J.A. BARROSO editor, Aspects of Mathematics and its Applications

ULTRACONTRACTIVE SEMIGROUPS AND SOME PROBLEMS IN ANALYSIS

E.B. DAVIES

Department of Mathematics, King's College, London, England

B. SIMON[1]

Division of Physics, Mathematics and Astronomy, Caltech, Pasadena, CA 91125, U.S.A.

Dedicated to Leopoldo Nachbin in recognition of his contributions to mathematics

We study L^2 to L^∞ properties of $\exp(-t\tilde{H})$, where $\tilde{H}$ is the Dirichlet form associated to a Schrödinger operator or to a Dirichlet semigroup. We use this study to obtain results about boundary behaviour of functions in suitable Sobolev spaces, and to obtain information of Brownian paths.

1. Introduction

One of the central themes of Leopoldo Nachbin's career has been the interplay of various aspects of abstract analysis with problems in concrete analysis. In this note, we want to sketch some results involving the relation of some abstract theory of L^p properties of semigroups and some concrete problems involving Schrödinger operators and Dirichlet Laplacians; complete details, refinements, etc. will appear elsewhere [8].

Here are three concrete problems we will address:

1.1. Sobolev Estimates up to the Boundary

Let Ω be a bounded open region in $\mathbb{R}^n$ and let H_Ω denote the Dirichlet Laplacian on $L^2(\Omega, \mathrm{d}x)$ which has compact resolvent. Let E_Ω be its smallest eigenvalue and ψ_Ω the corresponding eigenfunction, so ψ_Ω is determined by

$$H\psi_\Omega = E_\Omega\psi_\Omega \qquad (\psi_\Omega \geq 0)\,.$$

Let $W_p = \mathrm{Dom}(|H_\Omega|^{p/2})$ be the usual Sobolev space. It can be proved [8]

[1] Research partially supported by USNSF grant MCS-81-20833.

that if $\varphi \in W_p$ and $p > \frac{1}{2}n$ and if $\partial\Omega$ obeys a weak condition (each boundary point regular in the potential theory sense), then φ is a continuous function on clos Ω vanishing on $\partial\Omega$. We want to ask how fast φ vanishes, where, if necessary, we are willing to take p very large. If $\partial\Omega$ is smooth, it is easy to prove that any $\varphi \in W_p$ with p sufficiently large vanishes at least linearly in $\mathrm{dist}(x, \partial\Omega) \equiv d(x)$ and ψ_Ω vanishes exactly that fast (see e.g. [10]). That the situation for general Ω is more complicated is seen by the study of polyhedral regions (see e.g. [12]). The situation is especially easy to describe if $n = 2$ so Ω is a polygon: If $x_0 \in \partial\Omega$ is a vertex of interior opening angle α and if $x \to x_0$ along the bisector of that vertex, then ψ_Ω vanishes as $\mathrm{dist}(x, x_0)^m$ with $m = \pi/\alpha$. Some thought suggests that the correct rate of vanishing should be precisely that of ψ_Ω. Thus:

Problem (1.1). For what Ω and p is there an estimate of the form $|\varphi(x)| \leq c\| |H_\Omega|^{p/2}\varphi\|_2 \psi_\Omega(x)$?

Surprisingly, we know of no previous work, for general Ω, on this natural question.

1.2. *Conditioned Brownian Paths*

Given $x, y \in \mathbb{R}^n$ and $t > 0$, let $P_{x,y;t}$ be the probability measure on Brownian paths conditioned to begin at x and end at y at time t. Explicitly, P is a measure on continuous functions $b(s)$, $0 \leq s \leq t$, with $b(0) = x$, $b(s) = y$; the components of $b(s)$ are jointly Gaussian random variables with mean $m(s) = E_{x,y;t}(b(s)) = (1-t)x + ty$ and covariance $E_{x,y;t}((b_i(s) - m_i(s))(b_j(u) - m_j(u))) = d_{ij}s(1 - t^{-1}u)$ if $0 \leq s \leq u \leq t$ (see e.g. [16], [21]). Let $\Omega \subset \mathbb{R}^n$ be open and bounded and define

$$F_\Omega(x, y; t) = P_{x,y;t}(\{b \mid b(s) \in \Omega, \text{ for all } s, 0 \leq s \leq t\}),$$

the fraction of paths that stay in Ω. F_Ω should go to zero as either x or y approach $\partial\Omega$ (and will if $\partial\Omega$ has weak regularity; see [16]). Paths that don't leave Ω stay 'away' from $\partial\Omega$, so most paths that leave Ω should do so when s is near 0 or t, i.e. we expect that $F_\Omega(x, y; t)$ should go to zero as $F_\Omega(x, x; t)^{1/2} F_\Omega(y, y; t)^{1/2}$. Thus:

Problem (1.2). Let $D_\Omega(x; t) = F_\Omega(x, x; t)^{1/2}$. When is it true that for some $\alpha, \beta > 0$ and all $x, y \in \Omega$, we have that

$$\alpha D_\Omega(x; t) D_\Omega(y; t) \leq F_\Omega(x, y; t) \leq \beta D_\Omega(x; t) D_\Omega(y; t)?$$

We note that the upper bound is easy, since $(2\pi t)^{-n/2} \exp(-(x-y)^2/2t) \cdot F_\Omega(x, y; t)$ is the integral kernel of the positive operator $\exp(-\frac{1}{2}tH_\Omega)$.

1.3. *Ultracontractivity of Schrödinger Semigroups*

Consider a semigroup e^{-tA}, $t \geq 0$, of selfadjoint operators on $L^2(X, d\mu)$ with X a probability measure space and so that A obeys $\|e^{-tA}\varphi\|_p \leq \|\varphi\|_p$ for all $t \geq 0$, $1 \leq p \leq \infty$. (Such semigroups arise naturally as follows: Given H, a selfadjoint operator on $L^2(Y, d\nu)$, so that e^{-tH} is positivity preserving and so that $H\psi = E\psi$ for some ψ, a strictly positive, normalized vector in $L^2(Y, d\nu)$, then one can pick $X = Y$, $d\mu = \psi^2 d\nu$ and define $U : L^2(Y, d\nu) \to L^2(Y, d\mu)$ by $U\varphi = \psi^{-1}\varphi$. U is unitary and $A = U(H - E)U^{-1}$ obeys $e^{-tH_1} = 1$ and e^{-tA} is positively preserving. Such a semigroup is a contraction on all L^p spaces (see e.g. [17]). We will occasionally write $A = \tilde{H}$ and refer to the *$\tilde{H}$ construction*).

Given an L^p contractive semigroup, we say that it is *hypercontractive* if $\|e^{-tA}\varphi\|_4 \leq c\|\varphi\|_2$ for some $t > 0$ and *supercontractive* if $\|e^{-tA}\varphi\|_4 \leq c(t)\|\varphi\|_2$ for all $t > 0$. We introduce here the notion *ultracontractive* to mean that $\|e^{-tA}\varphi\|_\infty \leq c(t)\|\varphi\|_2$ for all $t > 0$. (We note that there is no point in replacing ∞ by some $p \neq 4$ in $(2, \infty)$; all $p < \infty$ yield a definition equivalent to supercontractivity.) If H is given of the form discussed parenthetically above, so that the corresponding $e^{-t\tilde{H}}$ (note the $\tilde{\ }$) is #-contractive, we say that e^{-tH} is *intrinsically #-contractive.*

Problem (1.3). Are any Schrödinger operators, $-\Delta + V$, intrinsically ultracontractive?

Here is some background on this problem. Since their introduction as tools in constructive quantum field theory (see e.g. [11], [21]), and especially after Gross' paper [13] on logarithmic Sobolev inequalities, hypercontractive estimates have provoked a large mathematical literature. In a recent bibliography on the subject, Gross [14] lists 51 papers! Nelson [15] showed that $-\Delta + cx^2$ is intrinsically hypercontractive in the initial paper on the subject, and he later showed that it is not intrinsically supercontractive. Eckmann [9], Rosen [19] and Carmona [4] studied the intrinsic #-contractive properties of general Schrödinger operators. They show that if $V(x)$ roughly goes to $|x|^a$, then one doesn't even have intrinsic hypercontractivity if $a<2$ and one has intrinsic supercontractivity if $a>2$. Apparently no one studied ultracontractivity because there was a belief that it couldn't hold for $(-\Delta+V)$. This belief, which we originally shared, seems to come from the fact that intrinsic ultracontractivity implies that for any eigenfunction φ of H, $\varphi\psi^{-1}$ is bounded. This is false for the harmonic oscillator, so it was uncritically assumed false in general. Indeed, a simple WKB argument shows that, in one dimension, if $V \sim |x|^a$ with $a>2$, then each $\varphi\psi^{-1}$ is bounded. We will see below that if $V \sim |x|^a$ with $a>2$, then $-\Delta+V$ is intrinsically ultracontractive. Indeed, ultracontractivity is the rule: If $V(x)=|x|^a(\log(|x|+2))^b$, then one has no intrinsic contractivity if $a<2$, $b\geq 0$; intrinsic hypercontractivity but not intrinsic supercontractivity if $a=2$, $b=0$; intrinsic supercontractivity but not intrinsic ultracontractivity if $a=2$, $0<b\leq 2$ and intrinsic ultracontractivity if $a>2$, $b\geq 0$ or $a=2$, $b>2$.

We should mention that one of us in [6], which was one motivation for us here, showed that very general one-dimensional Schrödinger operators on a finite interval with Dirichlet boundary conditions are intrinsically ultracontractive.

2. Some Abstract Theory

Let X be a locally compact, second countable Hausdorff space with regular Borel measure, ν, and let H be a semibounded self-adjoint operator on $L^2(X, d\nu)$ so that e^{-tH} has a jointly continuous integral kernel $a_t(x, y)$. Suppose also that

(i). $a_t(x,y)>0$ for all x, y and

(ii). $\mathrm{Tr}(e^{-tH})<\infty$ for all $t>0$.

Because of (ii), H has purely discrete spectrum $\{E_n\}_{n=0}^{\infty}$ with $E_0\leq E_1\leq E_2\leq\cdots$, where $E_0<E_1$ follows from (i) [18, XIII.12], and normalized eigenfunctions $\psi_n(x)$ with $\psi_0(x)>0$ for all x. It is not hard to see that

$$a_t(x,y)=\sum_{n=0}^{\infty} e^{-tE_n}\psi_n(x)\psi_n(y) \tag{2.1}$$

converges uniformly on compact subsets of X. Define

$$b_t(x)=\sqrt{a_t(x,x)}$$

(not to be confused with the function $d(x)=\mathrm{dist}(x,\partial\Omega)$ appearing in Problem (1.1)). Two estimates automatically hold

$$\psi_0(x)\leq e^{tE_0/2}b_t(x), \tag{2.2}$$

$$a_t(x,y)\leq b_t(x)b_t(y), \tag{2.3}$$

for (2.2) (which one of us has used extensively elsewhere [22]) is a consequence of setting $x=y$ in (2.1), and (2.3) is a consequence of the positivity of the operator e^{-tH}.

Theorem (2.1). *Under the above conditions, the following are equivalent:*

(i). e^{-tH} *is intrinsically ultracontractive.*

(ii). *For all* $0<t<\infty$, *there exists a* $c_t<\infty$ *with*

$$|(e^{-tH}f)(x)|\leq c_t\|f\|_2\psi_0(x),$$

where $\|\cdot\|_2$ *is the* $L^2(X, d\nu)$ *norm.*

(iii). *For all* $0<t<\infty$, *there exists a* c_t' *such that*

$$a_t(x,y)\leq c_t'\psi_0(x)\psi_0(y). \tag{2.4}$$

(iv). *For all* $t>0$, *there exists a* $c_t''<\infty$ *such that*

$$b_t(x)\leq c_t''\psi_0(x).$$

(v). *For all* $t>0$, *there exists a* $c_t''<\infty$ *such that*

$$a_t(x,y)\geq c_t'''b_t(x)b_t(y).$$

Before turning to a sketch of the proof, we note two things: First, that by (2.2), any (and hence all) of (i)–(v) imply

$$a_t(x,y)\geq c_t^{(iv)}\psi_0(x)\psi_0(y). \tag{2.5}$$

Second, the remarkable fact that an upper bound on a_t like (2.4) implies a lower bound like (2.5).

(Sketch of) **Proof.** (i)⇔(ii). Just involves disentangling the definition of intrinsic ultracontractivity.

(i)⇔(iii). The integral kernel, $\tilde{a}_t(x,y)$, of $e^{-t\tilde{H}}$ is easily seen to be $\tilde{a}_t(x,y)=e^{tE_0}\psi_0(x)^{-1}\psi_0(y)^{-1}a_t(x,y)$. Thus (iii) is equivalent to saying that $\tilde{a}_t$ is bounded. By the Dunford–Pettis theorem, this says that (iii) is equivalent to the assertion that $e^{-t\tilde{H}}$ is bounded from L^1 to L^∞ for all t. Given that $e^{-t\tilde{H}}$ is a contraction on each L^p, duality, interpolation and the semigroup property show that $e^{-t\tilde{H}}$ is bounded from L^1 to L^∞ for all t if and only if it is bounded from L^2 to L^∞ for all t.

(iii)⇒(iv). A triviality.

(iv)⇒(iii). Follows from (2.3).

(iii)⇒(v). Given t, pick a compact K, so that

$$\int_{X\setminus K}\psi_0(x)^2\,d\nu(x)\le\tfrac{1}{2}(c'_{t/3})^{-1}\exp(-\tfrac{1}{3}tE_0). \tag{2.6}$$

Then, using (iii),

$$\psi_0(x)\exp(-\tfrac{1}{3}tE_0)=\int_X a_{t/3}(x,y)\psi_0(y)\,d\nu(y)$$

$$\le\int_K a_{t/3}(x,y)\psi_0(y)\,d\nu(y)+c'_{t/3}\psi_0(x)\int_{X\setminus K}\psi_0(y)^2\,d\nu(y).$$

Using (2.6)

$$\int_K a_{t/3}(x,y)\psi_0(y)\,d\nu(y)\ge\tfrac{1}{2}\psi_0(x)\exp(-\tfrac{1}{3}tE_0). \tag{2.7}$$

Since K is compact and $a_t(x,y)\psi_0(x)^{-1}\psi_0(y)^{-1}$ is continuous and non-zero on all of $X\times X$, it has a strictly positive infimum, γ, on $K\times K$. Thus, by the semigroup property and (2.7)

$$a_t(x,y)\ge\int_{K\times K}d\nu(z)\,d\nu(w)\,a_{t/3}(x,z)a_{t/3}(z,w)a_{t/3}(w,y)$$

$$\ge\gamma\int_{K\times K}d\nu(z)\,d\nu(w)\,a_{t/3}(x,z)\psi_0(z)a_{t/3}(w,y)\psi_0(w)$$

$$\ge\tfrac{1}{4}\gamma\psi_0(x)\psi_0(y)\exp(-\tfrac{2}{3}tE_0).$$

(v)⇒(iv). We have that, using (2.2) and (v)

$$\psi_0(x)=e^{E_0t}\int_X[a_t(x,y)]\psi_0(y)\,d\nu(y)$$

$$\ge e^{E_0t}\int_X[c'''_t\exp(-\tfrac{1}{2}tE_0)\psi_0(y)b_t(x)]\psi_0(y)\,d\nu(y)$$

$$=c'''_t\,e^{tE_0/2}b_t(x).\ \square$$

With this theorem, we can reduce the solution of Problems (1.1) and (1.2) to statements about intrinsic ultracontractivity of Dirichlet semigroups.

2.1. *Problem (1.2), revisited*

Let a_t be the integral kernel of $e^{-tH_\Omega/2}$. Then, the Feynman–Kac formula [21] says that

$$F_\Omega(x,y;t)=[(2\pi t)^{-n/2}\exp(-(x-y)^2/2t)]^{-1}a_t(x,y),$$

$$D_\Omega(x;t)=[(2\pi t)^{-n/4}]^{-1}b_t(x).$$

Since $(x-y)^2$ is bounded on $\Omega\times\Omega$, we see that $F_\Omega(x,y)/D_\Omega(x)D_\Omega(y)$ is bounded above and below if and only if $a_t(x,y)/b_t(x)b_t(y)$ is bounded above and below. Thus:

Corollary (2.2). *For fixed Ω, Problem* (1.2) *has a positive solution for all t if and only if* $\exp(-tH_\Omega)$ *is intrinsically ultracontractive.*

2.2. *Problem (1.7), revisited*

The estimate

$$|\varphi(x)|\le c\,\||H_\Omega|^{p/2}\varphi\|_2\psi_0(x) \tag{2.8}$$

is equivalent (since H_Ω is invertible) to

$$|(|H_\Omega|^{-p/2}f)(x)|\le c\|f\|_2\psi_0(x).$$

Thus, since $\|e^{-tH}\|_2$ decays exponentially for large t and $H^{-p/2} = c_p \int t^{p/2-1} e^{-tH}\, dt$, we see that

Corollary (2.3). (2.8) *holds for all* $p > \alpha$ *if for* $0 < t < 1$:

$$|(e^{-tH_\Omega} f)(x)| \le d t^{-\alpha/2} \|f\|_2 \psi_0(x)\,. \tag{2.9}$$

We remark that if (2.8) holds for some p, then (2.9) holds for all $\alpha > p$, so Corollary (2.3) is 'almost' if and only if.

3. General Theory of Ultracontractivity of Dirichlet Forms

In this section, we combine known results of Gross [13] and Rosen [19] to reduce ultracontractive estimates to a single family of operator inequalities. We restate their results carefully, because there seems to be a tradition in the subject to misstate them. Eckmann [9] and Carmona [4] both misstate Gross' estimate because they copy this inequality exactly although they have changed the convention on one of his constants. Rosen [19] isn't explicit about his constants; when Carmona tries to be explicit, he makes two errors! Fortunately, these errors don't affect the main conclusions of those papers. For us, the behaviour of the constants is critical.

In the results below, we will not always state conditions on domains explicitly; these are discussed in detail in [4], [9], [19].

Gross' first important idea in [13] is the following:

Theorem (3.1). (*Gross* [13].) *Let* μ *be a probability measure. Let* A *be an operator on* $L^2(\Omega, d\mu)$. *Let* $f_p = |f|^{p-1}\operatorname{sign}(f)$ *and suppose that for some* $r \in (2, \infty]$ *and all* $p \in (2, r)$, *we have that, for all* $f \in \operatorname{Dom}(A)$:

$$\int |f|^p \log|f| \le c(p)\operatorname{Re}\langle Af, f_p\rangle + \Gamma(p)\|f\|_p^p + \|f\|_p^p \log\|f\|_p\,. \tag{3.1}$$

Suppose that

$$t = \int_2^r c(p)\frac{dp}{p}\,, \qquad M = \int_2^r \Gamma(p)\frac{dp}{p} \tag{3.2}$$

are both finite. Then (*Gross' estimate*):

$$\|e^{-tH} f\|_r \le e^M \|f\|_2\,. \tag{3.3}$$

Remark (3.2). The proof goes by letting $p(s)$ be defined by $s = \int_2^{p(s)} c(p)p^{-1}\,dp$ and differentiating $\|e^{-sH} f\|_{p(s)}^{p(s)}$. See Gross for details.

Remark (3.3). Gross' γ is related to our Γ by $\gamma = \Gamma/c$. He defines M by $M = \int_2^t \gamma(p(s))\,ds$ which can be seen to be equivalent to (3.2) by a change of variables.

Remark (3.4). Gross only states his result for $r < \infty$. Using $\|f\|_\infty = \lim_{p\to\infty}\|f\|_p$, the proof extends to $r = \infty$; the conditions $t, M < \infty$ are nontrivial if $r = \infty$.

Remark (3.5). (3.1) is called a *logarithmic Sobolev inequality*.

In our examples below, one has that (3.1) holds for any p and c, i.e. there is $\beta(p, c)$ with

$$\int |f|^p \log|f| \le c\operatorname{Re}\langle Af, f_p\rangle + \beta(p, c)\|f\|_p^p + \|f\|_p^p \log\|f\|_p\,. \tag{3.1'}$$

The second result in Gross [13] (quoted in a more explicit form due to Eckmann [9]) deals with a situation where Ω is an open subset of $\mathbb{R}^n$ and where

$$\langle Af, g\rangle = \int_\Omega \overline{\nabla f(x)} \cdot \nabla g(x)\, d\mu(x)\,, \tag{3.4}$$

on suitable f, g and with suitable domain hypotheses. A is called a *Dirichlet form*. If $H = -\Delta + V$ and $A = \tilde{H}$, then A is a Dirichlet form.

Theorem (3.6). (*Gross* [13], *Eckmann* [9].) *Let* A *obey* (3.4) *and suitable domain hypotheses. Suppose that* (3.1′) *holds for* $p = 2$ *and all* c *with* $\beta(2, c) \equiv b(c)$. *Then* (3.1′) *holds for all* $2 \le p_0 < \infty$ *where*

$$\beta(p_0, c) = \frac{2}{p_0}\, b\Big(\frac{2(p_0 - 1)}{p_0}\, c\Big)\,.$$

Remark (3.7). The basic idea is to replace $|f|$ by $|f|^{p_0/2}$ in (3.1′) for $p_0 = 2$. The left-hand side of (3.1′) becomes $\frac{1}{2}p_0 \int |f|^{p_0} \log |f|^{p_0}$. The first term on the right using $(\nabla|f|^{p_0/2})^2 = (\nabla|f|^{p_0-1})\nabla|f|(\frac{1}{2}p_0)^2/(p_0-1)$, becomes $\langle Af, |f|_p\rangle$ times $(\frac{1}{2}p_0)^2/(p_0-1)$. See [13] for details.

If we note that $2(p-1)/p$ varies from 1 to 2 as p varies from 2 to ∞ and use the fact that without loss we can suppose $b(2c) \leq b(c)$, we have by combining the last two theorems:

Theorem (3.8). *Suppose that A obeys* (3.4) *and suitable domain conditions and that* (3.1′) *holds for* $p = 2$ *with* $\beta(2, c) \equiv b(c)$. *Given t, suppose we can choose* $c(p)$ *so that*

$$t = \int_2^\infty c(p) \frac{\mathrm{d}p}{p}.$$

Then $\|\mathrm{e}^{-tH} f\|_\infty \leq \mathrm{e}^M \|f\|_2$ *where*

$$M = \int_2^\infty 2b(c(p)) \frac{\mathrm{d}p}{p^2}.$$

Example (3.9). $b(c) = Ac^{-k}$ at least for c small. We take (for t small) $c(p) = t(\log 2)/(\log p)^2$ and find $M = \mathrm{d}_k A t^{-k}$. This can be used to show suitable fractional powers of H generate supercontractive semigroups.

Example (3.10). $b(c) = \exp(c^{-a})$ at least for c small. Pick $c(p) = td(\alpha)/(\log p)^\alpha$ with $\alpha > 1$. Then $M < \infty$ if $a\alpha < 1$. Thus, if $a < 1$, we have ultracontractivity. It is interesting that the borderline is related to the borderline in Trudinger-type estimates; see [3].

Example (3.11). $b(c) = A_0 + A_1 \log(c^{-1})$. Take $c(p) = t(\log 2)/(\log p)^2$. Since $\int_2^\infty 2\, \mathrm{d}p/p^2 = 1$, we have

$$M = A_2 + A_1 \log(t^{-1}),$$

and

$$\|\mathrm{e}^{-tH}\|_{\infty,2} \leq bt^{-A_1}.$$

This is relevant for Problem (1.1) as we shall see.

One needs to ask when L^2 logarithmic Sobolev inequalities hold for Dirichlet forms. This is answered by an argument of Rosen [19] (extended by Carmona [4]).

Theorem (3.12). (*Rosen's lemma* [19].) *Let* $A = \tilde{H}$ *where* $H = -\Delta + V$ *on* $L^2(\mathbb{R}^n)$ *or* $H = -\Delta_\Omega$ *with* $\Omega \subset \mathbb{R}^n$. *Suppose that one has the operator inequality*

$$-\log|\psi_0| \leq \tfrac{1}{2}\delta H + g(\delta). \tag{3.5}$$

Then (3.1′) *holds for* $p = 2$ *with*

$$\beta(2, c) = g(\delta) + a_n - \frac{n}{4} \log \delta,$$

for a universal (n-*dependent*) *constant* a_n.

Remark (3.13). The proof just makes the constant explicit (and correct) in Carmona's version [4] of Rosen's argument [19]. a_n depends on the constant in the classical Sobolev inequality.

4. Getting our Act Together

The net result of the last section is that ultracontractivity of Schrödinger and Dirichlet semigroups is reduced to upper bounds on $-\log \psi_0$, i.e. lower bounds on ψ_0, and lower bounds on $H = -\Delta + V$. It is remarkable that (as we shall see) rather crude lower bounds on ψ_0 suffice; we say remarkable because ultracontractivity says that ψ_0/b_t is bounded above and also away from zero. Thus, for example, for an x^4 oscillator in one dimension where one knows that $b_t \sim cx^{-1}\exp(-dx^3)$, a lower bound $\psi_0 \geq c_1 \exp(-c_2 x^{4-\varepsilon})$, $\varepsilon > 0$, plugged into our machinery bootstraps to a lower bound by $c'x^{-1}\exp(-dx^3)$.

4.1. Schrödinger Semigroups (Solution of Problems (1.3))

If one looks at the argument of Carmona–Simon [5], one sees that the following is true:

Lemma (4.1). *If* $V(x) \leq c_1|x|^a + c_2$ *for some* $c_1 > 0$, c_2 *real, then*

$$\psi_0 \geq d_1 \exp(-d_2|x|^{1/2a+1})$$

for some $d_1, d_2 > 0$.

Indeed, Lemma (4.1) follows from an almost trivial path space estimate [8]. Given this lemma and Theorem (3.12), we have

Theorem (4.2). *Suppose that for some* $c_1, c_3 > 0$, c_2, c_4 *we have that*

$$c_3|x|^b + c_4 \leq V(x) \leq c_1|x|^a + c_2$$

where $\frac{1}{2}a + 1 < b$. *Then* $H = -\Delta + V$ *is intrinsically ultracontractive.*

Proof. Let $\alpha = b/(\frac{1}{2}a + 1)$. By Lemma (4.1),

$$-\log \psi_0 \leq c_5(V(x) + 2|c_4|)^{1/\alpha} \leq \tfrac{1}{2}\delta V(x) + d_1\delta + D_2\delta^{1-1/\alpha}.$$

By Example (3.9), and by Theorems (3.8) and (3.12), we obtain ultracontractivity. □

Note that $\frac{1}{2}a + 1 < b$ and $b \leq a$ imply $b > 2$. More refined estimates [8] show that if $c_2x^2 \log(|x| + 2)^b \leq V(x) \leq c_1x^2[\log(|x| + 2)]^b$ and $b > 2$, then one has ultracontractivity. More results on the Schrödinger case will appear in [8]. We emphasize that these results are multi-dimensional.

4.2. Dirichlet Semigroups (Problems (1.1) and (1.2))

We will prove ultracontractivity under suitable geometric hypotheses. Lest the reader think such hypotheses are unnecessary, we mention that there are examples of regions in $\mathbf{R}^2$ for which intrinsic ultracontractivity fails; for one can show that $b_t\psi_0^{-1}$ is unbounded (see [8]).

To verify (2.10), one needs lower bounds on ψ_0, i.e. upper bounds on $-\log \psi_0$, and lower bounds on H_Ω by functions of x. The latter problem is solved by a recent estimate of Davies [7]. Given Ω and $x \in \Omega$, and given a unit vector $\omega \in S^{n-1}$, let $d(x, \omega)$ be defined by:

$$d(x, \omega) = \inf\{|r| \mid x + r\omega \notin \Omega\},$$

and the quasidistance $q(x)$ by

$$\frac{1}{q(x)^2} = \int_{S^{n-1}} \frac{d\omega}{d(x, \omega)^2},$$

where $d\omega$ is the normalized invariant measure on S^{n-1}. Then

Theorem (4.3). (*Davies* [7].) *For any* Ω

$$n(4q^2)^{-1} \leq H_\Omega.$$

Remark (4.4). This is an elementary consequence of the inequality $\langle f, (4x^2)^{-1}f\rangle \leq \langle f', f'\rangle$ for $f \in \mathscr{C}_0^\infty(0, \infty)$. Combined with Agmon's method [2], this is useful for proving upper bounds on ψ, and critical in the example mentioned above where $\psi^{-1}d_t$ is unbounded.

To use this, we need

Definition (4.5). We say that Ω obeys an *exterior cone condition* if and only if there exists $\varepsilon > 0$, $\alpha > 0$ so that for each $x \in \partial\Omega$, there is a unit vector $e(x)$ with

$$\{y \mid 0 < |x - y| < \varepsilon, e \cdot (y - x) > \alpha|y - x|\} \subset \mathbf{R}^n \setminus \Omega.$$

A simple geometric argument shows that if Ω obeys an exterior cone condition, then $q(x) \leq ad(x)$ where a depends on α, ε and diam(Ω) and thus

Corollary (4.6). (*Davies* [7].) *If* Ω *obeys an exterior cone condition, then for a suitable constant* c

$$cd^{-2} \leq H_\Omega.$$

To get a lower bound on ψ_0, it is useful to define special cones as follows: Let $A \subset S^{n-1}$ be an open set; then given $x \in \mathbf{R}^n$ and ε, we define

$$C(x, A, \varepsilon) = \{y \mid 0 < |x - y| < \varepsilon, y - x/|y - x| \in A\}.$$

An elementary comparison argument [8] leads to:

Definition (4.7). Let A be an open subset of S^{n-1}. We say that Ω obeys an *A-interior cone condition,* if and only if there exists an ε, a $\delta > 0$, a $\beta > 0$, and for each $x \in \Omega$ with $d(x) < \delta$, a point $y(x) \in \partial\Omega$ and a rotation R_x so that

$$x \in C(y(x), R_x(A), \varepsilon) \subset \Omega,$$
$$\mathrm{dist}(x - y(x)/|x - y(x)|, S^{n-1} \backslash R_x(A)) > \beta.$$

Definition (4.8). Given $A \subset S^{n-1}$, let $\lambda(A)$ define the lowest eigenvalue of the Laplace–Beltrami operator on $L^2(A)$ with Dirichlet boundary conditions. Define $\alpha = \alpha(A) > 0$ by:

$$\alpha(\alpha + \nu - 2) = \lambda.$$

Theorem (4.9). [8] *Let Ω obey an A-interior cone condition. Then*

$$\psi_0 \geqslant c d^{\alpha(A)}.$$

Remark (4.10). If $\nu = 2$ and $A = \{(\cos\theta, \sin\theta) \,|\, 0 < \theta < \phi_0\}$, then $\lambda = (\pi/\phi_0)^2$ and $\alpha = \pi/\phi_0$ and the bound agrees with the one mentioned in our discussion of Problem (1.1).

If we combine Corollary (4.6), Theorem (4.9), Theorem (3.12), Example (3.11), and the inequality $-\log x \leqslant \varepsilon x^{-2} - \frac{1}{2}\log\varepsilon + b$ for suitable b, we find:

Theorem (4.11). *If Ω obeys an exterior cone condition and an A-interior cone condition, then e^{-tH_Ω} is intrinsically ultracontractive and*

$$\|e^{-t\tilde{H}_\Omega}\|_{\infty,2} \leqslant c t^{-[n/4 + \alpha(A)/2]}.$$

Remark (4.12). For cubes, one can use the method of images to compute exactly the divergence of $\|e^{-t\tilde{H}_\Omega}\|_{\infty,1}$ and to get a lower bound on $\|e^{-t\tilde{H}_\Omega}\|_{\infty,2}$ which has the same power behaviour as Theorem (4.11) if one takes A to be a quadrant. See [8] for a discussion of 'trumpet' conditions replacing cone conditions.

Given the discussion at the end of Section 2, we have solved Problems (1.1) and (1.2) under suitable cone conditions.

Acknowledgments

This work was done while we were both visiting The Australian National University (E.B.D. at the Research School of Physical Sciences, and B.S. at the Centre for Mathematical Analysis). We would like to thank Derek Robinson and Neil Trudinger for their hospitality, and we would like to thank Derek for his suggestion of the term 'ultracontractive'.

References

[1] S. Agmon, Lectures on elliptic boundary value problems (Van Nostrand, New York, 1965).
[2] S. Agmon, Lectures on exponential decay of solutions of second-order elliptic equations: Bounds on eigenfunctions of N-body Schrödinger operators (Princeton Univ. Press, Princeton, 1982).
[3] M. Aizenman and B. Simon, Brownian motion and Harnack's inequality for Schrödinger operators, Comm. Pure Appl. Math. 35 (1982) 209–273.
[4] R. Carmona, Regularity properties of Schrödinger and Dirichlet semigroups, J. Funct. Anal. 33 (1979) 259–296.
[5] R. Carmona and B. Simon, Pointwise bounds on eigenfunctions and wave packets in N-body quantum systems. V. Lower bounds and path integrals, Comm. Math. Phys. 80 (1981) 59–98.
[6] E.B. Davies, Hypercontractive and related bounds for double well Schrödinger operators, Quart. J. Math. (2) 34 (1983) 407–421.
[7] E.B. Davies, Some norm bounds and quadratic form inequalities for Schrödinger operators II, J. Operator Theory 12 (1984) 177–196.
[8] E.B. Davies and B. Simon, Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians, J. Funct. Anal. 59 (1984) 335–395.
[9] J.P. Eckmann, Hypercontractivity for anharmonic oscillators, J. Funct. Anal. 16 (1974) 388–406.
[10] D. Gilbarg and N. Trudinger, Elliptic partial differential equations of second order (Springer, Berlin, 1977).
[11] J. Glimm and A. Jaffe, Quantum physics (Springer, Berlin, 1982).
[12] P. Grisvard, Singularities for the problem of limits in polyhedrons, Sem. Goulaouc–Schwartz 1981/82, Ex. 8.
[13] L. Gross, Logarithmic Sobolev inequalities, Amer. J. Math. 97 (1975) 1061–1083.
[14] L. Gross, Papers on logarithmic Sobolev inequalities and hypercontractivity, unpublished, Aug. 1982.
[15] E. Nelson, A quartic interaction in two dimensions, In: Mathematical theory of elementary particles, eds. R. Goodman and I. Segal (MIT Press, Cambridge, MA, 1966) 69–73.
[16] S. Port and C. Stone, Brownian motion and classical potential theory (Academic Press, New York, 1978).

[17] M. Reed and B. Simon, Methods of modern mathematical physics, II. Fourier analysis, self-adjointness (Academic Press, New York, 1975).
[18] M. Reed and B. Simon, Methods of modern mathematical physics, IV. Analysis of operators (Academic Press, New York, 1977).
[19] J. Rosen, Sobolev inequalities for weight spaces and supercontractivity, Trans. Amer. Math. Soc. 222 (1976) 367–376.
[20] B. Simon, The $P(\phi)_2$ Euclidean (quantum) field theory, Princeton Ser. Physics (Princeton Univ. Press, Princeton, 1974).
[21] B. Simon, Functional integration and quantum physics (Academic Press, New York, 1979).
[22] B. Simon, Semiclassical analysis of low lying eigenvalues, II. Tunneling, Caltech preprint.