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Almost Periodic Schrödinger Operators: A Review*

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We review the recent rigorous literature on the one-dimensional Schrödinger equation, $H = -d^2/dx^2 + V(x)$ with $V(x)$ almost periodic and the discrete (= tight binding) analog, i.e., the doubly infinite Jacobi matrix, $h_{ij} = \delta_{i,j+1} + \delta_{i,j-1} + V_i\delta_{i,j}$ with V_n almost periodic on the integers. Two themes dominate. The first is that the gaps in the spectrum tend to be dense, so that the spectrum is a Cantor set. We describe intuitions for this from the point of view of where gaps open, and from the point of view of anomalous long time behavior. We give a theorem of Avron and Simon, Chulaevsky, and Moser that for a generic sequence with $\Sigma|a_n| < \infty$, the continuum operator with $V(x) = \Sigma a_n \cos(x/2^n)$ has a Cantor spectrum. The second theme involves unusual spectral types that tend to occur. We describe recurrent absolutely continuous spectrum, and show it occurs in some examples of the type just discussed. We give an intuition for dense point spectrum to occur, and some theorems on the occurrence of point spectrum. We sketch the proof of Avron and Simon, that for the discrete case with $V_n = \lambda\cos(2\pi\alpha n + \theta)$, if $\lambda > 2$ and α is a Liouville number, then for a.e. θ, h has purely singular continuous spectrum.

1. INTRODUCTION

In many years, flu sweeps the world. The actual strain varies from year to year; some years it has been Hong Kong flu, some years swine flu. In 1981, it was the almost periodic flu! There has been recent work on Schrödinger's operators with almost periodic potentials, by Avron and Simon [1–5], Bellisard and Testard [6, 7], Bellisard et al. [8], Chulaevsky [9], Johnson [10], Moser [11], Johnson and Moser [12], and Sarnak [13]. It has been an international outbreak; the above includes several Americans and Frenchmen, and an Israeli, a Russian, a German, and a South African. As with other strains of flu, there were earlier isolated outbreaks: We mention important earlier nonrigorous work by Az'bel [14], Aubry [15], Aubry and

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André [16], Hofstader [17], and Sokolov [18]; and rigorous results by Gordon [19], see Section 7 and by Dinaburg and Sinai [2], see Section 3, extended by Russmann [21]. Moreover, as we shall explain, a.p. potentials can be viewed as a kind of ergodic random potential, so that work of Benderskii and Pasteur [22] (extended by Shubin [23]), Pasteur [24, 25], Kunz and Soulliard [26] and Kirsch-Martinelli [27] become relevant. (But the deeper results of Goldshade, Molchanov, and Pasteur [28] are not applicable, and seem to require at least "mixing" randomness.)

We consider two classes of operators:

$$\frac{-d^2}{dx^2} + V(x) \qquad \text{on } L^2(-\infty, \infty)$$

with V a.p. on $(-\infty, \infty)$ and the discrete analog:

$$h_0 + V \qquad \text{on } l_2,$$

$$(h_0)_{ij} = \delta_{i,j+1} + \delta_{i+1,j}, \tag{1}$$

$$(V)_{ij} = V(j)\delta_{i,j}, \tag{2}$$

with V a.p. on Z (the $-2\delta_{ij}$ one would like to put in h_0 has been suppressed since it is just a multiple of the identity). Below, we review the definition and some properties of a.p. functions; think of them as uniform limits of finite linear combinations of unitary exponentials e^{ikx}.

Two main themes concern us here:

(1) There is a tendency for the spectrum to be a Cantor set, i.e., a closed set with no isolated points and whose complement is dense.

(2) Normally, in the strong coupling regime there is some dense point spectrum. This will only be true if the frequency module (defined below) has typical diophantine properties (as specified below). If the ratios of generators of the frequency module are Liouville numbers, then there tends to be singular continuous spectrum instead.

I was careful to use the words "tendency" and "normal" because these things are at best generic results. Just as there are very special periodic potentials on $(-\infty, \infty)$ whose associated Schrödinger operator has a spectrum with only finitely many gaps [29, 30], so there are a.p. potentials [30] with the same property. But just as a generic periodic potential has *all* its gaps open [31], I *believe* that in a sense to be made precise in Section 4, generic a.p. potentials have a dense set of gaps open.

I should emphasize that these properties have *not* yet been proven in any great generality; thus far, we only have results for some rather special examples, as I shall describe in this review.

To give a spectacular example, consider the l_2 operator with

$$V(j) = 3\cos(2\pi\alpha j),$$

where α is a parameter. We *believe* the following is the case. [0, 1] is the union of three disjoint sets S_1, S_2, S_3. S_1 is the rationals, and when $\alpha \in S_1$, $\sigma(h_0 + V)$ is an interval with finitely many subintervals removed, and the spectral measures are absolutely continuous. S_2 is an uncountable set of measure zero, and if $\alpha \in S_2$, then all spectral measures are singular continuous, and $\sigma(h_0 + V)$ is a Cantor set, albeit one which locally has strictly positive Lebesque measure. If $\alpha \in S_3$ (a set of full measure), then there is only a point spectrum which locally has strictly positive Lebesque measures. Thus, as an innocent looking parameter is varied in a simple Hamiltonian, rather spectacular spectral fireworks take place.

Actually, this potential is one of the few on which there are lots of results. More precisely, consider the family with two added parameters, λ and θ:

$$V(j) = \lambda \cos(2\pi\alpha j + \theta).$$

The quoted result for S_1 is true for all λ, θ and is standard periodic Schrödinger operator theory (see, e.g., [32, 33]). In Section 10, we will see [5] that if $\lambda > 2$, if α is a Liouville number, then for all θ, $h_0 + V$ has only singular continuous spectral measures. In Section 9, we will see [7] that if α has typical Diophantine properties, then for λ large, we have at least some point spectrum (for some θ) whose closure is locally uncountable.

Next, a brief summary of the notion of a.p. function, etc. Given a function, f, on $(-\infty, \infty)$, we let $f_t(x) \equiv f(x - t)$ be a translate of f. A continuous function, f, is called *almost periodic* (a.p.) if and only if $\{f_t\}_{t \in (-\infty, \infty)}$ has a compact closure, Γ, in the uniform norm. Γ (with the uniform topology) can be given the structure of a topological group in exactly one way, so that $t \to f_t$ is a continuous homomorphism of R into Γ. Since $R \to \Gamma$, by duality $\hat{\Gamma} \to \hat{R} = R$, so there is a distinguished subgroup of R, $\hat{\Gamma}$, called the *frequency module* of f. By the Peter–Weyl Theorem on Γ, f is a uniform limit of linear combinations of $e^{2\pi i \alpha x}$ with $\alpha \in \hat{\Gamma}$. Automatically, $\hat{\Gamma}$ is countable.

A good example is a *quasiperiodic function*, i.e., $f(x) = g(\alpha_1 x, \ldots, \alpha_n x)$, where g is a function on $\mathbb{R}^n$, of period 1 in each variable (i.e., g is essentially a function on T^n, the n torus). If the α's are rationally independent, and g has enough nonzero Fourier coefficients (to be precise, $\{k | \hat{g}(k) \neq 0\}$ must generate Z^n), then Γ is naturally isomorphic to T^n under $\theta_i \in T^n \leftrightarrow f_{\theta_i}$ with $f_{\theta_i}(x) = g(\alpha_1 x + \theta_1, \ldots, \alpha_n x + \theta_n)$ and $\hat{\Gamma}$ is generated by $\alpha_1, \ldots, \alpha_n$, i.e., $\hat{\Gamma} = \{m_1\alpha_1 + \cdots + m_n\alpha_n | m_i \in Z\}$. f is quasiperiodic if and only if $\hat{\Gamma}$ is finitely generated.

Another class of interest to us is the *limit periodic potentials*, i.e., functions f which are uniform limits of periodic potentials, albeit perhaps with longer and longer periods. A good example is

$$f(x) = \sum_{n=1}^{\infty} a_n \mathrm{con}(2\pi x/2^n); \qquad \sum |a_n| < \infty. \tag{3}$$

In this case, $\hat{\Gamma}$ is the set of dyadic rationals. f is limit periodic if and only if $\hat{\Gamma}$ has the divisor property, i.e., for all $\alpha, \beta \in \hat{\Gamma}$, there is $\gamma \in \hat{\Gamma}$ with α/γ and β/γ both in Z.

We say a rational number α has *typical Diophantine properties* if and only if for some C, k and all p, q, we have

$$\left|\alpha - \frac{p}{q}\right| \geqslant Cq^{-k-1}. \tag{4}$$

As the name implies, the set of such α's has full measure in R. An N-tuple $(\alpha_1, \dots, \alpha_N)$ will be called *typically Diophantine* if there is C and k with

$$\left|\sum_1^n m_i \alpha_i\right| \geqslant C\left(\sum m_i^2\right)^{-k/2}$$

for all integers m_i. α has typical Diophantine properties, if and only if $(\alpha, 1)$ is typically Diophantine.

α is called a *Liouville number* if it is irrational and (4) fails in the strong sense that there is C and there are integers p_n, $q_n \to \infty$ with

$$\left|\alpha - \frac{p_n}{q_n}\right| \leqslant Cn^{-q_n}. \tag{5}$$

Similar definitions hold for a.p. functions on Z.

If V is a.p., then rather than consider the single operator, $H_0 + V$ (with $H_0 = -d^2/dx^2$ or the h_0 of (2)), we should consider the family $H_0 + W$ as W runs through the hull of V. Putting Haar measure on Γ, we have a random set of operators. In this sense, we can think of a.p. potentials as a special set of random potentials. The process, $x \to W(x)$ is ergodic but not (weakly) mixing. Thus, in a certain sense, these are intermediate between periodic and random potentials.

At this point in time, it seems wisest to concentrate on the rather surprizing mathematical properties of the a.p. Schrödinger operators. However, we should mention, en passant, some potential applications:

(1) $-d^2/dt^2 + V(t)$ enters not only in quantum theory but also in the linear stability theory of classical mechanics. Thus these ideas may be relevant to stability of motion in the presence of several periodic motions with incommensurate periods. See [2] for speculations on the rings of Saturn.

(2) As Aubry [15] has emphasized, these ideas may be relevant to understanding electron transport in some one dimensional organic molecules.

(3) There exist "solid" alloys A_xB, where x appears to vary continuously with temperature. If x is irrational, the substrate must be a.p., not periodic.

(4) If one puts low-frequency phonons through a solid, to the extent that one can use an adiabatic one electron approximation, at any instant the distribution of nucleii will be a.p. if the phonon wave number is irrational.

(5) Similar ideas may describe light propagation through a.p. glass, which can probably be made with modern technology [34].

(6) One of the prettiest results in the KdV theory is the theorem of McKean and Trubowitz [35] that if the initial data is periodic in space, then the solution of KdV is a.p. in time. Symmetry suggests one should only need the initial data to be a.p. in space. Given the methods of [35], one needs to study a.p. Schrödinger operators to attack this problem.

(7) For reasons I won't go into here, there is a close analogy between a.p. Schrödinger operators and a two dimensional electron in a crystal and constant magnetic field with the flux through a unit cell an irrational number.

In Section 2, we present a general result, most of them quite soft, on the spectrum of general a.p. Schrödinger operators. Sections 3–6 discuss Cantor spectrum. Two distinct intuitions are described in Sections 3 and 5; Section 4 expands on Section 3 using the notion of integrated density of states (IDS) which plays a role in later developments also. Section 6 gives the one existing theorem on the occurrence of Cantor spectrum.* Section 7 proves Gordon's theory on the absence of point spectrum in certain cases; this is one of the few places we give complex details, in part, because the result is not available in English. In Section 8, we discuss an important formula of Thouless, which relates the IDS to the Lyaponov index. In Sections 9, 10 we discuss some special examples where very suggestive results have been found. In the last section, we present some open problems.

2. Some General Theorems

The first two results are very general; they automatically hold also in ν-dimensions, and are essentially specializations of results on random potentials (see, e.g., Pasteur [25]); the only difference is that things that hold a.e. in the random case, here hold everywhere.

THEOREM 2.1. *The spectrum, $\sigma(H_0 + W)$, of an a.p. Schrödinger operator is the same for all W in the hull, Γ, of a fixed a.p. potential V.*

THEOREM 2.2. *An a.p. Schrödinger operator has no discrete spectrum, i.e., no isolated eigenvalues of finite multiplicity.*

Since one dimensional second order difference or differential equations can have no more than multiplicity two (in fact, by Wronskian arguments at

*See Note added in proof.

most multiplicity one) eigenvalues, we have

COROLLARY 2.3. *A one dimensional a.p. Schrödinger operator has no isolated points in its spectrum; i.e., its spectrum is a perfect set.*

Another general one dimensional result (Pasteur [25]):

THEOREM 2.4 (One dim.). *Fix a real number E and hull* Γ. *Then* $\{W \mid E$ *is an eigenvalue of* $H_0 + W;\ W \in \Gamma\}$ *has measure* 0.

The next result is due to Kirsch and Martinelli [27]; some related results are earlier in Kunz and Soulliard [26].

THEOREM 2.5. *Let* Γ *be the hull of an a.p. function V. Then there exist fixed closed sets* S_1, S_2, S_3 *so that for almost all W operator* $H = H_0 + W$ *has point spectrum* ($\equiv$ *closure of the set of eigenvalues*) S_1, *a.c. spectrum* S_2 *and singular continuous spectrum* S_3.

We emphasize the almost in "almost all" above because, unlike Theorems 2.1 and 2.2, this result is only proven a.e. and is almost surely not true for all. Gordon [19] and subsequently Moser [36] and Johnson [37] constructed a.p. potentials V with an eigenvalue at $\inf \mathrm{sp}(H_0 + V)$. For these examples it is surely true that the point spectrum of $H_0 + W$ is empty for, e.g., $W \in \Gamma$.

As a simple consequence of Theorems 2.4 and 2.5, one has

COROLLARY 2.6 (Avron and Simon [5]). *The generic point spectrum* S_1 *is locally uncountable, i.e., if* $\lambda_0 \in S_1$, *then* $S_1 \cap (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ *is uncountable for all* $\epsilon > 0$.

Thus, if $S_1 \neq \varnothing$, it is thick in the sense of [1].

3. GAPS AND CANTOR SETS

To explain the first intuition behind Cantor spectrum, let f be a doubly periodic function of the form

$$f(x, y) = \sum_{n_1, n_2} a_{n_1, n_2} \exp[2\pi i(n_1 x + n_2 y)],$$

where a is real, $a_{-n} = a_n$, $a_n > 0$ for all n (this is chosen to avoid certain cancellations) and

$$|a_n| \leqslant Ce^{-A|n|}$$

for $A, C > 0$ (so f is analytic). Let $\alpha = p/q$ with p, q relatively prime, and let

$$V(x) = f(x, \alpha x),$$

which is periodic of period q. The usual analysis of periodic potentials (e.g., [32, 33]) shows that for λ small, $-d^2/dx^2 + \lambda V(x)$ will have gaps about the points $E_l = [2\pi(l/q)]^2$, indeed the size of the gap about E_l is exactly

$$2\lambda\left(\sum_{n_1 + n_2\alpha = l/q} a_{n_1 n_2}\right) + 0(\lambda^2).$$

It is natural to suppose that for α irrational, there is still a tendency for gaps to open about the points $[2\pi(n_1 + n_2\alpha)]^2$. Since these points are dense, we see that the resolvent set wants to be dense, i.e., the spectrum is a Cantor set. Note that since the total gap size wants to be $0[\lambda\Sigma|a_n|]$, the resolvent seems to want to be a semi-infinite interval and lots of small intervals whose total size is finite; thus the Cantor set is not one of measure zero (as is the conventional no middle third set) but rather one of infinite Lebesgue measure.

The first hint of this gap picture was in a paper of Dinaburg and Sinai [20] who proved the following deep result:

THEOREM 3.1 (Dinaburg–Sinai [20]). *Let*

$$f(t_1, \ldots, t_n) = \sum_{m_1, \ldots, m_n \in Z} a_m \exp[2\pi i(m_1 t_1 + \cdots + m_n t_n)]$$

be a function on T^n with

$$|a_m| \leqslant C\exp(-A|m|).$$

Let $\alpha_1, \ldots, \alpha_n$ have typical Diophantine properties. Then, there exist

$$\{\lambda_m\}_{m \in Z^n}, \{c_m\}_{m \in Z^n}, E_0$$

with

$$\sum |c_m| < \infty$$

so that for E in

$$S = \left\{ E \geqslant E_0 | \, |E - \lambda_m - (2\pi)^2 \left(\sum m_i \alpha_i\right)^2| \geqslant c_m \text{ all } m \right\}$$

the equation

$$Hu = 0,$$

$$H = \frac{-d^2}{dx^2} + V(x),$$

$$V(x) = f(\alpha_1 x, \ldots, \alpha_n x),$$

has only almost periodic solutions (and, in particular, only bounded solutions). Moreover, H has an absolutely continuous component of its spectrum on any positive measure subset of S.

Remarks. 1. Reference [20] exploits a KAM type argument.

2. Russmann [21] has extended and developed the results in [20].

3. As $|\Sigma m_i \alpha_i| \to \infty$ or as an overall constant in front of V goes to zero, the shifts λ_m go to zero.

4. Since $\Sigma |c_m| < \infty$, S has infinite measure.

5. This theorem does *not* say that $\sigma(H)$ is a Cantor set, although it is suggestive. First of all, S may not be a Cantor set since no control is given on the λ_m's (and indeed, the theorem is applicable to some special finite gap potentials [30] when we know $\sigma(H)$ is *not* a Cantor set). Moreover, the theorem does not assert that $\sigma(H) = S$; indeed, the use of KAM requires one to definitely exclude a bit of the spectrum.

6. In fact, there exists a function $c(E)$ on S so that Hu has solutions u_1 and $u_2 = \bar{u}_1$ with $u_1(x) = e^{ic(E)x}\psi(x)$ and $\psi(x)$ has its frequency spectrum in that of V.

4. The IDS and the Gap Labelling Theorem

If the gaps are dense, clearly they cannot be ordered by the integers (i.e., there is no "third gap"). It is remarkable that there appears to be a natural labelling of the gaps. This depends on an object which will be useful later. In the context of random potentials, this object was first proven to exist by Benderskii and Pasteur [22]. Interestingly enough, the approaches of Avron and Simon [5], Bellisard, Lima, and Testard [55], and Johnson and Moser [12] emphasize different aspects of this quantity: respectively as a density of states, as a C^*-algebraic quantity, and as a rotation number.

Let H be an a.p. Schrödinger operator on $(-\infty, \infty)$. Let $E_{(-\infty,\lambda)}(H)$ denote its spectral projections, and let $\mathcal{X}_{(a,b)}$ be the characteristic function of the interval $x \in (a, b)$. We define the *integrated density of states* (IDS) by

$$k(\lambda) = \lim_{L \to \infty} L^{-1}\operatorname{Tr}\left(\mathcal{X}_{(0,L)} E_{(-\infty,\lambda)}(H)\right). \tag{6}$$

A similar definition holds in the discrete case. The following combines results in [5, 12, 22]:

THEOREM 4.1. *The limit in* (6) *exists for every a.p. potential. k has the following properties*:

(1) $0 \leqslant k(\lambda)$ *and in the discrete case* $k(\lambda) \leqslant 1$.

(2) $k(\lambda)$ *is monotone in* λ *and continuous.*

(3) *k is independent of the choice of W in the hull of some V.*

(4) $\sigma(H)$ *is the set* $\{\lambda | \text{all } \epsilon > 0, k(\lambda + \epsilon) \neq k(\lambda - \epsilon)\}$ *of points of nonconstancy of* k.

Moreover, the limit in (6) *is the same if* H *is replaced by a sequence* H_L, *of operators equal to* H *on* $(0, L)$ *with some boundary conditions at* 0 *and* L.

Remarks. 1. One way of thinking of k that yields point (4) is the following: Pick any f on $(-\infty, \infty)$ with $\int |f|^2 \, dx = 1$. Let ρ_W be the spectral measure defined by

$$\rho_W(A) = \operatorname{Tr}\big(fE_A(H_0 + W)f\big).$$

Then, with dW being Haar measure on the hull

$$k(\lambda) = \int dW \rho_W(-\infty, \lambda). \tag{7}$$

2. The continuity (point (2)) is a reflection of the occurrence of λ as an eigenvalue only on a set of measure 0 and egn (7).

3. The existence of the limit and b.c. independence can be proven by an argument which is standard in the statistical mechanical literature. Using H_L, one has an operator with discrete spectrum and Tr(--) is just the number of eigenvalues in $(-\infty, \lambda)$; hence the name density of states.

Picking Dirichlet b.c., we see that in the continuous case $k(\lambda) = \lim_{L \to \infty} L^{-1} n_L(\lambda)$ when $n_L(\lambda)$ is the number of zeros in $(0, L)$ of the solution of $Hu = \lambda u$, $u(0) = 0$. From this, a simple argument implies:

COROLLARY 4.2 (Continuous case only). *Let* u *be real valued, and solve* $Hu = \lambda u$ *and let* $R(u, L)$ *be the number of times the vector* $(u(x), u'(x))$ *rotates about* $(0, 0)$ *in the interval* $(0, L)$ (*i.e.*, $(2\pi)^{-1}$ *times the change of the argument*). *Then the rotation number*

$$r(\lambda) = \lim_{L \to \infty} L^{-1} R(u, L) \tag{8}$$

exists (*and is independent of* u) *and*

$$r(\lambda) = \tfrac{1}{2} k(x). \tag{9}$$

To Johnson and Moser [12], (8) is the basic object, and (6) is proven using (8). Our ordering follows Avron and Simon [5]. There is a third, attractive and interesting way to look at $k(\lambda)$. Let R_0 be the C^* algebra generated by $\{f | f$ has the same frequency spectrum as $V\}$ and by the family of unitary translation operators and let $R = R_0''$, the corresponding Von Neumann algebra. In the discrete case, $H \in R_0$, and in the continuous case, H is affiliated to R_0 in the sense that $(H + i)^{-1} \in R_0$.

THEOREM 4.3 ([38]). *R is a type* II_∞ *algebra in the continuous case, and a type* II_1 *algebra in the discrete case.*

If "tr" is the corresponding trace, Shubin has proven the remarkable:

THEOREM 4.4 ([23]). *With the natural normalization in with* II_1 *case, and suitable normalization in the discrete case*:

$$k(\lambda) = \text{“tr”}\left(E_{(-\infty,\lambda)}(H)\right). \tag{10}$$

By Theorem 4.1 (4), $k(\lambda)$ is constant on any gap of $\sigma(H)$ and equal to a different constant on distinct gaps. We have the beautiful:

THEOREM 4.5 (Johnson and Moser [12]; Bellisard, Lima, and Testard [55]). *In the continuous case, in any gap, $k(\lambda)$ has a value in the frequency module of V.*

THEOREM 4.6 (Bellisard, Lima, and Testard [55]). *In the discrete case, suppose that the frequency module of V lies in $\{n_1\alpha + n_2 | n_1, n_2$ integer$\}$. Then, in any gap, $k(\lambda)$ lies in this same set.*

Remarks. 1. We emphasize that in the discrete case, even if the frequency module is $\{n\alpha\}$ (e.g., $\cos(2\pi\alpha n)$) $k(\lambda)$ will still have values in general in $\{n_1\alpha + n_2\}$ because in some sense, 1 is in the "frequency module of h_0."

2. The J–M proof depends critically on the continuous nature of $\mathbb{R}$ and does not obviously extend to the discrete case. J–M normalize k and $\hat{\Gamma}$ differently.

3. The restriction to the simple frequency module in Theorem 4.6 is due to the existing C^* algebra results, and eventually, the method should extend to more general cases.

4. Thus, if V has a frequency module generated by $\alpha_1, \ldots, \alpha_m$, we can label gaps by multiples of integers. We call these two theorems *Gap labelling* theorems.

The BLT proof comes from (1) the observation that if λ is in a gap we can find a continuous function f so that $E_{(-\infty,\lambda)} = f(H)$ and thus $E_{(-\infty,\lambda)}(H)$ is in R_0 rather than just in R and (2) the following:

THEOREM 4.7. (a) (A. Connes [39]). *In the case of Theorem* 4.5, *if $P \in R_0$ is a projection, then* "tr"(P) *is in the frequency module of V.*

(b) (Voiculescu and Pinsker [39, 40].) *In the case of Theorem* 4.6, *if $P \in R_0$ is a projection, then* "tr"(P) *is in* $\{n_1\alpha + n_2\}$.

The J–M proof [12] is a beautiful homotopy argument: Let $G(x, y; V)$ be the integral kernel of $(H - \lambda)^{-1}$ which exists and is continuous in V since $\lambda \notin \operatorname{spec}(H)$. By general principles, $G(x, x; V) \equiv F(x; V)$ is the product of two solutions of $Hu = \lambda u$, so by the argument before Corollary 4.2, $k(\lambda) = \frac{1}{2}\lim_{L\to\infty} L^{-1}$ (# of zeros of $F(x; V)$ on $[0, L]$). Moreover, it is not hard to show that where F is zero, F' is non-zero so that as in Corollary 4.2, $k(\lambda) = \lim_{L\to\infty} L^{-1}$ (# of times $H(x) = F'(x) + iF(x)$ rotates about 0). Now arg $H(0, W)$ defines a continuous function from Γ to T^1, the circle. Since Γ is an m-torus, such a map defines m winding numbers, $n_1, \ldots, n_m$. The result now follows from following through the various definitions.

The following can be used in place of the Green's function:

THEOREM 4.8 (Johnson [10]; continuous case). (a) *If* $\lambda \in \operatorname{spec}(H)$, *then for some* W *in the hull of* V, $(H_0 + W)u = \lambda u$ *has a solution in* $L^\infty(-\infty, \infty)$.

(b) *If* $\lambda \notin \operatorname{spec}(H)$, *then there is a continuous function* $W \mapsto P(W)$ *from* Γ *to rank one (not necessarily orthogonal) projections on* $\mathbb{R}^2$, *so that a solution of* $(H_0 + W)u = \lambda u$ *tends to zero at* $+\infty$ (*resp.* $-\infty$) *if and only if* $(u, u') \in \operatorname{Ran} P$ (*resp.* $\operatorname{Ran}(1 - P)$).

Remarks. 1. In (b) more is true; solutions either go to zero exponentially or blow up exponentially.

2. The proof is an easy combining of deep results of Sacher and Sell [41], the abstract eigenfunction expansion theory [42, 43] and its converse [44, 45].

3. The proof extends to the discrete case.

4. In the J–M proof, the integers n_i can be defined as winding numbers associated to the map $W \mapsto R$ on $P(W)$ from Γ to $\mathbb{RP}(2)$, the set of lines in $\mathbb{R}^2$.

The gap labeling theorem is an indication of how Cantor sets might come about:

THEOREM 4.9. *Fix* V a.p. *but not periodic. If for any* x *in the frequency module with* $x > 0$ (*continuous case*) *or* $0 < x < 1$ (*discrete case*), *there is a gap where* k *equals* x, *then* $\sigma(H)$ *is a Cantor set.*

Proof. We must show that any point, λ_∞ in $\sigma(H)$ is a limit of points not in $\sigma(H)$. Since the frequency module is dense in R and $x_\infty = k(\lambda_\infty) \geqslant 0$ (in $[0, 1]$ in the discrete case) we can find $x_n \to x_\infty$ with x_n in the frequency module. By assumption, we find λ_n not in $\sigma(H)$ so $k(\lambda_n) = x_n$. By passing to a subsequence, $\lambda_n \to \tilde{\lambda}_\infty$. Thus $k(\lambda_\infty) = k(\tilde{\lambda}_\infty)$. If $\tilde{\lambda}_\infty = \lambda_\infty$, then $\lambda_n \to \lambda_\infty$ so we are done. If $\lambda_\infty < \tilde{\lambda}_\infty$, $(\lambda_\infty, \tilde{\lambda}_\infty)$ is in the resolvent set since k is constant on that interval, so we are done. Similarly, if $\tilde{\lambda}_\infty < \lambda_\infty$, we have that $(\tilde{\lambda}_\infty, \lambda_\infty)$ is in the resolvent set.

In the periodic case, we know that generically all gaps are open ([31]). It is natural to conjecture that generically the hypotheses of Theorem 4.9 hold.

As a final result of interest about the IDS, we note:

THEOREM 4.10 (Avron and Simon [5]). *Fix* f *a continuous function on* T^m, *the* m-*torus. Let* $k(\lambda; \alpha_1, \ldots, \alpha_m)$ *denote the IDS for the potential*

$$V(x) = f(\alpha_1 x, \ldots, \alpha_m x).$$

k is continuous at all points where the $\{\alpha_j\}$ *are rationally independent* (*and generally discontinuous at points of rational dependence*).

5. Long Time Behavior and Cantor Sets

Avron and Simon [1] have pointed out another reason to expect Cantor sets. The point is that, in general, quantum particles get partially reflected from bumps even if the bumps have lower energy than the particles. Thus, an infinite sequence of bumps will generally trap quantum particles. In the periodic case, the first few bumps produce the coherences in the transmitted piece which allow almost perfect transmission in the latter bumps, and so there are many wave packets which move to infinity at finite velocity. In an a.p. potential, the particle will think it is moving in a periodic potential for a while, but eventually get reflected because things are not quite periodic, then move an even further distance, but then get reflected again. Thus, if there is escape to infinity, it is in a very anomalous way.

Indeed, if α is a Liouville number, the successive distances are very large and we have a behavior reminiscent of Pearson's examples [46], so the singular spectrum we find in Section 10 is not surprising.

But what of the cases where Dinaburg and Sinai tell us there is a.c. spectrum (Theorem 3.1)? How does one square that with anomalous long time behavior? The answer is that there is a refinement of a.c. spectrum.

Definition. ([1]) Let A be a self-adjoint operator. We say that φ is a *transient vector* for A if and only if $(\varphi, e^{-itA}\varphi) = 0(t^{-N})$ for all N. Any such φ is in $\mathcal{H}$ac, the absolutely continuous space and the closure of such φ is a subspace [1], called $\mathcal{H}_{\text{tac}}$, the *transient space*. $\mathcal{H}_{\text{ac}} \cap \mathcal{H}_{\text{tac}}^{\perp} \equiv \mathcal{H}_{\text{rac}}$, the *recurrent space*.

For the usual (e.g., periodic or N-body Schrödinger operators) case, $\mathcal{H}_{\text{rac}} = \{0\}$, but we have the elementary.

Theorem 5.1. ([1]) *If* $\sigma(A)$ *is a Cantor set, then* $\mathcal{H}_{\text{tac}} = \{0\}$.

Thus, Cantor spectrum provides a synthesis of the intuition of anomalous long time behavior and the a.c. spectrum of Dinaburg and Sinai. Indeed, in the next section, we will describe simple a.p. Schrödinger operators with $\mathcal{H}_{\text{rac}} = \mathcal{H}$.

6. The SCAM Theorems

In this section, we will describe a set of similar results obtained independently by Avron and Simon [3], Chulaevsky [9] and Moser [11] in 1980. We emphasized that the ordering of initials in the section title is purely for linguistic purposes.

The set of limit periodic potentials is a complete metric space, and, as such, there is a natural notion of generic [47], that of dense G_δ.

THEOREM 6.1 ([3, 9, 11]). *For a dense G_δ of limit periodic potentials, V, $\sigma(-d^2/dx^2 + V)$ is a Cantor set.*

Remarks. 1. It is *not* claimed to be a measure zero Cantor set; indeed, by construction [3], it is often not a zero measure set.

2. A similar result holds in the discrete case. Also, if we consider potentials given by (3) with the natural l^1 norm on $\{a_n\}$, the dense G_δ result remains true. One can therefore find V's which extend to entire analytic functions with Cantor spectrum.

THEOREM 6.2 ([3, 9]). *For a dense set of limit periodic V's, $\sigma(-d^2/dx^2 + V(x))$ is a Cantor set, and the spectrum is purely absolutely continuous.*

Remarks. 1. The set is only claimed to be dense, not G_δ. Indeed, we believe that in some regions the generic behavior will be some dense point spectrum.

2. Avron and Simon obtain no explicit estimate on V's which yield a.c. spectrum. Chulaevsky considers potentials of the form (3) and proves a.c. spectrum, if for all A, there is a C with

$$|a_n| \leqslant C\exp(-A2^n). \tag{11}$$

3. The operators guaranteed by this theorem have $\mathcal{H}_{\mathrm{rac}} = \mathcal{H}$ (in the notation of Section 5).

We cannot give the details of the proofs of these theorems. They are not hard; one can approximate V so well by periodic potentials that proving the right things in that case suffices.

7. GORDON'S THEOREM

Here we give with detailed proofs, a theorem proven by Gordon [19] in 1976. It asserts that certain a.p. potentials have no point spectrum. If one is only accustomed to the periodic case, where it is easy to see that there is no point spectrum, this result appears to be of limited interest, and it has thus received limited attention until recently. However, as a.p. potentials should often produce some dense point spectrum, (see Section 9), it is extremely interesting to know that they sometimes do not. Indeed, as we will see in Section 10, the theorem below has striking consequences.

DEFINITION. A potential, V, on $(-\infty, \infty)$ is called a *Gordon potential* if and only if there exist periodic potentials V_m, of periodic $T_m \to \infty$ so that for some $C > 0$

$$\sup_{-2T_m \leqslant x \leqslant 2T_m} |V(x) - V_m(x)| \leqslant Cm^{-T_m}.$$

A similar definition is used in the discrete case. Gordon [19] proved:

THEOREM 7.1 (Gordon's theorem, continuous case). *Let V be a Gordon potential on $(-\infty, \infty)$ and let u solve $-u'' + Vu = Eu$. Then*

$$\overline{\lim_{|x|\to\infty}} \left[|u'(x)|^2 + |u(x)|^2\right] / \left[|u'(0)|^2 + |u(0)|^2\right] \geqslant \tfrac{1}{4}. \tag{12}$$

The same method proves:

THEOREM 7.2 (Gordon's theorem, discrete case). *Let V be a Gordon potential on Z and let u solve $u(n+1) + u(n-1) + V(n)u(n) = Eu(n)$. Then*

$$\overline{\lim_{|n|\to\infty}} \left[|u(n)|^2 + |u(n+1)|^2\right] / \left[|u(1)|^2 + |u(0)|^2\right] \geqslant \tfrac{1}{4}. \tag{12'}$$

In the discrete case, an l^2 eigenfunction trivially has $u(n) \to 0$ as $|n| \to \infty$. In the continuous case, using Harnack's inequality, one can show [43] *that $|u|^2 + |u'|^2 \to 0$ if $u \in L^2$. Thus*:

COROLLARY 7.3. *A Schrödinger operator (continuous or discrete) with a Gordon potential has no point spectrum.*

Before giving the proof, let us note two examples:

EXAMPLE 1. If

$$V(x) = \sum_1^\infty a_n \cos(2\pi x/2^n)$$

and the a_n obey the Chulaevsky condition (11) (for any A, there is a C), then choosing

$$V_m(x) = \sum_1^{N_m} a_n \cos(2\pi x/2^n)$$

for suitable N_m, we see that V is a Gordon potential. Gordon's theorem fits in well with the result of Chulaevsky saying that there is only a.c. spectrum in this case.

EXAMPLE 2. On Z consider the potential

$$V(n) = \lambda \cos(2\pi\alpha n + \Theta)$$

with α a Liouville number. If

$$V_{p/q}(n) = \lambda \cos\left(2\pi \frac{p}{q} n + \Theta\right)$$

then

$$\sup_{-2q \leqslant n \leqslant 2q} |V(n) - V_{p/q}(n)| \leqslant 4\pi|\lambda|q\left|\alpha - \frac{p}{q}\right|$$

so choosing p/q suitably and using (5), we see that V is a Gordon potential.

We will prove Theorem 7.2; the proof of Theorem 7.1 is similar. Let $\Phi_m(n)$ be the column vector $(u_m(n), u_m(n+1))$, where u_m solves the Schrödinger equation with V replaced by V_m and with the initial condition $\Phi_m(0) = \Phi(0)$. Clearly for $n > 0$:

$$\Phi_m(n) = A_m(n) \cdots A_m(1)\Phi(0)$$

with similar results for $\Phi(n)$ and for $n < 0$, where $A_m(n)$ is the matrix:

$$A_m(n) = \begin{pmatrix} 0 & 1 \\ -1 & E - V_m(n) \end{pmatrix}. \tag{13}$$

Using the telescoping sum estimate

$$\|A_m(n) \cdots A_m(1) - A(n) \cdots A(1)\| \leqslant n\left[\sup_{m,j}\|A_m(j)\|\right]^{n-1} \times \sup_{1 \leqslant j \leqslant n} \|A_m(j) - A(j)\|$$

and the hypothesis on V, we see that

LEMMA 7.4.

$$\sup_{-2T_m \leqslant n \leqslant 2T_m} \|\Phi_m(n) - \Phi(n)\| \to 0 \qquad \textit{as } m \to \infty.$$

Clearly, the theorem follows from this estimate and

LEMMA 7.5.

$$\max(\|\Phi_m(T_m)\|, \|\Phi_m(2T_m)\|, \|\Phi_m(-T_m)\|, \|\Phi_m(-2T_m)\|) \geqslant \tfrac{1}{2}\|\Phi(0)\|.$$

Letting $B = A_m(T_m) \cdots A_m(1)$ and using periodicity, we see that Lemma 7.5 follows from

LEMMA 7.6. *Let B be any 2×2 invertible matrix, and let x be a unit vector. Then* $\max(\|Bx\|, \|B^2x\|, \|B^{-1}x\|, \|B^{-2}x\|) \geqslant \frac{1}{2}$.

Proof. Let $a_2B^2 + a_1B + a_0 = 0$ be the characteristic equation for B. Normalize the a's so that the a_j with maximum modulus is 1. Suppose that this maximum modulus a_j is a_1, i.e., $a_1 = 1$ and $|a_0|, |a_2| \leqslant 1$. Apply the basic equation to $B^{-1}x$ and find

$$x = -a_2 Bx - a_0 B^{-1}x \tag{14}$$

at least one of the vectors on the right of (14) must have norm $\frac{1}{2}$. A similar argument works on the other two cases.

Remark. The $\frac{1}{2}$ in the last lemma is optimal. Take $B = \frac{1}{2}\begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix}$ and $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

8. The Thouless Formula

Here we want to describe a result found by Thouless [48] in the context of discrete Schrödinger operators with random potentials. Its importance in the a.p. case was emphasized by Aubry and André [16]. Actually, the result does not hold in the form given in [48, 16] because various sets of measure zero are ignored and an interchange of limit and integral made. A careful proof was found by Avron and Simon [5]. Johnson and Moser [12] have a somewhat related result. Interestingly enough, Gordon's theorem [19] implies that for some simple looking potentials, the sets of measure zero ignored in [48, 16] really are present.

Given V and E, we define T_L or $T_L(V-E)$ to be the two by two matrix given by $T_L\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} u(L) \\ u'(L) \end{pmatrix}$, where u solves $-u'' + Vu = Eu$ with $u(0) = a$, $u'(0) = b$. In the discrete case $T_L = A(L) \cdots A(1)$ with A given by (13). ($L > 0$ and $T_L = [A(0) \cdots A(-L+1)]^{-1}$ if $L < 0$).

Definition. We say that $V - E$ has *Lyaponov behavior* with *Lyaponov index* $\gamma(E)$ if and only if

$$\gamma(E) = \lim_{|L| \to \infty} |L|^{-1} \ln \|T_L\|$$

exists.

Since T_L has determinant 1, we have

$$\gamma(E) \geqslant 0. \tag{15}$$

One particular reason for interest in Lyaponov behavior is the following

Theorem 8.1 (Osceledec [49]; see also Ruelle [50]). *If $A_1, \ldots, A_n, \ldots$ are 2×2 matrices with*

(1) $\lim_{n \to \infty} n^{-1} \ln \|A_n\| = 0$,

(2) $\lim_{n \to \infty} n^{-1} \ln \|A_n \cdots A_1\| = \gamma_+$,

(3) $\lim_{n \to \infty} n^{-1} \ln [\det(A_n \cdots A_1)] = \gamma_+ + \gamma_-$,

then, there exists a one dimensional subspace V of C^2 so that if $0 \neq \phi \in V$, *then*

$$\lim_{n\to\infty} n^{-1}\ln\|A_n \cdots A_1\phi\| = \gamma_-$$

and if $\phi \notin V$

$$\lim_{n\to\infty} n^{-1}\ln\|A_n \cdots A_1\phi\| = \gamma_+ .$$

In the a.p. case, T_L is precisely of the form $A_L \cdots A_1$ with (1) holding (since $\sup_n\|A_n\| < \infty$) and (3) holds with $\gamma_- = -\gamma_+ = -\gamma(E)$. Thus:

COROLLARY 8.4. *If $V - E$ has Lyaponov behavior, there are subspaces V_+ and V_- of C^2 (perhaps not distinct) with*

$$\lim_{t\to\pm\infty} |t|^{-1}\ln\|\phi(t)\| = \gamma(E)\ (\textit{resp } -\gamma(E))$$

if

$$\phi(0) \notin V_\pm\ (\textit{resp }\phi(0) \in V_\pm,\ \phi(0) \neq 0), \qquad \textit{where } \phi(t) = (u(t), u'(t))$$

[*in the discrete case* $\phi(n) = (u(n), u(n+1))$] *and u solves the Schrödinger equation $Hu = Eu$. In particular, if $\gamma > 0$, every solution is either exponentially growing or decaying.*

The remarkable fact is that not only is there Lyaponov behavior for many $V - E$ (this actually follows from the subadditive ergodic theorem), but that $\gamma(E)$ is intimately related to the IDS, $k(E)$, discussed in Section 4:

THEOREM 8.5 ([5]). *Let Γ be the hull of an* a.p. *function on $\mathbb{Z}$. Consider $\Gamma \times \mathbb{R}$ with Haar measure on Γ and Lebesgue measure on $\mathbb{R}$. Then for almost all $(V, E) \subset \Gamma \times \mathbb{R}$, $V - E$ has Lyaponov behavior with*

$$\gamma(E) = \int \ln|E - E'|\, dk(E'). \tag{16}$$

The same result holds in the continuous case, but (16) *is replaced by*

$$\gamma(E) = \gamma_0(E) + \int \ln|E - E'|\, d[(k - k_0)(E)], \tag{17}$$

where

$$\gamma_0(E) = [\max(0, -E)]^{1/2}; \qquad k_0(E) = \pi^{-1}[\max(0, E)]^{1/2}. \tag{18}$$

Equation (16) is the *Thouless formula.*

Remarks. 1. k_0 and γ_0 are just the IDS and Lyponov index in case $V = 0$.

2. Let $f(z)$ be the function on $\operatorname{Im} z > 0$

$$f(z) = \int \ln(z - E')\, dk(E')$$

in the discrete case and

$$f(z) = \sqrt{-z} + \int \ln(z - E')\, d\big[(k - k_0)(E)\big],$$

which is analytic in $\operatorname{Im} z > 0$. Since k is continuous, $\operatorname{Im} f(z)$ has the boundary value

$$\operatorname{Im} f(E + i0) = \pi k(E)$$

and (16) (resp. (17)) says that

$$\operatorname{Re} f(E + i0) = \gamma(E).$$

Thus the Thouless formula asserts that $\gamma(E) + i\pi k(E)$ is the boundary value of analytic function. Equation (16) is just a dispersion relation and (17) a once subtracted dispersion relation. Johnson and Moser [12] precisely prove that $\gamma + i\pi k$ (a slightly differently defined γ) is the boundary value of an analytic function.

3. Thus the Thouless formula says that essentially πk and γ are Hilbert transforms of one another. The proof of the Theorem uses the L^2-continuity of the Hilbert transform.

4. The mystery of this result is removed if one notes that if $u(x) \sim e^{i\alpha x}$ at infinity, then γ measures the imaginary part of α and k, as a rotation number, measures $\operatorname{Re} \alpha$. Alternatively (and this is the key step in the proof and is due to Thouless), matrix elements, t_L, of T_L are analytic functions given by Hadamard product formulae, i.e., $t_L(E) = \pi(E - E_i)$ (finite product in the discrete case; "renormalization" needed in the continuous case), so $L^{-1} \ln t_L(E) = \Sigma L^{-1} \ln(E - E_i)$ and *formally* the sum converges to $\int \ln|E - E'|\, dn(E')$, where $n(E)$ is the density of zeros of t_L. But zeros of t_L are eigenvalues with suitable boundary conditions.

9. Aubry Duality, Sarnak's Analysis, and the French Connection

The results thus far have either applied to all a.p. potentials or to a broad class of potentials like all limit periodic potentials. In this section, we collect some facts about some very special examples which allow more detailed analysis. We regard these examples as very instructive as to what should be expected in general situations.

The first general result concerns what we call the *almost Matthieu* operator

$$(Hu)(n) = u(n+1) + u(n-1) + \lambda\cos(2\pi\alpha n + \theta)u(n) \qquad (19)$$

named because of its similarity to the Matthieu operator

$$\frac{-d^2}{dx^2} + \lambda\cos(2\pi\alpha x + \theta) \qquad (20)$$

(by scaling and translation, one can take $\theta = 0$ and $\alpha = \pi^{-1}$ in (20) as is conventionally done but these parameters are nontrivial in (19)). If α is irrational, the hull corresponds to running θ through $[0, 2\pi)$.

THEOREM 9.1 (Aubry [15]). *Fix α irrational. Let $k(\alpha, \lambda, E)$ be the* IDS *for* (19) (*it is θ independent*). *Then*

$$k(\alpha, \lambda, E) = k(\alpha, 4/\lambda, 2E/\lambda). \qquad (21)$$

In particular:

$$\sigma(H(\alpha, \lambda, \theta)) = \tfrac{1}{2}\lambda\sigma(H(\alpha, 4/\lambda, \theta)). \qquad (22)$$

Remarks. 1. For a careful proof, see [5].

2. There are a variety of closely related connections between $\{H(\alpha, \lambda, \theta)\}_{\theta\in[0,2\pi)}$ and $\{(\lambda/2)H(\alpha, 4/\lambda, \theta)\}_{\theta\in[0,2\pi)}$ all going under the name *Aubry duality*.

3. Some insight is gotten by looking at $\alpha = p/q$, where p and q are relatively prime and at the operators on $l_2(0, q-1)$ with periodic boundary conditions, i.e., let

$$\big(H_q(\lambda)u\big)(n) = u(n+1) + u(n-1) + \lambda\cos(2\pi\alpha n)u(n)$$

(with $u(q) \equiv u(0)$, $u(-1) \equiv u(q-1)$. Let V: $l_2(0, q-1) \to l_2(0, q-1)$ by

$$(Va)(n) = \sum_{m=0}^{q-1} a(m)\exp(2\pi ipmn/q).$$

Because p and q are relatively prime, V is unitary and

$$\big(V^{-1}H_q(\lambda)Va\big)(n) = 2\cos(2\pi\alpha n)a(n) + \tfrac{1}{2}\lambda\big[a(n+1) + a(n-1)\big],$$

i.e., $V^{-1}H_q(\lambda)V = (\lambda/2)[H_q(4/\lambda)]$. Thus Aubry duality comes from the fact that under Fourier transform, the cosine and finite difference terms get interchanged. By taking suitable limits and using Theorem 4.10, one can turn the above into a proof of the theorem. We emphasize that V does *not* have a decent limit and for α irrational, $H(\alpha, \lambda)$ is not claimed to be unitarily equivalent to $\frac{1}{2}H(\alpha, 4/\lambda)$. Indeed, for α with typical Diophantine properties, it is likely that they are not unitarily equivalent in that we believe (see Section 10) that for $\lambda < 2$, $H(\alpha, \lambda)$ will have only a.c. spectrum and $H(\alpha, 4/\lambda)$ only point spectrum.

An interesting use of Aubry duality has been made by Bellisard and Testard [7]. First, by mimicking the Dinaburg–Sinai–Russman [20, 21] work they prove:

LEMMA 9.2 ([7]). *Let α have typical Diophantine properties. Then, for all sufficiently small λ, there is a nonempty, closed set S_λ with*

$$|S_\lambda| \to 4(|\cdot| = \textit{Lebesgue measure};\ 4 = |\sigma(h_0)|)\ \textit{as}\ \lambda \to 0$$

and a function $\Theta(E)$ on S_λ so that for $E \in S_\lambda$, there is a function $u(n)$ on Z with

$$u(n) = e^{i\theta n}\sum_m a_m \exp(2\pi i\alpha m n), \tag{23}$$

$$|a_m| \leqslant Ce^{-Dm}, \qquad D > 0, \tag{24}$$

$$u(n+1) + u(n-1) + \lambda\cos(2\pi\alpha n)u(n) = Eu(n). \tag{25}$$

In fact, as in Theorem 3.1, one can be somewhat explicit about the form of S. The interesting thing is to see what (23), (25) says about the a_m. It implies:

$$\tfrac{1}{2}\lambda[a_{m+1} + a_{m-1}] + 2\cos(2\pi m\alpha + \theta)(a_m) = Ea_m.$$

But (24) says that $a \in l_2$. Thus:

THEOREM 9.3 ([7]). *For λ sufficiently large, $H(\alpha, \lambda, \theta)$ has point spectrum for almost all θ.*

By Cor. 2.6, the point spectrum is locally uncountable. Gordon [19] had constructed examples where H has an eigenvalue but only for one value of θ.

There is a simple intuition [51] which explains why there is point spectrum for λ large. Consider a general potential b_n replacing $\cos(2\pi\alpha n +$

θ). Then $h_0 + \lambda b$ is for large λ, up to a constant, the matrix:

$$\begin{pmatrix} \ddots & & & & \\ \epsilon & b_{-1} & \epsilon & 0 & 0 \\ 0 & \epsilon & b_0 & \epsilon & 0 \\ 0 & 0 & \epsilon & b_1 & \epsilon \\ & & & & \ddots \end{pmatrix}$$

with $\epsilon = \lambda^{-1}$ small. When $\epsilon = 0$, there is clearly point spectrum with eigenvalues which are "localized" at a single point in n-space. When does the perturbation delocalize such states? In the two-by-two case

$$\begin{pmatrix} a & \epsilon \\ \epsilon & b \end{pmatrix}$$

the situation is easy to describe. If $\epsilon \ll |b - a|$, the eigenvalues look like $(1,0)$ and $(0,1)$ but if $\epsilon \gg |b - a|$, they look like $(1, \pm 1)$, i.e., they are delocalized. Thus, the natural condition for point spectrum to persist is

$$\epsilon^{|n-m|} \ll |b_n - b_m|$$

for all n, m. This fails if α is rational or Liouville no matter how small ϵ is, but it will hold for n bounded and α and θ with typical diophantine properties. A similar argument is valid for strongly coupled continuum models which at low energy look like the discrete model.

The next special example is strictly speaking not in the genre we have discussed, since the potential is not real, so H is not self-adjoint:

THEOREM 9.4 (Sarnak [13]). *Let $H(\lambda)$ be the operator on l_2*:

$$(H(\lambda)u)(n) = u(n+1) + u(n-1) + \lambda e^{2\pi i \alpha n} u(n),$$

where α has typical Diophantine properties. Then:

(a) *For $\lambda < 1$, $\sigma(H(\lambda)) = [-2, 2]$ and $H(\lambda)$ has no eigenvalues.*

(b) *For $\lambda > 1$, $\sigma(H(\lambda))$ is the curve in the complex plane which is the ellipse with center $(0,0)$ and semimajor axis $\lambda + \lambda^{-1}$. Moreover, $H(\lambda)$ has an infinite set of eigenvalues which are dense in $\sigma(H(\lambda))$.*

Remarks. 1. In (b), no claim is made about whether the corresponding eigenvectors are complete.

2. That $\sigma(H(\lambda))$ has no gaps for $\lambda < 1$ may seem counterindicative of the Cantor spectrum idea, but we note that the spectrum of $-d^2/dx^2 + \lambda e^{i\alpha x}$ is $[0, \infty)$ [52] even though we know real periodic potentials always produce some gaps.

3. The operator of this theorem is special, because taking Fourier transforms in trying to solve $(H(\lambda) - E)u = 0$ we find

$$\lambda \hat{u}(p + 2\pi\alpha) = [E - 2\cos p]\hat{u}(p)$$

and the equation is easily iterated. Sarnak [13] studies behavior of $\prod_0^n \{E - 2\cos(p + 2\pi\alpha)\}$ using the ergodic theorem.

The final special result involves a striking connection found by a French group [8] between solutions of a Kronig–Penny Hamiltonian

$$(H\varphi)(x) = E\varphi(x), \tag{26a}$$

$$H = \frac{-d^2}{dx^2} + \sum_{-\infty}^{\infty} b_n \delta(x - n), \tag{26b}$$

and a discrete operator:

$$(hu)(n) = \epsilon u(n), \tag{27a}$$

$$(hu)(n) = u(n + 1) + u(n - 1) + c_n u(n). \tag{27b}$$

THEOREM 9.5 ([8]; the *French connection*). *Let $E = k^2$ and suppose k is not a multiple of π. Let φ solve* (26) *and let $u(n) = \varphi(n)$. Then u solves* (27) *with*

$$\epsilon = 2\cos k, \tag{28a}$$

$$c_n = -[k^{-1}\sin k]b_n. \tag{28b}$$

Conversely, given any solution of (27) *with* (28) *holding, there is a (unique) solution φ of* (26) *with $\varphi(n) = u(n)$. Moreover, φ is in $L^p(-\infty, \infty)$ if and only if u is in l^p and φ has exponential growth (resp. decay) if and only if u does and at the same rate.*

Remarks. 1. In [8], a somewhat weaker result is found; namely, the matrix equations for $(u(n), u(n+1))$ *and* $(\varphi(n), \varphi'(n))$ are proven equivalent by a sequence of complicated matrix transformations. The relation $\varphi(n) = u(n)$ is not realized; this is a finding of Avron and Simon [51].

2. In the Avron–Simon view [51], this theorem is a straightforward calculation; namely, if φ solves (26), then in $(-1, 0)$,

$$\varphi(x) = a\cos kx + b\sin kx$$

and in $(0, 1)$

$$\varphi(x) = c\cos kx + d\sin kx.$$

We have

$$u(0) = a = c,$$
$$u(1) = c\cos k + d\sin k; \qquad u(-1) = a\cos k - b\sin k$$

and

$$\varphi'(0+) - \varphi'(0-) = k(d-b) = b_0 a$$

so

$$k(u(1) + u(-1)) = b_0 u(0)\sin k + 2ku(0)\cos k$$

as claimed.

3. In particular, $b_n = \mu\cos(2\pi\alpha n + \theta)$ corresponds to $c_n = \lambda\cos(2\pi\alpha n + \theta)$ with

$$\lambda = -[k^{-1}\sin k]\mu.$$

If $\lambda = 2$ is the critical value at which there is a shift from a.c. to pure point spectrum (see the next section), then this suggests that for $\mu < 2$, there is only a.c. spectrum, but at $\mu > 2$, these occur first one and then additional "bands" of point spectrum (determined by $|\mu k^{-1}\sin k| > 2$).

10. Almost Matthieu Operators Without A.C. Spectrum

In [16], Aubry and André state the following:

PSEUDO-THEOREM. *Fix α irrational; θ. Then, for $\lambda < 2$, the almost Matthieu operator* (19) *has only* a.c. *spectrum, and for $\lambda > 2$, H has only dense point spectrum with eigenfunctions which fall exponentially.*

We use the phrase "pseudo" not merely because the result is not rigorously proven, but because it is false as stated! Indeed, as Example 2 of Section 7 shows, if α is a Liouville number, there are no eigenvalues at all. Nevertheless, the idea of Aubry and André are sound ones, and by exercising some care, Avron and Simon [4, 5] have proven that:

THEOREM 10.1. *Fix α irrational and $\lambda > 2$. Then, for* a.e. *θ in* $[0, 2\pi)$ *(with respect to Lebesgue measure), the operator* (19) *has no absolutely continuous spectrum.*

Proof. Define γ by (16) and use Aubry duality (21) to see that

$$\gamma(E, \alpha, \lambda) = \gamma(2E/\lambda, \alpha, 4/\lambda) + \ln(\lambda/2) \tag{29}$$

so that, by (15)

$$\gamma(E, \alpha, \lambda) \geqslant \ln(\lambda/2) > 0 \qquad \text{if } \lambda > 2.$$

Thus, by Corollary 8.4 and Theorem 8.5, for a.e. θ, there is a set S_θ of measure zero (relative to Lebesgue measure), so that if $E \notin S_\theta$, every solution of $Hu = Eu$ is either exponentially growing or decaying both at $+\infty$ and $-\infty$. In particular, any polynomially bounded solution is in L^2. General theory [42, 43], says that for a.e. E with respect to a spectral measure, there are polynomially bounded solutions. Thus a spectral measure is supported by $S_\theta \cup P_\theta$, where P_θ is the (countable) set of eigenvalues. Since the Lebesgue measure of $S_\theta \cup P_\theta$ is zero, the spectral measure can have no a.c. piece.

Remarks. 1. Equation (29) is a result of Aubry and André [16]. In some sense, their error is that they ignored the possibility of the set S_θ.

2. Given $\gamma > 0$, the proof that there is no a.c. spectrum is close to one Pasteur gave in the random case [24]. Thus, this is a kind of "Pasteurized Aubry and André theorem."

Since Example 2 of Section 7 and Corollary 7.3 say that H has no point spectrum if α is a Liouville number, we have

COROLLARY 10.2 ([4, 5]). *If $\lambda > 2$, and α is a Liouville number, then, for a.e. θ, the almost Matthieu operator has purely singular continuous spectrum.*

Remarks. 1. Unfortunately, the proof is rather unilluminating in the sense that there is no reason given for singular continuous spectrum other than the absence of the other types. In terms of the intuition of Section 5, the Liouville numbers suggest motion where there are reflections only after very long distances, and so a behavior close to that in the Pearson example [46].

2. Note that the continuum polynomially bounded eigenfunctions, u, which exist here must be at energies E in S_θ showing that S_θ is nonempty. Moreover, these u do not go to zero at $\pm\infty$ contrary to one mythology which suggests singular continuous eigenfunctions correspond to $u \to 0$ but in a non-L^2 way.

We expect that if α has typical Diophantine properties, then the Aubry–André theorem is valid, i.e., there is an abrupt transition from a.c. to dense point spectrum at $\lambda = 2$.

11. PROBLEMS AND CONJECTURES

There have been a number of intriguing phenomena discovered or suspected in a.p. Schrödinger operators, but so far the majority of interesting results have involved restricted classes and/or weaker conclusion than one would like. This makes a section of problems particularly appropriate. We begin with the special almost Matthieu operators where one has the most information, but where much is lacking. "Problems" which don't end in a ? should be viewed as conjectures! Since the ordinary Matthieu operator has all its gaps open, we begin with

Problem 1 (The Ten Martini Problem). For all $\lambda \neq 0$, all irrational α and all θ, the operator (19) has a Cantor spectrum.

The name comes from the fact that Mark Kac [53] has offered ten Martinis to anyone who solves it. It is unclear what he will give for a partial solution like generic α but such results would be interesting. Actually, Kac said "has all its gaps there" so perhaps one should solve instead

Problem 2 (The Ten Martini Problem—Strong Form or should it be Dry Form). For all $\lambda \neq 0$, all irrational α, and all integers n_1, n_2, with $0 < n_1 + n_2\alpha < 1$, there is a gap for (19) on which $k(E) = n_1 + n_2\alpha$.

The next problems involve spectral properties of (19):

Problem 3. Let α have typical Diophantine properties. Prove that for $\lambda < 2$, the operator (19) has purely a.c. spectrum and for $\lambda > 2$, purely thick point spectrum.

Problem 4. What happens if α has typical Diophantine properties and $\lambda = 2$?

There seem to be four possibilities: (1) Purely singular continuous spectrum (2) Overlapping dense point spectrum and a.c. spectrum (3) Overlapping spectrum of all three types (4) Some other possibility. The first three possibilities are those consistent with the idea that point and a.c. spectrum are dual to each other and s.c. spectrum is self-dual. Since I see no reason for s.c. spectrum, I would vote for (2) if forced, but my preference is very weak.* Since s.c. spectrum appears to be self-dual, one expects:

Problem 5. Prove that if α is a Liouville number and $\lambda < 2$, there is purely singular continuous spectrum.

The next set of problems concern more general operators in one dimension.

Problem 6. Prove that a generic one dimensional a.p. Schrödinger operator has Cantor spectrum.

There are different versions of this problem. For example, one can consider all a.p. potentials or alternatively all a.p. potentials with a fixed given frequency module.

*See Note added in proof.

Problem 7. For a generic a.p. potential, V, $H_0 + \lambda V$ has *some* thick point spectrum for λ large.

The intuition in Section 9 suggests a more explicit form:

Problem 8. Fix α with typical Diophantine properties. Let f be a C^2 function on the two torus with a unique nondegenerate minimum. Then $H_0 + \lambda f(x, \alpha, x)$ has *some* thick point spectrum for λ large.

Next, it would be interesting to find more examples with s.c. spectrum.

Problem 9. Given a fixed monotone sequence $n_1, \ldots$ of integers and a sequence $a_m \in l_1$, let

$$H(a_m) = H_0 + \sum a_m \cos(x/2^{n_m}).$$

If n_m increases sufficiently rapidly, does H have purely s.c. spectrum?

Problem 10. If α is a Liouville number and f is generic, does $H_0 + f(x, \alpha x)$ have only s.c. spectrum?

The last three problems involve more than one dimension. The first is an analog of Thomas' theorem on the absence of eigenvalues in the periodic case [54]:

Problem 11. Prove that for any a.p. potential $V \neq$ constant and any E, $\{W | W$ in hull of V; E an eigenvalue of $H_0 + W\}$ has zero (Haar) measure.

Problem 12. What features of a.p. problems extend to N-dimensions?

We expect that the Cantor spectrum does *not* but thick point and singular continuous spectrum do.

Problem 13. What features of a.p. problems extend to the Hamiltonian of an electron in two dimensions, with irrational magnetic field and periodic potential?

So there is still lots to do. Perhaps this article will succeed in spreading the a.p. flu!

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Note added in proof. In the year plus ε between when this article was written and publication, there has been additional progress, including solutions of some of the listed problems. We mention:

1. A weak solution of Problem 1 has been found by Bellisard–Simon (*J. Funct. Anal.*, to appear), who prove there is a Cantor spectrum for a dense G_δ of all pairs (λ, α).

2. There is numerical evidence that for $\lambda = 2$, α irrational, $\sigma(H)$ has zero Lebesgue measure suggesting possibility (1) is the correct answer to Problem 4.

3. W. Craig and B. Simon (in preparation) have solved Problem 11 affirmatively in the discrete case.

4. W. Craig (Caltech preprint), using KAM theory, has constructed a variety of weakly almost periodic sequences so that $h_0 + V$ has only dense point spectrum. Extending Craig's work, Poschel (ETH preprint) and Bellisard, Lima, Scoppola (Marseille preprint) have constructed many additional examples, including some limit periodic examples (Poschel).

5. Prange *et al.* (Maryland preprint) have found explicitly the eigenvalues in a particularly weakly almost periodic potential.

6. Bessis *et al.* (Saclay preprint) have found an a.p. tridiagonal matrix (but not just 1's off diagonal) with exactly computable spectrum.

7. Johnson (U.S.C. preprint) (see also Craig–Simon (Caltech preprint)) has proven spec(H) must have positive logarithmic capacity.

8. M. Hermann (Ecole Polytechnique preprint) has found an alternate and very elegant proof that $\gamma > 0$ for the $\lambda > 2$ almost Mathieu operator. His proof extends to a much larger class of potentials.

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