

D. Stark Effect

The RS series for the resonances in hydrogen are asymptotic [60] and Borel summable [56] about imaginary field. The same is true for complex atoms [43,103]. There is enough analyticity to reduce the asymptotics of a_n to those of $\operatorname{Im}E(\beta + i0)$ [43]. For hydrogen, these asymptotics have been rigorously controlled [8].

E. Double-Well Oscillator

The levels split in two with each having the same asymptotic RS series [5]. The splittings are given rigorously by WKB tunneling formula [81,86].

F. 1/R Expansion

For general molecular systems, the series are asymptotic [84]. For H_2^+, the splittings have been rigorously computed [86].

Here are some of the major open questions in this area from the point of view of rigorous results:

(a) *Double wells: Asymptotics of a_n:* Is there any precise meaning one can give to something like Eqs. (6.2) and (6.3)? If not, can one rigorously obtain asymptotics of a_n by some kind of instanton analysis?

(b) *Double wells: Summation:* Is there any natural procedure to obtain the eigenvalues (there are two of them) from the perturbation theory? It would be very impressive to obtain the Born–Oppenheimer curves for $H_2^+$to 20 place accuracy!

(c) *Zeeman: Asymptotics of a_n:* Can one make Avron's analysis [10] rigorous? This would be very interesting since it would require mathematical control of multidimensional WKB of the type thus far only available for one-dimensional WKB by the use of ODE methods.

(d) *Stark for complex atoms:* Can one control the width of atoms other than hydrogen as $F \downarrow 0$? This would also require multidimensional WKB.

(e) *Lipatov method for field theories.* Can one prove rigorous results on the asymptotics of the perturbation coefficients in a field theory? The simplest object (the analog of $\operatorname{Tr}\{\exp[-TH(\beta)]\}$ controlled in Ref. 2) would be

$$\int d\mu_0(\phi)\, \exp\left(-\beta \int_{|x|<1} :\phi^4(x):d^2x\right),$$

with $d\mu_0(\phi)$ the Euclidean field Gaussian measure [2] and :-: is Wick ordering.

(f) *Average Feynman diagrams:* Can one prove the BW formula rigorously by their approach in Ref. 104?

(g) *BW formulas for the coefficients in* n^{-1} *expansion:* We have heard from Silverstone at this conference the coefficients b_k in the improved equation (1.6),

$$a_n \approx (\sqrt{6}/\pi^{3/2})(-1)^{n-1}3^n\Gamma(n+\tfrac{1}{2})(1+b_1n^{-1}+\cdots+b_jn^{-j}+\cdots),$$

themselves seem to obey a BW formula. Can one prove anything rigorously about this? Since the b_j are given in terms of diagrams [105], maybe the approach of Ref. 104 would be useful.

At the conference, intriguing ideas were presented concerning problems (a) (by Damburg and by Silverstone), (b) (by Zinn-Justin), and (d) (by Avron).

8. Some Personal Reminiscences

Having spent much of my professional career so far as a kind of wunderkind, it is a new (and on the whole pleasant) experience to be in a position where I can (perhaps immodestly) regard myself (together with Carl Bender and Tai Wu) as one of the "grand old men" of the subject of a conference. I hope I will therefore be indulged, especially in light of the retrospective nature of part of this conference, if I tell a few of the incidents in my own part of the story.

Around 1967 both Arthur Wightman at Princeton and Carl Bender and Tai Wu at Harvard, unaware of the other's interest, began to think about the question of whether the perturbation series for the eigenvalues of $p^2+x^2+\beta x^4$ had more to do with the eigenvalues than merely being asymptotic, whether other perturbation series (for example, about $\beta=\infty$) might be useful and the related analyticity questions.

I should emphasize that in both cases the motivation was to think of this as a model quantum field theory. I know Wightman and I assume Bender and Wu were not really concerned with accurate calculations of the eigenvalues; indeed, one can get extremely accurate eigenvalues with variational methods. This point was lost by some of the people who wrote later; one of the few penalties of having worked in this area has been the refereeing of papers which find a method which can get eigenvalues to only two or three places by some uncontrolled technique with my paper and the Bender–Wu papers quoted to show why the subject is interesting!

Wightman gave the subject as a Ph.D. thesis problem to Arnie Dicke, a graduate student at Princeton and Bender used this as his thesis problem. By early 1968 when I got involved I was working on a different problem under Wightman's supervision, also one on convergence of field theory perturbation series [18]. I got involved originally because of a technical problem Dicke and Wightman were having justifying Symanzik scaling for complex coupling.

At the time I was trying to absorb some of the things in Kato's famous book on perturbation theory [3] and was regarded as a local expert on the subject although I was only an expert in the relative sense (considering others in Princeton) rather than any absolute sense. When I initially tried to read Kato, I cursed it soundly: Here he was telling me all about the problems of general operators while I had the feeling that

no one could possibly care about anything but self-adjoint operators which were clearly much simpler. However, once I got into the complex coupling business and later when I worked on complex scaling, I blessed the book!

The solution of the complex Symanzik scaling problem was simple: One implements the scaling for real parameter and then rather than trying to implement the scaling for complex parameters, one only invokes analyticity of certain subsidiary functions. That is, rather than argue that the operators $p^2+x^2+\beta x^4$ and $e^{-2i\theta}(p^2+e^{4i\theta}x^2+e^{6i\theta}\beta x^4)$ are unitarily equivalent (they are not), one deduces equality of eigenvalues by invoking analyticity of the eigenvalues of $p^2+\alpha x^2+\beta x^4$ in α and β and the equality if $e^{i\theta}$ is replaced by a real number.

This is of course one part of the ideas introduced by Combes under the rubric "dilation analyticity," now usually known as complex scaling. This illustrates the "missed opportunities" aspects of the history; places where looking back, one can see that I failed to pursue a direction that would have been promising. This is not to say that I can imagine having made the Combes discovery by pursuing this line. Rather, while I worked in both areas, I am embarrassed to say that I did not appreciate the close connection until the work of Graffi and Grecchi [56] roughly five years after the Combes work. And this realization overcame some psychological barriers I had in trying to understand how the Bender–Wu method could be made precise leading to my joint work with Harrell [8].

It is not surprising that Wightman and Wu had professional connections but perhaps a trifle surprising that Bender, Dicke, and I did; after all graduate students at different schools often don't know one another. But I had been an undergraduate at Harvard when Carl arrived as a graduate student and we took a number of courses together. Moreover, a fellow graduate student of Carl's named Kenny Klein had known Arnie Dicke as an undergraduate. In fact, most of the Bender–Wu results (from Ref. 1) were learned in Princeton first by Kenny telling Arnie. One should not think that there was a situation of tremendous competition between Harvard and Princeton in this work. In the first place, Bender and Wu had results six months to a year before we had any interesting results. More importantly, the level of rigor and thus the methods were very different and to some extent the concerns differred: Bender and Wu worked mainly on the series itself and we on how the series related to the eigenvalues. The main overlap was in the analyticity properties where there is no question that the intuition developed by Bender and Wu using an uncontrolled approximation was invaluable in the rigorous work we did.

One example involved the Bender–Wu singularities. The initial hope at Princeton was that perhaps the large β expansion (2.2) had infinite radius of convergence, thereby making up for the zero radius of convergence of the RS series. The Bender–Wu singularities, if they really existed independently of the WKB approximation, destroyed this idea and their existence became important to us. In 1968–1969, Wightman was on leave at the IHES near Paris. He discussed the anharmonic oscillator at several places he visited and in particular at CERN where H. Epstein, V. Glasser, J. J. Loeffel and especially A. Martin made valuable comments during his visit and later. In particular, Martin gave a simple proof that $E(\alpha,1)$ could not be an entire function: If $(p^2+\alpha x^2+x^4)\varphi=E(\alpha,1)\varphi$, then

$$\operatorname{Im}E(\alpha,1)=\operatorname{Im}\alpha\left(\int x^2|\varphi(x)|^2\,dx\Big/\int|\varphi(x)|^2\,dx\right)$$

so that $E(\alpha, 1)$ is a Herglotz function, i.e., Im$E > 0$ when Im$\alpha > 0$. Every such entire function is linear but it is not hard to see that $E(\alpha, 1)$ is not linear. Thus E cannot be entire and Martin had proven the existence of at least some singularities.

Fortunately, when Wightman wrote me about Martin's result, and I went to the library to look up the proof of the fact that every entire Herglotz function is linear, I could not understand the proof which appealed to the Herglotz representation theorem. I had to find my own proof which extended to show that if $f(\alpha)$ is analytic near ∞, and if Im$f > 0$ when Im$\alpha > 0$, then

$$f(\alpha) = \sum_{-\infty}^{1} a_n \alpha^n,$$

and from this one could see that $E(\alpha, 1)$ could not be analytic near infinity, i.e., it had to have infinitely many singularities.

Encouraged by Dicke and Wightman, I continued studying the analytic properties. In the early spring of 1969, I got a letter from Wightman which began "The specter of Padé is haunting Europe. *S* Matricists of the world unite" (I always gave him high Marx for this suggestion). During 1968–1969, Daniel Bessis in Saclay had systematically been applying Padé to partial wave amplitudes in the perturbation theory for realistic field theories and he found that if coupling constants were arranged to get a few particle poles right, other poles came right (thus partially verifying the vision of G. Chew enshrined in the then popular "*S*-matrix theory"). Bessis had a large number of very talented young theoretical physicists working with him; three of them, J. Zinn-Justin, S. Graffi, and V. Grecchi, eventually played important roles in the ideas I have reviewed in this paper. Saclay is just down the valley from Bures where the IHES is situated and Wightman got a good dose of the potential of Padé. He wrote suggesting that I try to compute Padé for the anharmonic oscillator (I heard later that Sidney Coleman had asked Bender during his Ph.D. oral exam if he had considered computing Padé approximants from his series but the suggestion was never followed up).

I know almost nothing about computers (and what I learned then was somewhat like the German I learned for my language exam in grad school; having crammed it in rapidly, I quickly forgot it all!). I was fortunate in several ways. First, Bender and Wu had already computed the a_n up to $n = 75$ and secondly the Padé table is given by very simple determinantal formulas. With good determinant subroutines, the program is rather trivial. Thirdly, I had a good friend named Rick Bauer who was a fellow graduate student and who helped me to write the program. Finally, at the time graduate students could compute for free on a program called WHAT FOUR (or WHATNOT or something like that) which allowed one to run for up to six minutes at a time (real time in a time sharing system). I discovered that I could compute diagonal Padé's, up to [20,20] within the allotted time. This was lucky since I later heard that around [23,23], roundoff error tended to pile up giving nonsensical answers. The numbers were spectacular; one had rapid monotone convergence to answers in agreement with the variational numbers. (I later learned that Reid [95] and Rousseau [35] had done a similar calculation but not having the Bender–Wu numbers not to such high order.)

I had beginner's luck with the program. The first one I wrote worked beautifully

after one hitch. Initially, I put in the a_n all as positive numbers without putting in the alternating signs but once I caught this it computed perfectly.

I misunderstood a result quoted in Baker's review article [25] and using the non-theorem that resulted, I found a "proof" of the convergence of Padé which the referee ripped to shreds. That summer (1969), I attended a summer school at Brookhaven where Nick Khuri was one of the lecturers. He was so taken with my numbers, he made sure I visited Rockefeller to show the results to Andre Martin who was passing through. He and Loeffel then were able to provide most of a proof which was hammered out in some correspondence between them and Wightman which resulted in our joint announcement. Loeffel, who I did not meet for some years afterwards, was the first of my coauthors whom I only met subsequent to our collaboration.

Having met success with Padé, I took Hardy's book [24] out of the library. However a quick perusal only showed me methods devoted to marginally divergent series and I missed his discussion of Borel summing completely. During 1969–1970 I preprinted my long paper [21] and sent it to various people and fortunately S. Graffi and V. Grecchi got a Xerox copy from someone, probably Bessis.

In the summer of 1970 I traveled to Europe for the first time mainly to be a minor speaker at a Padé festival Bessis was running in Cargèse and to be a student at a summer school at Les Houghes. While I was in Cargèse, Grecchi sought me out. He and Graffi had done some Borel sum calculations for the anharmonic oscillator and he came to me with that, with a Xerox of the relevant part of Hardy's book and with a sketch of how to use my results to verify the summability. There was one step that they wanted me to fill in. When I was able to, we agreed to write a three-authored paper. He and Graffi were supposed to visit me in Les Houches but due to some passport irregularity Grecchi was stopped on the Italian side of the Mont Blanc tunnel so only Graffi showed up. This time, I had met both coauthors, but not together!

I should mention a few facts about the Bender–Wu formula before closing this section. Perhaps the most impressive element in the history is the constant $\sqrt{6}/\pi^{3/2}$ in front. The Γ and 3^n are almost trivial to read off from the a_n and when they were done Bender and Wu had a constant to a few figures. They somehow concluded this constant was $\sqrt{6}/\pi^{3/2}$; amazingly this was the right answer!

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