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Fifteen Problems in Mathematical Physics

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Abstract Presentation and discussion of a number of important open problems in mathematical physics.

0 Introduction

When the editors of this volume asked me to contribute, I had mixed feelings. Since I had recently written several long review articles, I was very reluctant to write another. One the other hand, I had fond remembrances of the scattering theory meetings I attended at Oberwolfach in 1971, 1974 and 1977, meetings which clearly had an important positive influence on the field. In thinking of the rather special character of Oberwolfach and its vitality, I realized an article which looks towards the future belonged among those rigthfully celebrating the past. The editors responded very warmly to my suggestion of an article on open problems in mathematical physics: hence this article. By looking towards the future, I also was able to survey broad areas of mathematical physics; unfortunately, Oberwolfach has intersected mathematical physics mainly in scattering theory and in classical mechanics, but I hope the future sees conferences in areas like quantum field theory, statistical mechanics and mathematical aspects of condensed matter physics!

It is with some misgivings that I set out in writing this article. Broad problem survey articles bring to mind Hilbert's famous article [1]. I am no Hilbert, and I certainly don't want anyone to think I feel any comparison is possible except using Lev Landau's logarithmic scale. Nevertheless, I have borrowed some of Hilbert's devices. While many of the problems stated are quite explicit and precise, some are so vague as to be close to ludicrous. Also, even more than Hilbert, I use the device of grouping several problems into "one", but when I do that, I have labeled them A, B,... Indeed, my 15 problems are really 32, explicitly:

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Problem 1 A: Almost always global existence for Newton's equations
1 B: Existence of non-collisional singularities in the Newtonian N-body problem
Problem 2 A: Ergodicity of gases with soft cores
2 B: Approach to equilibrium
2 C: Asymptotic abelianness for the quantum Heisenberg dynamics
Problem 3 : Turbulence and all that
Problem 4 A: Fourier's heat law
4 B: Kubo formula
Problem 5 A: Exponential decay of $\nu = 2$ classical Heisenberg correlations
5 B: Pure phases at low temperatures in the $\nu \geqq 3$ classical Heisenberg model
5 C: GKS for classical Heisenberg models
5 D: Phase transitions in the quantum Heisenberg model
Problem 6 : Existence of ferromagnetism
Problem 7 : Existence of continuum phase transitions
Problem 8 A: Formulation of the renormalization group
8 B: Proof of universality
Problem 9 A: Asymptotic completeness for short range N-body quantum systems
9 B: Asymptotic completeness for Coulomb potentials
Problem 10 A: Monotonicity of ionization energy
10 B: The Scott correction
10 C: Asymptotic ionization
10 D: Asymptotics of maximal ionized charge
10 E: Rate of collapse of Bose matter
Problem 11 : Existence of crystals
Problem 12 A: Existence of extended states in the Anderson model
12 B: Diffusive bound on "transport" in random potentials
12 C: Smoothness of $k(E)$ through the mobility edge in the Anderson model
12 D: Analysis of the almost Mathieu equation
12 E: Point spectrum in a continuous almost periodic model
Problem 13 : Critical exponent for self-avoiding walks
Problem 14 A: Construct QCD
14 B: Renormalizable QFT
14 C: Inconsistency of QED
14 D: Inconsistency of φ_4^4
Problem 15 : Cosmic censorship

In deciding what is mathematical physics, I have generally tried to follow two basic rules: (1) Problems like "quantize gravity", where it is clear that the basic underlying physics is not understood, have not been included even if their solution is likely to involve a lot of mathematics. (2) Problems in "pure mathe-

matics", even quite close to mathematical physics (like operator algebras) have generally not been included (which forces me to give some explanation in connection with Problem 13).

In an undertaking like this, I have benefited greatly from advice and information I received from a number of colleagues whom I consulted. I would like to thank Jürg Fröhlich, Bob Geroch, Jim Glimm, Anatoly Katok, Joel Lebowitz, Elliott Lieb, John Mather, Roger Penrose, Derek Robinson, Don Saari, Alan Sokal, Arthur Wightman and most especially, Tom Spencer, for their aid.

1 Existence for Newtonian Gravitating Particles

Newton's equations for N-particles of masses $m_1, \dots, m_N$ interacting gravitationally in units where $G = 1$ are

$$m_i \ddot{\vec{r}}_i = \sum_{j \neq i} m_i m_j (\vec{r}_j - \vec{r}_i) |\vec{r}_i - \vec{r}_j|^{-3} \tag{1.1}$$

It is obvious that already for $N = 2$, (1.1) can fail to have solutions global in time for suitable initial conditions, e.g. $\dot{r}_1 = \dot{r}_2 = 0$. For $N = 2$, it is easy to see that the set of initial conditions leading to a collision is a subset of those conditions of total angular momentum zero, so the set of initial conditions for which global existence fails has measure zero if $N = 2$.

Problem 1 A (Almost always global existence for Newton's equations). Prove that the set of initial conditions for which (1.1) fails to have global solutions has measure zero in $\mathbb{R}^{6N}$.

We show our general feeling for what we believe is the answer, but we should emphasize that some excellent mathematicians believe that there may be an open set of initial conditions leading to non-global solutions.

To be more precise, the problem of singularities of (1.1) is connected with some pair colliding, i.e. we say a global solution fails to exist if at some finite time, T, $\lim_{t \uparrow T} [\min_{i \neq j} |\vec{r}_i(t) - \vec{r}_j(t)|] = 0$. It is easy to see that the set of initial conditions, NE, leading to this is an F_σ so if NE has measure zero, its complement is automatically a dense G_δ.

We call a singular time, T, a *collision* if, for each i, $\lim_{t \uparrow T} \vec{r}_i(t)$ exists (and is a finite point). A *binary collision* is one where only pairs of $\vec{r}_i(T)$ are equal. The set of all initial conditions leading to a collision we will call C, and its complement in NE we call NC. The subset of C leading to binary collisions is denoted BC.

Painlevé [2] appears to have been the first person to have seriously discussed these questions and, in particular, he proved that NC is empty if $N = 3$. Much more recently, Saari [3] proved that NC has measure zero if $N = 4$. The analogous problem is open for $N \geqq 5$ and, as we shall see, is the key question.

Birkhoff [4], applying a result of Sundman [5], showed that BC has measure zero, and in 1972—73, Saari proved [6]:

Theorem 1.1 C has measure zero and is Baire first category for any N.

This result does not immediately imply the same for various invariant subsets of lower dimension (i.e. $I \cap C$ has zero measure in the appropriate measure on an invariant subset of lower dimension). For I the set of configurations lying in a fixed plane, this is proven by Saari in the same references, and for the manifold of fixed angular momentum, it is a result of Urenko [7]. Saari's proof depends on an interesting and detailed analysis of precisely what happens at a collision.

Theorem 1.1 reduces an affirmative solution to problem 1 A to showing that NC has measure zero, and in particular, Saari's later result that NC has measure zero if $N=4$ settles Problem 1 A in that case.

One general fact is known about NC, namely:

Theorem 1.2 (Sperling [8], based on ideas of von Zeipel [9]). For a solution whose initial conditions is in NC,

$$\lim_{t \uparrow T} \sum |\vec{r}_i(t)|^2 = \infty \tag{1.2}$$

As we have remarked, it is known (Painlèvé) that NC is empty when $N=3$. There is no proof that it is not always empty, but there are strong indications it is not. First, for particles on a line there is an obvious way continuing through a binary collision (have the particles bounce off each other in their mutual center of mass frame). Mather and McGehee [10] found an initial configuration of 4 particles on the line which, if continued through binary collisions by this rule, have a time T which is an accumulation point of binary collisions, and (1.2) holds.

Recently, J. Gerver [11] produced a simple mechanism for non-collisional singularity in $N=5$. He imagines a situation of 3 very massive particles at the edges of an isosceles triangle. A light "moon" is rotating about the particle, S, at the distinguished vertex and it is the "falling" of this moon into S that serves as the "engine" pumping energy into the system. A fifth particle travels more or less around this triangle. It moves essentially in a hyperbola as it swings around each vertex with the edges of the triangle being the asymptotes of the hyperbola. As it passes each vertex is gives an "outwards" kick to each particle. As it passes by S it picks up enough energy from the "engine" (i.e. the moon of S ends up in a smaller orbit after the passage of particle 5 trough the area of S) to enable it to continue its circuit around the enlarged triangle. Scaling arguments show that as the triangle gets bigger the circuit time of particle 5 decreases geometrically, and in finite time the triangle becomes infinitely larger. Gerver presents a number of detailed calculations to support this picture. Since he doesn't present a complete proof, we have

Problem 1 B (Existence of non-collisional singularities in the Newtonian N-body problem) Show NC is non-empty for some N and suitable m_i.

We caution the reader that Mather has made significant progress on making a rigorous proof of Gerver's scenario.

Next, a reason we tend to believe NE has measure zero. In quantum mechanics, it is a theorem of Kato [12] that global solutions of the Schrodinger equation with Coulomb potentials exist. Since quantum mechanics tends to only smooth out sets of measure zero, one expects that NE has measure zero. No doubt this reasoning will infuriate classical mechanics. In any event, if NE turns out to have an open subset, the *classical* limit of the corresponding quantum theory will be very interesting.

The quantum analog of Problem 1 A is, as we have noted, solved. Indeed, there is an enormous and more or less complete literature on the solubility of the Schrodinger equation summarized in Reed-Simon, Vol. II [13]. With the recent paper of Leinfelder-Simader [14] who solved one interesting open question in this area, only one basic selfadjointness question remains:

Jörgen's Conjecture Let $W(x) \geqq V(x)$ on R^ν and let M be a finite union of closed submanifolds in R^ν. Suppose that $-\Delta + V$ is essentially selfadjoint on $C_0^\infty(R^\nu \setminus M)$ and bounded below. Then $-\Delta + W$ is essentially selfadjoint on $C_0^\infty(R^\nu \setminus M)$.

We note there are counterexamples if the assumption the $-\Delta + V$ is bounded below is dropped (see Pg. 155—156 [13]).

We have been careful not to include this among our list of problems. It has intrinsic interest but its importance is primarily technical. It is significant in part because it is over 10 years old and several technically strong people have worked on it without success.

2 Open Questions in Ergodic Theory

The founding fathers of statistical mechanics, especially Boltzmann and Gibbs, realized that the deepest aspect of thermodynamics from a microscopic point of view was the "zeroth law", that bulk systems rapidly approach equilibrium states parametrized by a few macroscopic parameters. By 1930, the standard wisdom was that the key notion is a proof that the classical dynamics on the constant energy manifolds of phase space is ergodic (see e.g. Avez-Arnold [15] for a discussion of the basic notions of ergodic theory). It is ironic that Sinai's celebrated result that the hard sphere gas is ergodic was announced [16] at approximately the same time that the KAM theory developed, for one important consequence of KAM is that many classical systems will not be ergodic: There will be an invariant subset of phase space consisting of a union of invariant tori of positive total measure.

It has been 20 years since Sinai's announcement, and a complete, detailed proof has not yet appeared except for the (nontrivial) case of two particles [17].

A partial sketch for $N = 3,4,5$ appears in [18]. Recently, Sinai and Chernoff [19] have proven that the Kolomogorov-Sinai entropy of a hard sphere gas is positive, and even that the entropy per particle is positive in the thermodynamic limit. (While these results are mathematically independent of ergodicity, the ideas in their proof are presumably an important aspect of a possible proof of ergodicity.)

Despite the blow that KAM gives to the 1930's wisdom, it is an interesting question to extend Sinai's proof beyound the hard sphere gas: His system in its simplest form involves N particles in a cubic box bouncing elastically off the walls and each other.

Problem 2A (*Ergodicity of gases with soft cores*) Find a class of repulsive smooth potentials for which the N-particle dynamics in a box (with, say, smooth wall potentials) is ergodic.

The expected lack of ergodicity for systems with interacting potentials which are not strictly repulsive requires a convincing revised standard wisdom to explain the approach to equilibrium. One idea advocated by Wightman [20] among others is that there is one ergodic component of such systems which, in the limit as the volume goes to infinity (with constant particle density), occupies a larger and larger fraction of phase space.

Problem 2B (*Approach to equilibrium*) Verify the above scenario to justify approach to equilibrium of large systems with forces which are attractive at suitable distances, or else find an alternate scenario which doesn't rely on strict ergodicity in finite volume.

We want to emphasize that the studies of ergodicity of the dynamics of infinite partial systems (see e.g. [21]), while interesting mathematically, does *not*, in our opinion, address this issue. The ergodicity of the infinite particle non-interacting gas shows that this kind of ergodicity comes from the fact that one puts equilibrium into the system at infinity by the choice of underlying measure and that equilibrium "diffuses" into finite regions.

We also note that neither the standard wisdom or the above candidate for a revised standard wisdom addresses the basic question of why the approach to equilibrium in the real world is on a time scale so short compared to typical recurrence times in the system.

Finally, we should say a few words about approach to equilibrium in quantum systems which is very difficult for many reasons, e.g. in finite volume the systems tend to have discrete spectrum and thus almost periodic behavior in time. There has been some interesting study of approach to equilibrium in infinite quantum lattice systems, but even here much more is unknown than known. One basic question involves the notion of asymptotic abelianness under time translation. The notion was originally introduced under space translation where it is an obvious feature very useful in the abstract study of such systems (see e.g. Ruelle [22] and Bratelli-Robinson [23]). The relevance of these ideas in quantum systems is discussed in Chapter 6 of Bratelli-Robinson [23]. Unfortunately, the only examples where it is known there is asymptotic abelianness are quasi-free states and the closely related one-dimensional $X-Y$ model (on the even algebra). For simplicity we state the next problem for a definite model, but any non quasi-free, intrinsically non-abelian multidimensional model would be interesting.

Problem 2C (*Asymptotic Abelianness for the Quantum Heisenberg Dynamics*) Prove (or disprove) that the multidimensional quantum Heisenberg model has asymptotically abelian dynamics.

3 Long Time Behavior of Dynamical Systems

Problem 3 (*Turbulence and all that*) Develop a comprohensive theory of the long time behavior of dynamical systems including a theory of the onset of, and of fully developed turbulence.

This problem is so general as to be verging on the absurd. We include it in part to indicate our strong feeling that this is an area which is not only fashionable but important as well. We state it in this form because it seems the field is not yet at a level of maturity where one can focus on certain crucial questions; rather, the first problem is to formulate the really significant questions. For some recent reviews of some of the more spectacular developments in the area, see Feigenbaum [24] or the book of Collet-Eckmann [25].

As for the question of turbulence, there has been considerable progress in understanding the onset of turbulence (see e.g. Ruelle [26] or Eckmann [27]), but our understanding of fully developed turbulence is far from fully developed.

The connection between turbulence and the Navier-Stokes equation is not clear, but there may well be one. In this regard, we should note that the existence theory for this important equation is not completely satisfactory; see Foias-Tenam [28] for a review.

4 Transport Theory

At some level, the fundamental difficulty of transport theory is that it is a steady state rather than equilibrium problem, so that the powerful formalism of equilibrium statistical mechanics is unavailable, and one does not have any way of precisely identifying the steady state and thereby computing things in it.

A second difficulty concerns the fact that most transport is a diffusion phenomena and there is no satisfactory derivation of diffusion from an underlying microscopic dynamics except in some limit in which a physically fixed scale (like particle sizes) is varied rather than a physically varied scale (like system sizes).

To explain this diffusion remark in an example, consider a very crude model of a linear system with particles moving between a wall at 0, and another at L. We characterize the fact that we imagine the wall at 0 having temperature T_0 and the one at L having temperature T_1 by saying that upon collision with the wall, all 0 the particles always come off with velocity $v_0 \sim \sqrt{T_0}$ and upon collision with the wall at T_1 with velocity $v_1 \sim \sqrt{T_1}$. As L varies we imagine increasing the number of particles moving back and forth to keep their density fixed. If we change L and assume the particles are non-interacting, a simple calculation shows that the rate of energy transport between the two walls is unchanged although Fourier's heat law says that it should go as $(\Delta T) L^{-1}$. If one, by fiat, imagines that particle interaction causes a diffusion of heat so that transit times go as L^2 not L, then the rate of heat transfer has the proper L^{-1} behavior.

The connection with diffusion links these transport questions with the material discussed in Section 12.

In the problem below, we would allow a model which brought temperature in even with as bad a caricature as the above crude model.

Problem 4 A (*Fourier's Heat Law*) Find a mechanical model in which a system of size L has a temperature difference ΔT between its ends and in which the rate of heat transfer in the infinite L limit goes as L^{-1}.

There are also serious foundational questions in quantum transport. A basic formula in condensed matter physics is the Kubo formula for conduction; see e.g. [29] for discussion. Not only are the usual derivations suspect, but van Kampen [30], among others, has seriously questioned its validity on physical grounds.

Problem 4 B (*Kubo Formula*) Either justify Kubo's formula in a quantum model, or else find an alternate theory of conductivity.

5 Heisenberg Models

Lattice models of statistical mechanics have been fruitful testing grounds for ideas in the theory of phase transitions. The last 15 years have seen remarkable progress in the rigorous study of these models, especially the Ising model. For each site α in $\mathbb{Z}^\nu$ we imagine a spin $\vec{\sigma}_\alpha$ taking values in S^{D-1}, the unit sphere in D-dimensions. Given $\Lambda \subset \mathbb{Z}^\nu$, a finite subset, we define

$$H_\Lambda = - \sum_{<\alpha\gamma> \in \Lambda} \vec{\sigma}_\alpha \cdot \vec{\sigma}_\gamma \tag{5.1}$$

the sum being over all nearest neighbor pairs in Λ. Given a parameter $\beta =$ inverse temperatures, we form a probability measure on $(S^{D-1})^\Lambda$ by

$$\langle f \rangle_{\Lambda,\beta} = \int f(\sigma_\alpha)\, e^{-\beta H_\Lambda(\sigma_\alpha)} \prod_{\alpha \in \Lambda} d\mu_0(\sigma_\alpha)/Z_\Lambda \tag{5.2a}$$

$$Z_\Lambda = \int e^{-\beta H_\Lambda(\sigma_\alpha)} \prod_{\alpha \in \Lambda} d\mu_0(\sigma_\alpha) \tag{5.2b}$$

where $d\mu_0(\sigma_\alpha)$ is the usual invariant measure on S^{D-1} (if $D=1$, so $S^{D-1} = \{\pm 1\}$, $d\mu_0(\sigma_\alpha) = \frac{1}{2}[\delta(\sigma_\alpha + 1) + \delta(\sigma_\alpha - 1)]$. By $\langle f \rangle_\beta$ we mean suitable limits as Λ approaches Z^ν. $D=1$ is called the Ising model, $D=2$ the plane rotor and $D=3$ the classical Heisenberg model. These models are quite different because their symmetry groups are quite distinct: In $D=1$ a discrete group, in $D=2$ an abelian continuous group and in $D=3$ a non-abelian continuous group.

Problem 5 A (*Exponential decay of $\nu = 2$, $D = 3$ correlations*). Consider the two dimensional classical Heisenberg model ($\nu = 2$, $D = 3$). Prove that for any β, $\langle \sigma_\alpha \cdot \sigma_\gamma \rangle_\beta$ decays exponentially as $|\alpha - \gamma| \to \infty$.

Here is some background on this problem: If $\lim_{|\alpha-\gamma|\to\infty} \langle \sigma_\alpha \cdot \sigma_\gamma \rangle \neq 0$, one says the model has long range order (LRO), an indication of multiple phases (see Ruelle [22], Griffiths [31], Israel [32], or Simon [33]). For $D=1$ (Ising), there is LRO when β is sufficiently large so long as $\nu \geq 2$ (Peierls [34]); for $D \geq 2$, it is known there is LRO for β large if $\nu \geq 3$ (Fröhlich et al. [35]), but if $\nu = 2$, there is no LRO for any β (Mermin-Wagner [36]). Dyson [37] gave an intuitive argument that when $\nu = 2$, $D \geq 2$ and β is large, $\langle \sigma_\alpha \cdot \sigma_\gamma \rangle_\beta$ should only have power decay; in the '70's it was realized in the non-rigorous theoretical physics literature (see e.g. [38]) that due to a renormalization group intuition, one should expect that there is this power decay when $D=2$ but not if $D \geq 3$. Recently, Fröhlich-Spencer [39] have proven that if $D=2$, $\nu=2$ there is only power decay if β is large. The important open question above concerns whether the situation is different if $D \geq 3$, $\nu = 2$. Because of the connection with "infrared freedom", an important notion in Q.C.D., this problem has importance in quantum field theory.

The next problem concerns the structure of the set of "pure phases" ($\equiv$ extreme points of the set of translation invariant DLR states); we will not give the precise definition on this notion: See Ruelle [22], Israel [32] or Simon [33]. The symmetry group acts on the set of equilibrium states.

Problem 5 B (*Pure phase at low temperatures*). Prove that at large β and $\nu \geq 3$, the set of equilibrium states for the $D=3$ model forms a single orbit under $SO(3)$ which is the sphere S^2.

This result says that at fixed low temperature, the phases are characterized by a single unit vector describing, say, the direction of the magnetization. The analogous result for $D=1$ was proven by Gallavotti-Miracle Sole [40] and for $D=2$ by Fröhlich-Pfister [41]. It is likely that a solution of this problem will either involve developing new correlation inequalities for these types of models, or else one will understand these phenomena without correlation inequalities.

Problem 5C (GKS for classical Heisenberg models). Consider the model with $D = 3$, arbitrary ν. Let f, g be finite products of the form $(\sigma_\alpha \cdot \sigma_\gamma)$. Is it true that

$$\langle fg\rangle_{\Lambda,\beta} \geqq \langle f\rangle_{\Lambda,\beta}\langle g\rangle_{\Lambda,\beta} \tag{5.3}$$

for all Λ, β.

Actually, one wants this for more general ferromagnets than nearest neighbor coupling. (5.3) for $D=1$ is the famous inequality of Griffiths [42], Kelly and Sherman [43]. It was extended to $D=2$ by Ginibre [44] who obtained it from a generalized set of inequalities (Ginibre's inequalities). Shortly before his death, Sherman announced a proof of Ginibre's inequality, and therefore GKS for general D, but his notes seemed to contain an error. In fact, Sylvester [45] has recently proven that Ginibre's inequality is false for $D \geqq 3$. This leaves the GKS situation open; it is generally believed they are true. Many other inequalities and many applications would immediately follow.

The final of our Heisenberg-model problems involves the quantum model. The phase space $(S^{D-1})^\Lambda$ is replaced by a Hilbert space $\mathbb{C}^{2^{|\Lambda|}}$ thought of as $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ ($|\Lambda|$ times). $\sigma_{\alpha i}$ is the operator which is a tensor product of τ_i in the α factor and 1 in the others. Here τ_i are the standard Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

H_Λ is still given by (5.1) but (5.2) is replaced by

$$E(f) = \mathrm{Tr}(f\, e^{-\beta H_\Lambda})/Z_\Lambda \tag{5.4a}$$

$$Z_\Lambda = \mathrm{Tr}(e^{-\beta H_\Lambda}). \tag{5.4b}$$

Problem 5D (Phase transition in the quantum Heisenberg model). Prove that for $\nu \geqq 3$ and β large, the quantum Heisenberg model has LRO in the sense that

$$\lim_{|\alpha-\gamma|\to\infty} \langle \sigma_\alpha \cdot \sigma_\gamma\rangle_\beta \neq 0.$$

A positive solution of this problem was announced by Dyson, Lieb and Simon [46], but they made an error. For the antiferromagnet, i.e. $\beta < 0$ and $|\beta|$ very large, Dyson, Lieb and Simon [47] prove that

$$\lim_{|\alpha-\gamma|\to\infty} |\langle \sigma_\alpha \cdot \sigma_\gamma\rangle| \neq 0.$$

Quite likely, this problem is connected with problem (vi) below.

Here are some other interesting open questions in lattice models: we suppose some familiarity with terminology (see [22, 31, 32, 33]).

(i) Let J be a non-negative function on Z^ν. The model given by (5.2) with

$$H_\Lambda = -\sum_{\alpha\gamma\in\Lambda} J(\alpha-\gamma)\, \sigma_\alpha \cdot \sigma_\gamma$$

and $D=1$ is called the general ferromagnetic Ising model. If $J(\alpha)=0$ for all but finitely many α, one calls the model finite range. One defines

$$p(\beta) = \lim_{|\Lambda|\to\infty} |\Lambda|^{-1} \ln Z_\Lambda(\beta).$$

$p(\beta)$ is convex and so automatically differentiable for all but finitely many β. One expects that $p(\beta)$ is actually C^1 for these models. Prove it. This is important because of results of Lebowitz [48].

(ii) The one dimensional Ising model with $J(\alpha) = |\alpha|^{-2}$ for $\alpha \neq 0$ is especially interesting. An argument of Thouless [49] (made partially rigorous by Simon-Sokal [50]) suggests that the magnetization of this model is discontinuous in β. It is known (Fröhlich-Spencer [51]) that the magnetization is nonzero for β large. Prove the magnetization is discontinuous.

(iii) Consider the basic nearest neighbor model with $D=1$, $\nu \geqq 3$. Define

$$\beta_c^{(1)}(\nu) = \inf\{\beta \mid \lim_{|\alpha-\gamma|\to\infty} \langle\sigma_\alpha \cdot \sigma_\gamma\rangle \neq 0\} \quad \text{and}$$

$$\beta_c^{(2)}(\nu) = \sup\{\beta \mid \langle\sigma_\alpha \cdot \sigma_\gamma\rangle \leqq C_1\, e^{-C_2|\alpha-\gamma|} \textit{ for some } C_1, C_2\}.$$

Clearly $\beta^{(1)} \geqq \beta^{(2)}$. Prove they are equal.

(iv) There are interesting questions concerning the existence of equilibrium states ($\equiv$ DLR states) which are not translation invariant. For $D=1$, $\nu=2$, Aizenman [52] proved these don't occur. Dobrushin [53] proved for $D=1$, $\nu \geqq 3$, there are such states. Define $\beta_r(\nu)$, the roughening temperature, to be the inf over all β for which there exist nontranslation invariant states; van Beijeren [54] proved that $\beta_r(\nu) \geqq \beta_c^{(1)}(\nu-1)$. A basic question is that a "roughening transition occurs", i.e. $\beta_r(3) < \beta_c^{(1)}(3)$. There is reason to believe (see Fröhlich et al. [55]) that $\beta_r(\nu) = \beta_c^{(1)}(\nu)$ if $\nu \geqq 4$. Prove or disprove this.

(v) Do plane rotors have nontranslation invariant states? If they do, the states will be quite different from those in the Ising case. See [55, 41] for further discussion.

(vi) Find additional methods for proving phase transitions occur when there is continuous symmetry ("spontaneously broken continuous symmetry"). At this point, all we have are reflection positivity methods (Fröhlich, Spencer, Simon [35]; Fröhlich et al. [56]) which are quite rigid in terms of when they apply and the "scales of contours" method of Fröhlich-Spencer [39] which seems to be restricted to $D=2$.

6 Ferromagnetism

Mathematicians who have been exposed to the Ising model often delude themselves that in understanding that, they have understood the reason for magnetism. While it is true that the lesson of that model, namely that local interactions can cooperatively produce long range order, is an important aspect of ferromagnetism, it is not the only one nor the most mysterious one.

The point is that the Ising model postulates an interaction which tends to make neighboring spins point parallel. These spins which are associated with the magnetic moments of neighboring atoms produce bulk magnets by aligning in parallel. It is true that magnetic dipoles have direct interactions with each other, but the magnitude of such interactions is so small that they would set temperature scales much lower than those associated with real magnets (and they don't have the proper $\sigma_1 \cdot \sigma_2$ form to boot!).

The mysterious aspect of magnetism is what produces the strong effective spin aligning interaction. There is a standard explanation due to Heisenberg based on the Pauli principle: Since electron-electron interactions are repulsive, their spatial wave function wants to be as antisymmetric as possible (tending to keep them apart), so by the Pauli principle, their spin wave function is as symmetric as possible, which produces a tendency for parallel spins.

While this picture is quite possibly the correct one, it is far from proven: Indeed, in one space dimension, it is false! Lieb and Mattis [57] have shown that the total electron spin of the ground state of an even number of electrons in one dimension is zero!

Problem 6 (Explanation of Ferromagnetism). Verify the Heisenberg picture of the origin of ferromagnetism (or an alternative) in a realistic quantum system or in a suitable model.

7 Continuum Phase Transitions

Phase transitions are one of the more striking phenomena in nature. While there has been considerable rigorous understanding in the case of lattice systems, there has been virtually none on continuum models—a phase transition has been proven in only one rather artificial model [58].

To state the problem precisely, we quickly review some basic statistical mechanics. Because it uses more familiar quantities, we work in the canonical ensemble; technically, the grand canonical ensemble is often easier to deal with (see Ruelle [22]). We fix a pair potential, v obeying

(1) (stability) For some C and all $x_1, \ldots, x_N \in R^3$:

$$\sum_{1 \leq i < j \leq N} v(x_i - x_j) \geqq -CN$$

(2) (temperedness) $|v(x)| \leqq C(1+|x|)^{-3-\varepsilon}$

Given a finite volume, Λ, in R^3, a number N and an inverse temperature β, we define the partition function, Z_Λ and free energy F_Λ by

$$F_\Lambda(\beta, N) = -\ln Z_\Lambda$$

$$Z_\Lambda = \int_{x_i \in \Lambda} \prod_{i=1}^{N} d^3 p_i d^3 x_i \, e^{-\beta H(p_i, x_i)}$$

$$H(p_i, x_i) = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i<j} v(x_i - x_j).$$

One can show that if ϱ, a value of density, is fixed and if Λ approaches R^3 in a suitable way and $N/\Lambda \to \varrho$, then $|\Lambda|^{-1} F_\Lambda(\beta, N)$ has a limit, called $f(\beta, \varrho)$. This function is concave in β, so the one-sided derivatives exist at all points. A (first order in β) phase transition corresponds to f failing to be C^1:

Problem 7 (Existence of Continuum Phase Transitions). Show that for suitable choices of v, and for ϱ sufficiently large, f is non-C^1 at some β.

A reasonable v to think about is a function like the Lenard-Jones potential $v(r) = ar^{-12} - br^{-6}$ which gets very large and positive for r small but has a small negative well in which particles can stick.

Alternatively, instead of looking for a phase transition in β, one can pass to grand canonical ensemble and look for a transition in fugacity where the density jumps.

8 Rigorous Renormalization Group

One of the most celebrated developments in theoretical physics during the past 15 years is surely the "renormalization group theory of critical phenomena" of Fisher, Kadanoff and Wilson (see e. g. [59]). The basic idea of shifting scales as one approaches a critical point via a nonlinear map of Hamiltonians and obtaining information from the fixed points of that map is being applied in a variety of situations, e. g. the work of Feigenbaum [24] and parts of the philosophy are often present in work which doesn't embrace the full machine, e. g. the spirit of the renormalization group hovers over the recent work of Fröhlich-Spencer [39].

In some of these analog studies, the nonlinear maps are on well defined spaces and there has been considerable progress on a rigorous mathematical analysis, e. g. the work of Collet, Eckmann and Landford [60] on the Feigenbaum theory. The original Wilson theory is on functions of inifinitely many variables and it is far from clear how to formulate the maps in a mathematically precise way (let alone then analyze their fixed point structure); indeed, there are various no-go theorems [61] to certain obvious ways one might try to make a precise formulation. To make the following problem precise, we specialize to lattice systems:

Problem 8A (Formulation of the Renormalization Group). Develop a mathematically precise version of the renormalization transformations for ν-dimensional Ising-type systems.

It may turn out that this problem can be finessed and one can get out renormalization group type results without a complete formalism. In this regard, see the work of Gawedski-Kupiainen [62].

It is often claimed that the renormalization group "explains" universality. It seems to me that it does not; rather, it assumes universality! For the kind of local analysis done in the renormalization group framework doesn't explain why the fixed points found seem to have "basins of attraction" which are all (or at least most) of the space of interactions. Thus:

Problem 8B (Proof of Universality). Show that the critical exponents in the three dimensional Ising models with nearest neighbor coupling but different bond strengths in the three directions are independent of the ratios of these bond strengths.

9 Asymptotic Completeness for Atomic Scattering

We begin with a brief description of multiparticle systems. See [63], Section XI.5, for more details. Consider n quantum mechanical particles of masses $m_1, \ldots, m_n$ moving in ν-dimensions. After removing the center of mass motion, the wave functions live naturally on $\{(x_1, \ldots, x_n) \in R^{\nu n} | \sum m_i x_i = 0\} = X$ (isomorphic to $R^{\nu(n-1)}$). Place the metric $d(x, y) = [\sum m_i (x_i - y_i)^2]^{\frac{1}{2}}$ on X and let H_0 be $(-\frac{1}{2})$ times the Laplace Beltrami operator in this metric. Here is an equivalent definition: Given any $(x_1, \ldots, x_n) \in R^{\nu n}$, let $R(x) = (\sum m_i)^{-1} \sum m_i x_i$ and let $\pi(x) = x - (R, R, \ldots, R)$. Given f a function on X, let $\pi^*(f)$ be the function on $R^{\nu n}$, given by $\pi^*(f)(x) = f(\pi(x))$. Then H_0 can be defined by

$$[-\sum (2m_i)^{-1} \Delta_{x_i}] \, \pi^*(f) = \pi^*(H_0 f)$$

H_0 is the kinetic energy of these particles.

Pick functions V_{ij} on R^ν. For the time being, let us suppose that

$$|V_{ij}(x)| \leq C(1 + |x|)^{-1-\varepsilon} \tag{9.1}$$

for some $\varepsilon > 0$. We write V_{ij} for the function on X given by $V_{ij}(x_i - x_j)$. On X we define

$$H = H_0 + \sum_{i,j} V_{ij}.$$

Let a be a partition of $\{1, \ldots, n\}$, i.e. a family of $\#(a)$ disjoint subsets whose union is all of $\{1, \ldots, n\}$. If i and j are in the same subset of a, we write $(ij) \subset a$. If these are in destinct subsets, we write $(ij) \not\subset a$. Elements of a are called clusters: Define

$$H(a) = H_0 + \sum_{(ij) \subset a} V_{ij}$$

$H(a)$ describes a situation of particles interacting within clusters but not between cluster. Given a, we can pick coordinates for X in two classes, x^a and x_a. The x^a describe difference of centers of mass of different clusters and the x_a coordinate differences within a cluster. Corresponding to such a decomposition, $L^2(X) = \mathscr{H}_a \otimes \mathscr{H}^a$ where $\mathscr{H}_a$ is functions of the x_a. $H(a)$ then decomposes to

$$H(a) = H_a \otimes I + I \otimes T^a$$

T^a is independent of V and describes the kinetic energy of relative motion of the clusters. H_a describes internal motion of the clusters. Let P_a denote the projection in $\mathscr{H}_a$ onto the point spectrum of H_a. Ran P_a is the sum of products of bound states for each cluster. Let $P(a) = P_a \otimes I$. Thus $P(a)$ describes the projection in $L^2(X)$ onto functions which are sums of products of bound states in the clusters and free motion of their centers of mass.

Theorem 9.1 Suppose $\nu \geq 3$ and that (9.1) holds. Then

$$s - \lim_{t \to \mp\infty} e^{itH} e^{-itH(a)} P(a) = \Omega_a^{\pm} \tag{9.2}$$

exist. Moreover, if $a \neq b$, then Ran $\Omega_a^+ \perp$ Ran Ω_b^+. $\varphi \in$ Ran Ω_a^+ if and only if there exists η so that

$$\|e^{-itH} \varphi - e^{-itH(a)} P(a) \eta\| \to 0 \text{ as } t \to -\infty \tag{9.3}$$

Remarks 1. See [63; XI.5] for a proof. The result is claimed there for $\nu = 1,2$ also but the proof is in error; in $\nu = 1,2$ one requires some information on decay of the eigenfunctions of the H_a.

2. (9.3) says that as $t \to -\infty$, the interacting state $e^{-itH} \varphi$ looks asymptotically like bound clusters moving relatively freely.

3. We have included the case a_1 where a has one cluster for which $\Omega_{a_1}^{\pm} = P(a_1) =$ projection onto the bound clusters of H.

Problem 9A—1st Form (Asymptotic Completeness for Short Range N-body Systems).

Under the hypotheses $\nu \geqq 3$, (9.1) proves that

$$\bigoplus_a \text{Ran } \Omega_a^+ = L^2(X)$$

As already remarked, Ran $\Omega_{a_1}^+ = \mathscr{H}_{p.p.}$, the point spectral subspace for H, and it can be shown that if $a \neq a_1$, Ran $\Omega_a^+ \subset \mathscr{H}_{a.c.}$ the absolutely continuous space for H. Thus problem 9A is often stated as

Problem 9A — 2nd Form

(i) Prove $\mathscr{H}_{s.c.}$, the singular continuous space is empty.

(ii) Prove $\bigoplus_{a \neq a_1}$ Ran $\Omega_a^+ = \mathscr{H}_{a.c.}$

The limits in Thm. 9.1 fail to exist in the case where $V_{ij}(x) \sim |x|^{-1}$ at ∞. There is a modification of the wave operators (9.2) due to Dollard, for which the limits $\Omega_a^{D,\pm}$ exist. These are described, e.g. in [63], Section XI.9.

Problem 9B (Asymptotic Completeness for Coulomb Potentials). Under the hypotheses,

$\nu = 3$, $V_{ij}(x) = e_{ij}|x|^{-1}$, prove

$$\bigoplus_a \text{Ran } \Omega_a^{D,+} = L^2(X).$$

Of course, one wants to allow sums of Coulomb and short range potentials.

For $n=2$, these problems were solved over 20 years ago at least if $|V(x)| = 0(|x|^{-\nu-\varepsilon})$ with the sharpest results due to Agmon-Kuroda and Enss (see [64] and [65]). For $n=3$ and $|V(x)| = 0(|x|^{-2-\varepsilon})$ and an extra assumption (no resonances in two body subsystems), Faddeev [66] solved the problem; see Ginibre-Moulin [67], Thomas [68], Howland [69], Kato [70], Yajima [71], Sigal [72] and Hagedorn-Perry [73] for additional information. The Coulomb 3-body problem was solved by Mercuriev [74] and by Enss [75]. Enss also treated the general $(0(|x|^{-1-\varepsilon}))$ 3-body problem. Mourre [76] has announced general 3-body results also. For a suitable class of analytic potentials and a suitable sense of genericity, Hagedorn [77] (for $n=3,4$) and Sigal [78] for general n have solved the problem, but genericity plays a central role.

The "half" of asymptotic completeness that requires $\mathscr{H}_{s.c.}$ is empty was solved by Balslev-Combes [79] for Coulomb potentials and for a wide class of short range V's by Perry et al. [80] using ideas of Mourre [81]. The basic open question concerns completeness without requiring genericity or analyticity.

This important open question has been studied to some extent since 1960, and very actively for the last ten years. At the risk of jinxing the solution, it seems to me like a good bet that it will be solved in the next five years and probably sooner: Both Enss' and Mourre's three-body methods appear promising for N-bodies and Agmon's long range two-body work [82] may be useful in the N-body problem.

10 Quantum Potential Theory

Basic to atomic and molecular physics is the binding energy of a quantum mechanical system of electrons interacting with one or more nuclei. To be explicit, fix N and consider two classes of operators on $L^2(\mathbb{R}^{3N})$, with $x \in \mathbb{R}^{3N}$ written as $x = (x_1, \ldots, x_N)$. For any fixed Z define

$$H_N(Z) = \sum_{i=1}^{n} \left(-\Delta_i - \frac{Z}{|x_i|}\right) + \sum_{1 \leq i < j \leq N} 1/|x_i - x_j| \tag{10.1}$$

and for Z_0, k and $R_1, \ldots, R_k \in R^3$ we define

$$H_N^{(k)}(R_1, \ldots, R_k; Z_0) = \sum_{i=1}^{N} -\Delta_i + \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|} + \sum_{1 \leq \alpha, \beta \leq k} \frac{Z_0^2}{|R_\alpha - R_\beta|} - \sum_{\substack{1 \leq \alpha \leq k \\ 1 \leq i \leq N}} \frac{Z_0}{|x_i - R_\alpha|} \tag{10.2}$$

Note that the 3rd term in (10.2) is a constant depending only on the parameters R_i and not an operator on $L^2(\mathbb{R}^{3N})$. (10.1) is the Hamiltonian of an atom in the approximation of infinite nuclear mass and (10.2) that of a molecule in Born-Oppenheimer approximation. We define

$$E_B(N; Z) = \inf \text{spec}(H_N(Z))$$
$$E_B^{(k)}(N; R_1, \ldots, R_k; Z_0) = \inf \text{spec}(H_N^{(k)}(R_1, \ldots, R_k; Z_0)).$$

The B stands for "Boson" since the operators are taken on $L^2(\mathbb{R}^{3N})$ and ignore the Pauli principle. For fermion electrons one should restrict $H_N(Z)$ (and $H_N^{(k)} \ldots$) to $\mathscr{H}_{\text{phys}}$ the subset of $L^2(\mathbb{R}^{3N})$ of all functions $f(x_1, \ldots, x_N)$ antisymmetric under interchange of the coordinates (actually, because of the fact that electrons have two spin states, we should take f to be a sum of functions transforming under permutations as representations with at most two columns in their Young tableaux). The inf of the spectrum of restricted operators we will call E without any subscript. These are the physically relevant objects so we do not give them a subscript F.

The total binding energies are basic physical objects. While several significant properties are known (see especially Thms. 10.1, 2, 3 below), it is shocking how little we know about $E(N; Z)$ and $E^{(k)}(N; R_1, \ldots, R_k; z_0)$. This is shown by the first open problem. Define

$$(\Delta E)(N, Z) = E(N-1, Z) - E(N, Z)$$

the energy it takes to remove electron N. It is a consequence of the HVZ theorem ([83], Section XIII.5) that $(\Delta E)(N, Z) \geqq 0$.

Problem 10A (Monotonicity of the Ionization Energy). Prove that

$$(\Delta E)(N-1, Z) \geqq (\Delta E)(N, Z)$$

for all N, Z.

This is just the fact, almost obvious, that it takes more energy to remove inner electrons than outer ones. Since in removing electron $(N-1)$ there is one fewer electron to repel, and since the Pauli principle only makes things better this should be true. It seems to be remarkably difficult to prove. Indeed, it is false if one requires it for nuclei with all possible finite masses (rather than our infinite mass assumption) and one allows for electron spin [84]. The inequality to be proved says that $E(N, Z)$ for Z fixed is convex in N.

A weaker result that would be of interest would be to prove: "If $\Delta E(N, Z) = 0$, then $\Delta E(N+1, Z) = 0$". This result would be relevant in connection with the Ruskai-Sigal theorem (Thm. 10.2 below).

To state the next open problem, we need to recall

Theorem 10.1 (Lieb-Simon [85]) $\lim_{Z \to \infty} E(Z, Z)/Z^{7/3}$ exists.

In fact, the limit is given by a "Thomas-Fermi" energy, e_{TF}. See Lieb [86] for further information and insight.

Problem 10B (The Scott Correction). Prove that $\lim_{Z \to \infty} (E(Z, Z) - e_{TF} Z^{7/3})/Z^2$ exists and is the constant found by Scott [87].

If one drops the electron-electron repulsion, one can find the new $E(Z, Z)$ exactly and compute the $0(Z^2)$ term exactly and see it corresponds to the fact that Thomas-Fermi fails to get the inner electrons correctly. Since the electron repulsion shouldn't matter for the inner electrons, Scott [87] conjectured that the Z^2 term is the same as for the non-interacting case. Recent physicists' arguments which seem difficult to make rigorous for the Scott correction can be found in Bander [88] and Schwinger [89]. We remark that the obvious asymptotic series one might conjecture on the basis of the last theorem and problem, namely

$$E(Z, Z) \sim a_1 Z^{7/3} + a_2 Z^2 + a_3 Z^{5/3} + a_4 Z^{4/3} + \cdots$$

is almost surely *not* correct: There may be a $Z^{5/3}$ term but after that there are almost surely oscillations at the $Z^{4/3}$ level.

Physically, the quantity $E(Z, Z)$ as a total binding energy is not so interesting. The ionization energy $(\Delta E)(Z, Z)$ is much more interesting.

Problem 10C (Asymptotics of Ionization Energy). Find the leading asymptotics of $(\Delta E)(Z, Z)$ for large Z.

Lieb-Simon [85] suggest that $(\Delta E)(Z, Z)$ goes to a constant, but even on an intuitive level, that is not clear. Indeed, it isn't clear to me whether the leading power ($\alpha = \lim_{Z \to \infty} \ln \Delta E(Z, Z)/\ln Z$) is 0, positive or negative!

For the next problem, we need to recall

Theorem 10.2 (Ruskai [90], Sigal [91]). For every Z, there is an N_0 so that $(\Delta E)(N, Z) = 0$ if $N \geqq N_0$.

This says that one cannot bind arbitrarily many electrons to a nucleus. Let $N(Z)$ be the smallest N_0 for which the above is true. Zhislin [92] (see also Simon [93]), showed that $(\Delta E)(N, Z) > 0$ if $N \leqq Z$, so $N(Z) \geqq Z$ and thus

$$\varliminf_{Z \to \infty} N(Z)/Z \geqq 1.$$

Moreover, Sigal [91] has proven that

$$\varlimsup_{Z \to \infty} N(Z)/Z \leqq 2 \qquad (10.3)$$

It is quite reasonable to think this 2 can be replaced by 1.

Problem 10D (Asymptotics of Maximal Ionized Charge). Prove that $\lim_{Z \to \infty} N(Z)/Z = 1$.*)

Sigal's argument (for 10.3) in [91] uses the Pauli principle. In fact, if Problem 10D has a positive solution, it must use the Pauli principle, since Benguria-Lieb [94] have proven that if $N_B(Z)$ is defined using E_B in place of E, then $\lim_{Z \to \infty} N_B(Z)/Z > 1$.

For the last formal problem, we need to recall what is the most significant result known about Coulomb energies, "the stability of matter".

Theorem 10.3 (Dyson-Lenard [95], Lieb-Thirring [96]). For a universal constant:

$$E^{(k)}(N, R_1, \ldots, R_k; Z_0) \geqq -C(1 + Z_0^{7/3})[N + k].$$

*) See Note added in proof.

For Z_0 bounded this was first proven by Dyson-Lenard [95]; Lieb-Thirring [96] not only simplified the proof considerable, but found a value of C within the best possible value by an order and half in magnitude (Dyson-Lenard [95] had a constant off by many orders). This result is important since it implies that bulk matter doen't contract as more particles are added (see e.g. Lieb [86]); it is the starting point of a proof of the existence of good thermodynamics for Coulomb systems (Lebowitz-Lieb [97]).

This result depends critically on the Fermi nature of the electrons. Indeed, define

$$E_B(k, N; z_0) \equiv \inf_{R_i} E_B^{(k)}(N, R_1, \ldots, R_k; z_0).$$

Then Lieb [98] has proven that

$$-DN^{5/3} \leqq E_B(N, N; 1) \leqq -CN^{5/3}.$$

Let $\tilde{E}_B(k, N, Z_0)$ be the analogous object where now the "protons" are given a finite mass and so both "electrons" and "protons" (viewed as bosons) are treated quantum mechanically. Then, Dyson [99] has proven that for a suitable $C>0$:

$$\tilde{E}_B(N, N; 1) \leqq -CN^{7/5}$$

The best lower bound known is

$$\tilde{E}_B(N, N; 1) \geqq -DN^{5/3}.$$

Problem 10E (Rate of Collapse of Bose Matter). Find suitable C_1, C_2, and α so that

$$-C_1 N^\alpha \leqq \tilde{E}_B(N, N; 1) \leqq -C_2 N^\alpha.$$

One suspects that $\alpha = 7/5$. Since electrons in nature are not bosons, one might think that this problem is of purely mathematical interest. In fact, since Dyson's trial function is of BCS type, a real understanding of this problem could improve our understanding of superconductivity.

The reader can consult Lieb's Lausanne lecture [100] for a list of other interesting open Coulomb problems. In connection with the Lieb-Thirring proof of Thm. 10.3, we should mention the open question of finding the best constant in the Cwickel-Lieb-Rosenbljum bound: See Simon [101], pgs. 96—97 and Glaser-Martin [102] for further discussion of this and related problems.

11 Existence of Crystals

It is an observed fact of nature that most materials occur in a crystalline state at low temperatures. Yet there is no proof or even a very convincing argument to show that even at zero temperatures ensembles of quantum mechanical atoms want to form crystals. Clearly to avoid boundary effects, one must take an infinite system; any finite system will not be a strictly crystalline form. Moreover, as a first problem, one should imagine infinite nuclear masses.

Thus, we should fix an integer, z_0 (a nuclear charge), and take $N = kz_0$ and consider the function $E^{(k)}(N; \vec{R}_1, \ldots, \vec{R}_k; z_0)$. We denote by $\{\vec{R}_i^{(k)}\}_{i=1}^{(k)}$ a minimizing configuration for this function (it is not automatic, indeed, not proven that such a minimum exists, i.e. that the minimum isn't taken for some $|\vec{R}_i - \vec{R}_j| = \infty$; presumably, for suitable z_0, such a minimum does exist).

Of course, the minimizing configurations is not unique; it is invariant under a common Euclidean motion of the nuclei or under permutation of indices and there could be additional non-uniqueness. For this reason, we are careful to deal with "a choice" below.

Here is one possible statement which would show at zero temperature atoms with atomic number z_0 form a crystal. There is a choice of minimizing configurations $R^{(k)}$, so that (i) $R_j^{(k)}$ converges to some $R_j^{(\infty)}$ as $k \to \infty$ for each fixed j. (ii) For any R_0, there is a J so that $|R_j^{(k)}| \geqq R_0$ if $j > J$. (iii) The $R_j^{(\infty)}$ lie in a lattice, i.e. a subset of R^3 left invariant by a subgroup of translations isomorphic to Z^3. Condition (ii) is included to prevent one nucleus from "getting lost" in the limit due to mislabeling.

Problem 11 (Existence of Crystals). Prove the above statement or another suitable version of the existence of crystals for some z_0.

We note that the classical analog of this result is unknown. There is, however, an interesting series of papers on this classical question by Radin [103] and a paper of Duneau-Katz [104].

Of course, if one solves Problem 11, the next thing is to worry about finite but low temperature, then melting, then...

12 Random and Almost Periodic Potentials

In this section we want to discuss $-\Delta + V$ on $L^2(R^\nu)$ and its discrete analog

$$((hu)(n) = \sum_{|\delta|=1} u(n+\delta) + V(n)\,u(n))$$

on $l^2(Z^\nu)$ where V is either a stochastic process with strong mixing properties ("random potentials") or an almost periodic function. This is an area of considerable current interest to me, and so I may be accused of lacking perspective in including the five problems listed here. It seems to me that the first problem below is very significant from any viewpoint; perhaps (but I think not) the second, third and fourth are too specialized; the fifth is included so the reader can help me win a bet.

To be precise about random potentials, one can consider a particular model known as the Anderson model. Choose the $V(n)$ to be independent, identically distributed random variable with distribution uniformly in $(-\lambda, \lambda)$ (λ is a number known as the coupling constant). The following results are proven: In any dimension, ν, the spectrum $\sigma(h)$ is almost surely $[-2\nu-\lambda, 2\nu+\lambda]$ (e.g. Kunz-Souillard [105]) and if $\nu=1$, almost surely h has only dense pure point spectrum ("localized states") (see [105] and Delyon et al. [106]; also Goldshade et al. [107] for the case of $-d^2/dx^2+V(x)$ with suitable random V). In the physics literature the belief is that the same result holds if $\nu=2$ (although until roughly 5 years ago this was not the belief) but when $\nu \geqq 3$ it is believed that one has only dense point spectrum when $\lambda \geqq \lambda_0 > 0$, but for $\lambda < \lambda_0$ there is a region $[-a(\lambda), a(\lambda)]$ of absolutely continuous spectrum ("extended states") with dense point spectrum in $\pm[a(\lambda), 2\nu+\lambda]$. Fröhlich and Spencer [108] have recently obtained some results in the region where there is supposed to be localized states and they will probably succeed in proving dense point spectrum soon when either λ is large or $|e|$ is near $2\nu+\lambda$. This leaves the region of extended states.

Problem 12A (Existence of Extended States in the Anderson Model). Prove that in $\nu \geqq 3$, for λ small, there is a region with absolutely continuous spectrum, and determine whether this is false when $\nu=2$.

We mention here the interesting results of Kunz-Souillard on extended states in the Anderson model on a Bethe lattice (which in some sense has $\nu=\infty$) of which so far only an announcement exists [109].

At first sight, one might think that since the $\lambda=0$ operator has a.c. spectrum, extended states shouldn't be so hard since one just has to find a simple perturbation argument. That this is not the case is shown by the expectation that when $\lambda \neq 0$ the a.c. spectrum should be associated with diffusive motion; explicitly, when $\lambda=0$, $(\delta_0, (e^{itH_0}\vec{N}e^{-itH_0})^2\,\delta_0) \sim 0(t^2)$ where δ_0 is the element of $l^2(Z^\nu)$ which is 1 at $\vec{0}$ and 0 elsewhere and $(\vec{N}u)(\vec{n}) = \vec{n}\,u(\vec{n})$ while we expect that:

Problem 12B (Diffusive Bound on "Transport" in Random Potentials). For the Anderson model (and more general random potentials) prove that

$$\operatorname{Exp}(\delta_0, (e^{itH}\vec{N}e^{-itH})^2\,\delta_0) \leqq c(1+|t|)$$

This result is clearly connected with our discussion in Section 4. We note that it is easy to prove the analogous bound if ct is ct^2; indeed, that is true for any bounded V (see e.g. Radin-Simon [110]) and that when there are extended states, it is believed that expectations grow as $D_\pm t$ for t large.

There is one last aspect of the Anderson model we want to mention. A basic object is the integrated density of states (e.g. [111]), $k(E)$. In the Anderson model, there are two places we might worry about lack of smoothness in k; at the edges of the spectrum where k is certainly non-analytic and at the mobility edge (indicated as $\pm a(\lambda)$ above). At the edges, a result known as Lifschitz tails (see e.g. [112]) suggests k is C^∞ so the following problem is really about the mobility edge:

Problem 12C (Smoothness of k(E) Through the Mobility Edge in the Anderson Model).
Is $k(E)$ a C^∞ function of E in the Anderson model at all couplings?

Of course, there are a myriad of other questions about the mobility edge with more physics (e.g. behavior of the diffusion constant); we list the above as the simplest one.

Our last pair of problems involves the case of almost periodic potentials. The simplest example in many ways is the almost Mathieu equation on $l^2(Z)$

$$(hu)(n) = u(n+1) + u(n-1) + \lambda\cos(2\pi\alpha\eta+\theta)\,u(n) \tag{12.1}$$

where λ, θ and α are parameters. It is an idea of Sarnak [113] that the spectral properties should depend on Diophantine properties of α: if

$$\left|\alpha - \frac{p}{q}\right| \geqq Cq^{-k}$$

for some C and k, we call α a Roth number and if there is an infinite sequence q_k with

$$\left|\alpha - \frac{p_k}{q_k}\right| \leqq \exp(-kq_k),$$

we call α a Liouville number. (The Roth numbers have full Lebesgue measure while the Liouville numbers are a dense G_δ!) Here is the belief about the spectrum of (12.1) (see e.g. [114]):

(a) If α is a Liouville number and $\lambda \neq 0$, then for a.e. θ, the spectrum is purely singular continuous.

(b) If α is a Roth number and $|\lambda| < 2$, the spectrum is purely absolutely continuous for a.e. θ.

(c) If α is a Roth number and $|\lambda| > 2$, the spectrum is purely dense pure point.

(d) If α is a Roth number and $|\lambda| = 2$, $\sigma(h)$ has Lebesgue measure zero and the spectrum is purely singular continuous.

All that has been proven about this model is: (i) (a) is true if $|\lambda| > 2$ [111] (ii) In case (a), there is at least no point spectrum [115, 111] (iii) When α is Roth there is at least some a.c. spectrum when $|\lambda|$ is very small and some point spectrum when $|\lambda|$ is very large [116] (iv) If $\lambda > 2$, there is at least no a.c. spectrum [111].

Problem 12D (Analysis of the Almost Mathieu Equation). Verify the picture (a)—(d) above.

Our final problem is the only one involving the continuum case $-\Delta+V$. As noted above, it is known that for λ large and α suitable, (12.1) has some point spectrum. Here is a continuous analog of that:

Problem 12E (Point Spectrum in a Continuum Almost Periodic Model). Show that for α, λ, μ suitable

$$-\frac{d^2}{dx^2}+\lambda\cos(2\pi x)+\mu\cos(2\pi\alpha x+\theta) \tag{12.2}$$

has some point spectrum for a.e. θ.

I pick this problem among all possible continuum problems because two excellent mathematicians have bet me that (12.2) has no point spectrum. I don't give their names to spare them public embarrassment (not caused by their choosing to disagree with me, but by the fact that, in this case, they are wrong!).

13 Self-Avoiding Random Walks

We want to first describe a mathematical problem which is easy to describe, and then we will briefly explain why it is included in a list of problems in mathematical physics. Consider the lattice Z^d of integral points in d-dimensions. (We abandon our usual v here because in this subject v is usually used for the object in (13.1)). A self-avoiding walk (SAW) of length n is a sequence of $n+1$ *distinct* points $R(0),\ldots,R(n)\in Z^d$ so that $R(0)=0$ and $|R(i+1)-R(i)|=1$. This differs from ordinary random walks in the requirement that the R's be distinct (hence self-avoiding). Let $k(n)$ denote the number of SAW of length n, labeled

$$\{R_i^{(n)}(j)\,|\,j=0,\ldots,n;\ i=1,\ldots,k(n)\}$$

and we define the mean displacement by:

$$D(n)=\langle R^{(n)}(n)^2\rangle^{\frac{1}{2}}\equiv[k(n)^{-1}\sum_i R_i^{(n)}(n)^2]^{\frac{1}{2}}.$$

One expects that (more or less) $D(n)\sim Cn^\nu$ as $n\sim\infty$. Essentially nothing is known about ν which we might define by

$$\nu=\lim_{n\to\infty} n^{-1}\ln D(n) \tag{13.1}$$

(it is *not* known that the limit exists). Intuitively, the self-avoiding property should force the path to grow faster than in ordinary random walks where $\nu=\frac{1}{2}$

so one certainly expects that

$$\nu\geqq\tfrac{1}{2} \tag{13.2}$$

but even this is unknown. Indeed, a few moment reflection on "trapping in cul de sacs" will indicate the problems. A proof of (13.2) would be *very* interesting and probably represent real progress.

This subject is reviewed in [117, 118].

Computer calculations suggest that if $d=2$, $\nu=3/4$, if $d=3$, $\nu\cong.59$(some prefer $\nu=3/5$ and don't believe .59), and if $d\geqq4$, $\nu=1/2$. That ν seems to be 1/2 if $d\geqq4$ is believed connected with the fact that Brownian motion is non-selfintersecting if $d\geqq5$ and has only "logarithmic" selfintersections if $d=4$ (see e.g. [101] and reference therein).

Problem 13 (Critical Exponents for self-Avoiding Walks). Prove that $\nu=1/2$ for $d\geqq4$ and $\nu>1/2$ for $d\leqq3$.

So much for the simple statement of this problem. Why is this problem here? There are many reasons:

(1) ν is in many ways the most elementary example of a critical exponent and problem 13 is an expression of the fact that in high dimension these exponents are supposed to agree with mean field theory, which in this case is the Gaussian value $\nu=1/2$. Critical exponents are important in the theory of phase transitions so this section is related to Section 8; indeed, there is an analog of universality; ν is supposed to be dependent only on dimension of the lattice and not on its exact form (e.g. in 3-dimensions, the SAW on the cubic lattice and the face centered lattice are believed to be the same).

(2) The SAW model is related to elementary models of polymers; indeed, much of the work on SAW has been done by polymer people and both review articles mentioned above appear in Advances in Chemical Physics.

(3) There is supposed to be connection between SAW and the Ising model. Actually, it seems to me that this is at a deep level only through Fisher's bounds [119] and, in particular, the real relevance to the Ising model is only the analogy.

(4) Symanzik [120] had a vision of φ^4 field theories which have been a fertile source of intuition and which relates φ^4 field theories to SAW. Indeed, Brydges et al. [121] have made use of a random walk expansion of φ^4 theories which relates them to random walks in which self-intersections are not forbidden but are suppressed relative to SAW's. If one writes out the formalism for n-component φ^4 and then formally sets n to 0, SAW's result! (This is a remark of de Gennes [122]). Progress on understanding SAW could help us understand quantum field theory.

See Westwater [123] for additional information on this and related problems.

14 Quantum Field Theory

A list of problems like this one written 10 or even 30 years ago would surely have included the mathematically consistent construction of quantum electrodynamics (QED). We will see what happened to that problem below, but it is clear that quantum field theory remains a basic element of fundamental physics and a continual source of inspiration to mathematicians.

The most spectacular development in theoretical physics of the past 10 years has been the formulation of a generally accepted model of strong interaction physics, a model of quarks interacting through a non-abelian Yang-Mills gauge fields (whose quanta are called gluons). This model is normally called quantum chromodynamics (QCD). As with most quantum field theories, the theory is written down by physicists by giving a formal Lagrangian and there are numerous infinities only eliminated formally; that is, one is quite far from a mathematically precise set of objects.

Problem 14A (Construct QCD). Give a precise mathematical construction of quantum chromodynamics.

For a discussion of the model formally (actual class of models depending on the number of quarks and of various groups), see e.g. [124].

The past 15 years have seen the development of the first mathematically consistent quantum field theories in two and three space-time dimensions. This area, known as constructive quantum field theory, is nicely summarized in the book of Glimm and Jaffe [125]; see Seiler [126] for a discussion of mathematical aspects of Yang-Mills field theories. All the models constructed lie in a class known as "super renormalizable" since their infinities are rather mild. There is another class of formal field theories known as "renormalizable", of which QCD is the most interesting but also one of the more complex technically. It is possible to imagine someone constructing a renormalizable theory but being unable to handle QCD because of some of the difficulties intrinsic to Yang-Mills fields or to fermions. Thus, the following is interesting:

Problem 14B (Renormalizable QFT). Construct any non-trivial renormalizable but not superrenormalizable quantum field theory.

With regard to QED, for many years this was believed to be the fundamental theory of electrons and photons. The impressive agreement between experiment and QED was used as an argument that the formal theory had an underlying mathematically consistent formulation. This is no longer believed to be the case, at least among an overwhelming majority of the theoretical high energy physics community. Rather, it is believed that QED by itself is *not* consistent; rather, there is a consistent (non-abelian gauge) unified theory of weak and electromagnetic interactions, but the differences of the perturbation series of this consistent theory and QED are very small at low energies, explaining the agreement with experiment. This should be taken as a warning to those

who argue that a theory that seems to agree with nature must be mathematically consistent and it is pointless to prove such an "obvious" fact. In any event, this leads to:

Problems 14C (Inconsistency of QED). Prove that QED is not a consistent theory.

Alan Sokal has dubbed the discipline of proving certain field theories are not consistent "destructive field theory". There are some results in this new area. Fröhlich [127] (using ideas from Brydges et al. [121]) and Aizenman [128] have shown that there is no non-trivial limit of lattice cutoff φ^4 theories in space time dimension $d > 4$, if one only renormalizes with mass and coupling constant renormalization. This is *not* a verification of the phenomena responsible for the putative inconsistency of QED, where it is believed perturbation theory is misleading because of lack of infrated stability. For φ_d^4, $d > 4$, formal perturbation theory suggests that renormalization of higher degree than four will be required. Thus an analogous result for $d = 4$ where the heuristics for QED are also valid, would be especially interesting. Fröhlich and Aizenman have results in $d = 4$ but they suffer from various loopholes, e.g. at this point a φ_4^4 with finite field strength renormalization has not been ruled out. There is also a loophole suggested by Gallavotti-Rivasseau [129] which, while an intriguing possibility, is probably not going to save φ_4^4. This leads us to propose:

Problem 14D (Inconsistency of φ_4^4). Prove that a non-trivial φ_4^4 theory does not exist.

15 Cosmic Censorship

The reader who has tired at the length of this article may well wish that the title of this section had been applied sooner.

Classical general relativity is a discipline whose death has been prematurely claimed by too many theoretical physicists. It remains healthy and vigorous, in part because of input from astrophysics (such as the identification of probable black holes and the identification of an effect of gravitational radiation) and, in part, due to a frequent injection of fertile mathematical ideas (such as those of Hawking and Penrose and, more recently, of Schoen-Yau and Witten). By any reasonable definition of the term, it is clear that much of classical general relativity is "mathematical physics". It is unfortunate that a horizon seems to separate general relativists and other mathematical physicists. I am not alone among mathematical physicists in knowing less about the subject than I should. I have included a problem from general relativity here to express my belief in the unity of mathematical physics, but I must confess a feeling of "I hope I got it right". In any event, the reader should consult various articles of Penrose [130] for more information, and his article [131] for additional problems in general relativity.

Problem 15 (Cosmic Censorship). Formulate and then prove or disprove a suitable form of cosmic censorship.

Very roughly speaking, cosmic censorship says that for Einstein's equations coupled to matter obeying "realistic" evolution equations (such as Maxwell's equations or suitable Yang-Mills equations), "naked singularities" do not "generically" occur. It would be interesting to prove the result even for vacuum solutions of the Einstein's equations (i.e. those with no matter).

Cosmic censorship deals with the deep and thorny issue of singularities in general relativity. The first "singularity" in general relativity was the Schwarzschild singularity: If the Schwarzschild solution (the field of a static, spherically symmetric source) is continued, in the absence of matter, to a distance (which, for usual bodies, is far within the matter producing the field) called the Schwarzschild radius, there appears to be a singularity. We say "appears" because it was realized later (by Eddington, Lemaitre and Synge) that the singularity was not one of the geometry but rather of the coordinate system used: e.g., in another coordinate system found by Kruskal, one can continue past the Schwarzschild radius until a true singularity appears. We say "true singularity" because a suitable curvature scalar, a coordinate independent object, diverges there.

While the Schwarzschild "singularity" is not a singularity of the geometry, it has important geometric and physical significance: It is a horizon in that no light rays from inside it can pass out to infinity. In this way, it prevents us from "seeing" the true singularity which would presumably be the ultimate psychedelic experience.

It is not easy to get explicit solutions of the Einstein equations because of the many components and variables, and for that reason, most known solutions have very high symmetry. For a while, there was a belief that the true singularity in the Schwarzschild solution (which can arise in finite "time" from non-singular Cauchy data if matter collapses to a point) might be an artifact of the symmetry, and that most solutions very near to Cauchy data leading to a singularity might well be free of singularities. A basic discovery of Penrose and Hawking [132] was that this was not the case but that solutions near one with a Schwarzschild-solution-type (true) singularity have some type of singularity. This stability result for black holes is very significant, given the apparent occurence of black holes in the cosmos.

The Hawking-Penrose theorem says we must learn to live with singularities or else rely on some quantum effect to save us. Upon some reflection, a singularity like that in the Schwarzschild solution is not so difficult to live with because it doesn't live next door, i.e. we don't see it. A "naked singularity" is, roughly speaking, one with the property that light rays form points arbitrarily near it can escape to infinity. These are much more disturbing from a physical point of view. One cannot conjecture that naked singularities never occur, since

one does in a solution called the Taub-NUT solution, which Misner [133] has dubbed a "counterexample to almost anything". However, this solution has a high symmetry and one can conjecture that naked singularities tend to become clothed by horizons under most small perturbations. This is the content of Cosmic Censorship. Given the history of the Hawking-Penrose theorem, one might well suspect that the idea that naked singularities are associated with symmetries is wrong; however, recent results of Isenberg-Moncreif [134] tend to support the notion that naked singularities imply symmetry.

There are other examples of naked singularities among the Weyl axisymmetric class. To eliminate such examples, it may be necessary to make some kind of hypothesis of initial conditions which are non-singular and "realistic".

Note added in proof

Problem 10D has been solved by E. Lieb, I. Sigal, B. Simon and W. Thirring (in prep.), who prove that $\lim_{Z\to\infty} N(Z)/Z = 1$ but no effective control on the rate of convergence is obtained. Can one prove that $N(Z) - Z$ is bounded?

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