4.1. BOUNDEDNESS OF CONTINUUM EIGENFUNCTIONS AND THEIR RELATION TO SPECTRAL PROBLEMS

We will describe a set of problems for matrices acting on $\ell^2(\mathbb{Z})$. There are analogous problems for $\ell^2(\mathbb{Z}^\nu)$ and for suitable elliptic operators on $L^2(\mathbb{R}^\nu)$. Let A be a bounded self-adjoint operator on $\ell^2(\mathbb{Z})$ whose matrix elements obey $a_{ij} \equiv (\delta_i, A\delta_j) = 0$ if $|i-j| \geq K$. A fundamental result asserts the existence of a measure $d\rho(E)$, a function $n(E)$ taking the values $0,1,\ldots,\infty$ (infinity allowed) with $n(E) \geq 1$ $(d\rho)$-a.e. E and $n(E)=0$ if $E \notin \operatorname{supp} \rho$ and for each E, $n(E)$ linearly independent sequences $u_\alpha(E;n)$; $\alpha = 1,\ldots,n(E)$ (not necessarily in ℓ^2) so that (a) $|u_\alpha(E;n)| \leq C(1+|n|)$ (b) $\sum a_{ij} u_\alpha(E;j) = E\, u_\alpha(E;i)$; (c) Let $\mathcal{H}' = L^2(\mathbb{R}; \mathbb{C}^{n(E)}; d\rho)$, i.e. functions, f, on $\mathbb{R}$ with $f(E)$ having values in $\mathbb{C}^{n(E)}$ (where $\mathbb{C}^\infty = \ell^2$) and let C_0 denote sequences in $\ell^2(\mathbb{Z})$ of compact support. Define U taking C_0 into $\mathcal{H}'$ by $(Ug)_\alpha(E) = \sum_m \overline{u_\alpha(E;m)}\, g(m)$. Then U extends to a unitary map of $\ell^2(\mathbb{Z})$ onto $\mathcal{H}'$; (d) $U(Ag) = E(Ug)$.

These continuum eigenfunction expansions are called BGK expansions in [1] in honor of the work of Berezanskii, Browder, Gårding, Gel'fand and Kac, who developed them in the context of elliptic operators. See [1,2,3] for proofs. These expansions don't really contain much more information than the spectral theorem. The most significant additional information concerns the boundedness properties of u ; see [4,5] for applications.

Actually, the general proofs show that $(1+|n|)$ in part (a) can be replaced by $(1+|n|)^\alpha$ for any $\alpha > \frac{1}{2}$. Indeed, one shows that for any $g \in \ell^2$, one can arrange that for $(d\rho)$-a.e. E $g(\cdot)\, u_\alpha(E,\cdot) \in \ell^2$. If one could arrange a set, S, of good E's where $gu \in \ell^2$ for all $g \in \ell^2$ with $\rho(\mathbb{R} \setminus S) = 0$, then on S, $u \in \ell^\infty$. This leaves open:

QUESTION 1. <u>Is it true that for</u> $(d\rho)$-<u>a.e.</u> E, <u>each</u> $u_\alpha(E,\cdot)$ <u>is bounded</u>?

There is a celebrated counterexample of Maslov [6] to the boundedness in the one dimensional elliptic case. As explained in [1], Maslov's analysis is wrong, and it is not clear whether his example has bounded u's a.e. We believe the answer to question 1 (and all other yes/no questions below) is affirmative, but for what we have to say below, a weaker result would suffice:

QUESTION 2. <u>Is it at least true that for</u> $(d\rho)$-<u>a.e.</u> E <u>and all</u> α : $\frac{1}{2N+1} \sum_{|n|\leq N} |u_\alpha(E,n)|^2$ <u>is bounded</u>?

QUESTION 3. <u>Is it true that</u>

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{|n|\leq N} |u_\alpha(E,n)|^2 \equiv k(\alpha,E)$$

<u>exists</u>? The $\overline{\lim}$ we will denote by $\bar{k}(\alpha,E)$.

Given a subset M, of $\{(E,\alpha) : E \in \mathbb{R}, \alpha \leq N\}$ we define

$$(P(M)g)(n) = \sum_\alpha \int_{\{E:(E,\alpha)\in M\}} u_\alpha(E,n)(Ug)_\alpha(E)d\rho(E)$$

where a suitable limit in mean may need to be taken. Define

$$M_1 = \{(E,\alpha) : u_\alpha(E,\cdot) \in \ell^2\}$$

$$M_2 = \{(E,\alpha) : k(\alpha,\bar{E}) = 0 \quad \text{but} \quad (E,\alpha) \notin M_1\}$$

$$M_3 = \{(E,\alpha) : \bar{k}(\alpha,E) \neq 0\}\ .$$

Obviously, $P(M_1)$ is the projection onto the point spectrum of A.

QUESTION 4. <u>Is it true that</u> $P(M_2)$ <u>is the projection onto the singular continuous space of</u> A <u>and</u> $P(M_3)$ <u>the projection onto the absolutely continuous spectrum of</u> A ?

Among other things this result would imply that in the Jacobi case (where the number K of the third sentence in this note is 2), the singular spectrum is simple.

In higher dimensions, one can see situations where A separates (i.e. $\ell^2(\mathbb{Z}^\nu) = \ell^2(\mathbb{Z}^{\nu_1}) \otimes \ell^2(\mathbb{Z}^{\nu_2})$ and $A = A_1 \otimes I + I \otimes A_2$) where A has a.c. spectrum with eigenfunctions decaying in ν_2 dimensions but of plane wave form in the remaining ν_1-dimensions.

One can also imagine a.c. spectrum from combining singular spectrum for A_1 and A_2. In either case $k=0$ for lots of continuum a.c. eigenfunctions.

QUESTION 5. Is there a sensible (i.e. not obviously false) version of Question 4 in the multidimensional case?

There are examples [7] of cases where A has only point spectrum but there is an eigenfunction with $k(\alpha, E)>0$ (since it occurs on a set of ρ-measure zero, it isn't a counterexample to a positive answer to Question 4). Does the second part of Question 4 have a positive converse?

QUESTION 6. Is it true that if $Au = Eu$ has a bounded eigenfunction with $\bar{k}>0$ for a set, Q, of E's of positive Lebesgue measure, then A has some a.c. spectrum on Q ?

QUESTION 7. What is the proper analog of Question 6 for singular continuous spectrum?

REFERENCES

1. S i m o n B. Schrödinger semigroups. - Bull.Amer.Math.Soc., 1982, 7, 447-526.
2. Б е р е з а н с к и й Ю.М. Разложение по собственным функциям самосопряженных операторов. Киев, Наукова думка, 1965 (Transl.Math. Mono., v.17, Amer.Math.Soc., Providence, R.I., 1968).
3. К о в а л е н к о В.Ф., С е м ё н о в Ю.А. Некоторые вопросы разложения по обобщенным собственным функциям оператора Шредингера с сильно сингулярными потенциалами. - Успехи мат.наук, 1978, 33, вып.4, 107-140 (Russian Math.Surveys,1978,33, 119-157).
4. P a s t u r L. Spectral properties of disordered systems in one-body approximation. - Comm.Math.Phys.,1980, 75, 179.
5. A v r o n J., S i m o n B. Singular continuous spectrum for a class of almost periodic Jacobi matrices. - Bull.Amer.Math.Soc., 1982, 6, 81-86.
6. М а с л о в В.П.Об асимптотике обобщенных собственных функций уравнения Шредингера. - Успехи мат.наук,1961,16, вып.4, 253-254.
7. S i m o n B., S p e n c e r T. unpublished.

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