# REGULARITY OF THE DENSITY OF STATES FOR STOCHASTIC JACOBI MATRICES: A REVIEW

Barry Simon
Division of Physics, Mathematics and Astronomy
California Institute of Technology
Pasadena, CA 91125

## 1. The Density of States, the Lyaponov Exponent and Their Relation

In this paper, we will discuss stochastic Jacobi matrices which are operators on $\ell^2(\mathbb{Z}^\nu)$. Indicate elements of this Hilbert space by $u(n)$ with $n\in\mathbb{Z}^\nu$. The free (kinetic) energy operator is given by:

(1) $$(H_0u)(n) = \sum_{|j|=1} u(n+j)$$

and we will consider operators, $H = H_0+V_\omega$ , where $\omega$ is a label in a probability measure space. The potential, V, is a family of random variables forming an ergodic process. To be explicit, we let $(\Omega,\mu)$ be a probability measure space with a family, $T_1,\dots,T_\nu$ of commuting, measure preserving transformations which generate an ergodic action. Pick f, a measurable function on $\Omega$, and define

(2) $$V_\omega(n) = f(T_1^{n_1}\cdots T_\nu^{n_\nu}\omega)$$

For simplicity, we will normally suppose that f is bounded, although many results only require the minimal regularity property

(3) $$\int \ln[|f(\omega)|+1]\,d\mu(\omega) < \infty$$

We will occasionally discuss unbounded f's, in which case we will freely use those results which hold in the more general setting.

Two special subclasses of stochastic Jacobi matrices have received especial attention: The situation where V is an almost periodic function and $\Omega$ is just the hull of V (see, for example, the appendix of Avron-Simon [1] for background on almost periodic functions), and the situation where V is a family of independent, identically distributed random variables, a setup known as the Anderson model. We will let $d\kappa$ denote the probability density of V in this case.

These families of operators have received considerable attention in the theoretical physics literature. The Anderson model is supposed to be a caricature of the effect of impurities on electron motion in solids, and of electron motion in amorphous materials, like glass. The almost periodic models may describe certain alloys and the recently discovered quasicrystals. The mathematical physics literature has discussed both these models and their continuum analogs where $\ell^2(\mathbb{Z}^\nu)$ is replaced by $L^2(\mathbb{R}^\nu)$ and H becomes a differential operator. We will occasionally mention results that are not known to extend to the continuum case, but in the interests of simplicity, we will restrict our discussion to the discrete Jacobi matrix case. This way, one can avoid getting bogged down in technical subtleties; indeed, these technical problems can often be nontrivial, so that much more is known currently about the discrete case than about the continuum case.

The deepest and most interesting feature of these models concerns the spectral properties of the operators. There are recent reviews of these things in the book of Cycon et al. [14], and the lecture notes of Carmona [8] and Spencer [43]. The density of states, which we discuss here, is a less interesting object, but one which has evoked a



considerable literature because it is a basic object of some use in the deeper analysis, and simply related to directly measurable quantities in physical systems.

Let $H_{\omega,L}$ denote the restriction of H, thought of as an infinite matrix, to indices with $|i_1|,\dots,|i_\nu| \le L$. Thus H is a matrix of dimension $(2L+1)^\nu$. The *integrated density of states* (ids) is defined by

$$(4) \qquad k(E) = \lim_{L\to\infty}(2L+1)^{-\nu}\#(\text{of e.v. of } H_{\omega,L} \le E)$$

That the limit exists is a result going back to Benderskii-Pastur [6]. There have been numerous refinements of this existence theorem which is essentially a consequence of the ergodic theorem. The following result is proven in Avron-Simon [1]. It discusses the "typical spectrum" using another consequence of the ergodic theorem, namely, that there is a subset, W, of $\Omega$ of full measure and a subset, $\Sigma$, of $\mathbb{R}$ so that the spectrum of $H_\omega$ is $\Sigma$ whenever $\omega \in W$. The definition (4) is not so convenient as an initial definition since the ergodic theorem allows a set of measure zero where the limit fails to exist, and because the set of E is uncountable, this can cause problems. This explains why we deal with vague convergence: The separability of the continuous functions allows one to deal with one set of full measure. One thus defines a measure $dk_{\omega,L}$ to be the point measure giving weight $(2L+1)^{-\nu}$ to each eigenvalue of $H_{\omega,L}$. Degenerate eigenvalues are given multiple weight, so that $dk_{\omega,L}$ is a probability measure. We will also define $\chi_L$ to be the projection onto those vectors in $\ell^2$ supported in the region where of these proofs is to relate the eigenvalues of H (or more properly, the restrictions to [0,L-1] and [1,L]) to the vanishing of matrix elements of T(E), to note that these matrix elements are monic polynomials in E, and so write them in terms of eigenvalues. [2] handles real E using the theory of Hilbert transforms, while [12] uses subharmonic functions. There is a second approach to the Thouless formula using Weyl m-functions: See Johnson-Moser [20], Kotani [23] and Simon [36].

One should emphasize that k(E) is a bad indication of the spectral properties of H. As we will see in Section 7, there exist two distinct families of stochastic Jacobi matrices which have identical ids's, but so that one family has pure point spectrum with probability one, and the other has singular continuous spectrum with probability one. The ids does determine the absolutely continuous spectrum because of the Thouless formula and the following theorem of Kotani [23]:

Theorem 1.3 For any one-dimensional stochastic Jacobi matrix, the set of real E for which the Lyaponov exponent vanishes is the essential support of the absolutely continuous spectrum for a typical H.

Even here, the determination from k(E) is a global one, and doesn't really have much to do with the density of states. If the conjecture that the higher dimensional Lloyd model has some extended states is correct, then in higher dimensions, one would know that the absolutely continuous spectrum is not determined by the ids.



**2. Continuity of k**

In this section, we will discuss the idea beyond the very simplest proof of the following fundamental result:

Theorem 2.1 The ids, k(E), is continuous for any stochastic Jacobi matrix.

This result was proven in the one-dimensional case by Pastur [34]. The higher dimensional result was proven by Craig-Simon [13], who proved a stronger result which we will discuss in the next section. An elementary proof in the higher dimensional case was subsequently found by Delyon-Souillard [16].

The key idea in both the Pastur and Delyon-Souillard proofs is to exploit formula (6). To prove that k is continuous, one needs only show that

$$(2L+1)^{-\nu}\,\mathrm{Tr}(\chi_L P_{(E)}(H_\omega)) \to 0 \tag{10}$$

In the one-dimensional case, this is trivial since $P_{(E)}(H_\omega)$ is at most two-dimensional. In the higher dimensional case, Delyon-Souillard make use of the fact that, for projections P and Q

$$\mathrm{Tr}(PQ) \le \dim\ Q[\mathrm{Ran}\ P] \tag{11}$$

so that one needs only show that the restriction of the set of $\ell^2$-eigenfunctions to a finite box forms a space whose dimension grows at a rate small compared to the volume of the box. In fact, Delyon-Souillard obtain a bound which only grows as the surface area of the box.

It is an annoying and unfortunate fact that there is still no proof of continuity of the ids in the continuum case except in one-dimension, where the Pastur argument goes through. This is the most important open question in the study of the ids for stochastic Schrödinger operators.

## 3. Log-Hölder Continuity of k

Craig-Simon [13] proved the following extension of Theorem 2.1:

Theorem 3.1 The ids, k(E), is log-Hölder continuous for any stochastic Jacobi matrix.

A function, f, is called log-Hölder continuous if and only if there is a constant, C, so that

$$\left|f(x)-f(y)\right| \leq C(\ln\left|x-y\right|^{-1})^{-1}$$

for all x,y with $\left|x-y\right| \leq \frac{1}{2}$.

This theorem depends on the following elementary lemma:

Lemma 3.2 Let dq be a measure of compact support with distribution function q. If

$$\int \ln\left|x-y\right| dq(y) \geq 0$$

then q is log-Hölder continuous.

The idea is that one cannot lose log-Hölder continuity at a point E' without dq being so concentrated that the integral diverges to $-\infty$ at E'. Given the lemma, the Thouless formula immediately implies Theorem 3.1 in the one-dimensional case; this was already noted in [12]. This is because the Lyaponov exponent, as a limit of positive quantities (the matrix T has determinant 1, and thus norm at least 1) is positive. The general case is proven by showing that the integral is positive also in the multi-dimensional case, for the integral can be shown to be



the limit of the average of the non-negative Lyaponov exponents for strips.

There is a sense in which Theorem 3.1 is essentially optimal, for given $\varepsilon$, Craig [11] has constructed examples of almost periodic functions (actually, only in a weak sense; see Pöschel [35] for strictly almost periodic examples) for which there are points, E, with

$$\lim_{\delta\downarrow 0} \left|k(E+\delta)-k(E)\right|/[\ln \delta^{-1}]^{1+\varepsilon} = \infty$$

Moreover, we will see that there exist random potentials which yield a k which is not Hölder continuous of any prescribed strictly positive order.

## 4. The One-Dimensional Anderson Model: Positive Results

To go beyond Theorem 3.1 and find smoothness properties of k, one must be prepared to make some special hypotheses, as the discussion at the end of the last section makes clear. It is clear that one should not look for much smoothness in the case of almost periodic potentials, for it is a general phenomenon (see the discussion in section 9) that the spectra of such operators tend to be Cantor sets, that is, closed, nowhere dense sets. The corresponding k cannot be $C^1$ because its derivative is zero on the complement of the spectrum, which is dense. It is therefore natural to look at the Anderson model. In general, LePage [25,26] has proven the following result:

Theorem 4.1 The density of states, k(E), associated to any one-dimensional Anderson model, is Hölder continuous of some strictly positive order.

In the next section, we will mention examples of Anderson models whose ids fails to be Hölder continuous of any given prescribed order. Thus, one must make some additional assumptions on the input measure, $d\kappa$, in order to be certain that k has greater regularity properties. Given that the k associated to V = 0 is not $C^1$ but has a divergent first derivative at E = 2 and -2, one might naively expect that k cannot be too smooth but, in fact, the randomness is smoothing. Not only is k $C^\infty$ if $d\kappa$ is $C^\infty$, but under some minimal regularity assumptions on $d\kappa$, k is already $C^\infty$. This phenomenon was first proven to occur by Simon-Taylor [41], whose results applied to the case originally studied by Anderson, where

$$d\kappa(x) = \frac{1}{b-a} \chi_{[a,b]}(x)dx$$

Subsequently, Campinino-Klein [7] and March-Snitzman [30] proved results which complement and/or extend the results of [41]. The following is proven by Campinino-Klein:

<u>Theorem 4.2</u> Consider the one-dimensional Anderson model with input distribution $d\kappa$. Suppose that $d\kappa$ has moments of all orders, and its Fourier transform $m(p) = \int e^{-ipx} d\kappa(x)$ obeys

$$|m(p)| \le C(1+|p|)^{-\alpha}$$

for some $C, \alpha > 0$. Then the ids, k(E), is $C^\infty$.

The smoothness of k associated to random operators viz-a-viz the singularities of the free case is illuminated by the fact that the singularities in the free case are at the edge of the spectrum, where the random case has the Lifschitz tail behavior to be discussed in



Section 8.

## 5. The One-Dimensional Anderson Model: Negative Results

There is an Anderson-type model of especial interest in providing counterexamples for regularity results that one might conjecture. This is what might be called the Bernoulli-Anderson model, where the input measure, $d\kappa$, is a pure point measure with two point support, i.e.

$$d\kappa = \theta\delta_a + (1-\theta)\delta_b$$

We will call this the Bernoulli-Anderson model with parameters $a, b, \theta$. Halperin [19] studied a closely related continuum model and showed nonregularity of k. His argument can easily be made rigorous, and this was done by Simon-Taylor. The result is:

<u>Theorem 5.1</u> The Bernoulli-Anderson model with parameters $a, b, \theta$ has a k(E) which is not Hölder continuous of any order greater than

$$\alpha_0 = 2|\log(1-\theta)|/\text{Arc}\cosh(1 + \tfrac{1}{2}|a-b|)$$

Note that $\alpha_0$ goes to zero as $|a-b| \to \infty$, or as $\theta \downarrow 0$, showing that one cannot improve on Theorem 4.1 without making a restriction on $d\kappa$, which will eliminate the Bernoulli-Anderson model. The idea behind the proof is quite simple. One finds certain energies about which the finite volume eigenvalues are clumped. These are eigenvalues for the operator obtained by surrounding a finite array of a's and b's by a sea of b's. Since the corresponding eigenfunctions decay exponentially, one can show that the system in an enormous box will have one eigenvalue exponentially near the infinite volume eigenvalue for each large sub-box of the big box containing the finite array surrounded by b's.

The Bernoulli-Anderson model has evoked considerable interest in the physics and chemical physics literature. This is partly because it models a binary alloy and partly because, before the advent of high speed computers, it was about the only model where one could reasonably compute the ids numerically. There are a number of features of the ids of the model which are hinted at by numerical and theoretical studies, but not yet rigorously proven:

(1) It is likely that, for suitable values of the parameters, the Bernoulli-Anderson model has an ids, k, for which dk has a singular continuous component; see Simon-Taylor [41].

(2) Luck-Nieuwenhuizen [29] have an analysis of the structure of k(E) at the energies described above (eigenvalues for the operator obtained by surrounding a finite array of a's and b's by a sea of b's), which suggests, but does not rigorously prove, the precise nature of the singularities at these energies.

(3) There are certain "special energies" at which the density of states is supposed to vanish; see Endrullis and Englisch [17].

## 6. The Higher Dimensional Case

Much less is known about the Anderson model in dimension greater than one. All indications are that the ids gets better behaved as dimension increases, so it is not an unreasonable conjecture that, for any Anderson model in dimension greater than one, the ids is $C_\infty$. Unfortunately, all the results proven thus far have involved showing that dk is about as well behaved as the input measure $d\kappa$, and there are



no $C^\infty$ results for cases where $d\kappa$ has compact support. One of the nicest results is the following one proven by Wegner [45]:

<u>Theorem 6.1</u> Let k be the ids of an Anderson model whose input measure, $d\kappa$, is absolutely continuous with bounded Radon-Nikodym derivative. Then k is Lipschitz continuous.

In addition to the ideas of Wegner, there is an alternate proof using ideas of Simon-Wolff [42] on averages of the spectral measures under rank one perturbations.

There is also a result of Constantinescu, Fröhlich and Spencer [10] which says that if $d\kappa$ is absolutely continuous with a Radon-Nikodym derivative which is analytic in a sufficiently wide strip, then k is real analytic either in the region where $|E|$ is large or for large coupling constant.

## 7. Cauchy Models

There is one class of stochastic Jacobi matrices which is useful because one can compute the ids precisely. These are models where V has a Cauchy distribution with restrictions to be made precise on the correlations between V's at distinct sites. The first model of this type for which the ids was computed is the Anderson model with a Cauchy density for $d\kappa$, i.e.

$$d\kappa(x) = \frac{\lambda}{\pi}\frac{dx}{x^2+\lambda^2}$$

This model is known as the Lloyd model [28]. Much more recently, Grempel et al. [18] computed the ids in the almost periodic potential with

$$(12) \qquad V(n) = \lambda \tan(\pi\alpha n+\theta)$$

a model which has come to be called the Maryland model, after the place where Grempel et al. worked. They found the remarkable fact that the ids was the same in the two models for the same value of $\lambda$, and in particular, the ids in the Maryland model is independent of the frequency $\alpha$ so long as it is irrational. Motivated by this discovery, Simon [37] proved the following:

<u>Theorem 7.1</u> Let k be the ids for a stochastic Jacobi matrix whose potential has the form:

$$V(n) = \sum_{j=1}^{J} \alpha_j \tan(\beta_{j,n}+\theta)$$

where $\alpha_j$, $\beta_{j,n}$, $\theta$ are random variables with the two restrictions that $\theta$ is uniformly distributed and $\alpha_j \geq 0$, $\Sigma\alpha_j = \lambda$. Then

$$k(E) = \int \frac{\lambda}{\pi} \frac{1}{\lambda^2+(E-E')^2} k_0(E')dE'$$

where $k_0(E)$ is the ids for the free model in the corresponding dimension.

The Lloyd model corresponds to the case where $J = 1$ and the $\beta_j$ are uniformly and identically distributed, independent random variables.

Simon [38] has proven that there are some values of $\alpha$ for which the Hamiltonian corresponding to (12) has point spectrum, and other values for which the Hamiltonian has singular continuous spectrum. Since the ids in the two cases are the same, one sees that the ids cannot distinguish between point and singular continuous spectrum.



## 8. Lifschitz Tails

It is not hard to show that (see Kunz-Souillard [24]) the spectrum of a typical H for the $\nu$-dimensional Anderson model is given by

$$(13) \qquad \mathrm{spec}(H) = \mathrm{spec}(H_0) + \mathrm{supp}(d\kappa)$$

Suppose that $\mathrm{supp}(d\kappa) = [a,b]$ so that $a - 2\nu$ is the bottom of the spectrum of H and thus, by Theorem 1.1, $k(E) = 0$ for $E < a - 2\nu$. In cases where k is smooth, it must go to zero as E approaches $a - 2\nu$ from above faster than any polynomial. The rate at which it goes to zero was first determined by E.M. Lifschitz [27], so that this region is known as the Lifschitz tail. The leading behavior is given by the formula

$$(14) \qquad k(E) \sim \exp[-(E-a-2\nu)^{-\nu/2}]$$

Lifschitz provided a simple intuition about why this formula holds: For a state to have energy only $\varepsilon$ above the minimum value, both its kinetic and potential energies must be small. Since the kinetic energy of a state of extent L is of order $L^{-2}$, we must have that

$$\varepsilon \sim L^{-2}$$

For the potential energy to be of order $\varepsilon$, most of the sites in this box must have a potential value very close to the minimum value a, and this will have a small probability of order exp(-S) where S is the number of sites in the box, i.e. $S \sim L^{\nu} = \varepsilon^{-\nu/2}$.

The earliest proofs of Lifschitz's result tended to use rather sophisticated arguments from the theory of large deviations. More recently, proofs have been given closer to the spirit of Lifschitz's original arguments; these proofs exploit Dirichlet-Neumann bracketing;

see, for example, Kirsch-Martinelli [21], Simon [39] and Mezincescu [31]. (14) is proven in the sense that

(14') $$\lim_{E\downarrow a+2\nu} [\ln(E-a-2\nu)]^{-1}\ln[\ln k(E)^{-1}] = -\nu/2$$

under the requirement that $\kappa(a,a+\delta)$ doesn't vanish faster than polynomially as $\delta\downarrow 0$.

These Lifschitz tails which occur at the outer edges of the spectrum are occasionally called "external Lifschitz tails". There has also been study of the situation where $\mathrm{supp}(d\kappa) = [a,b]\cup[c,d]$ with $c - b > 4\nu$. In that case, there is a gap in the spectrum of H, and one expects that the approach of $k(E)$ to its value in the gap as E approaches the gap from within the spectrum has Lifschitz behavior. These "internal Lifschitz tails" have been proven to occur by Mezincescu [32], and subsequently by Simon [40].

This isn't the end of the story concerning Lifschitz tails: For random plus periodic potentials, Kirsch-Simon [22] have proven that there are external Lifschitz tails, but no proof of internal Lifschitz tails has been found.

## 9. Gap Labeling

There is a final aspect of the density of states special to the almost periodic case that we should mention, especially since it suggests that "normally" the ids will not be $C^1$ in these cases. The *frequency module* of an almost periodic function on $\mathbb{Z}$ is defined as follows: Any almost periodic function has an average:



$$Av(f) = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{|n|\le L} f(n)$$

The frequency module of an almost periodic function is the additive subgroup of $\mathbb{R}$ generated by 1 and those frequencies, $\alpha$, for which $Av(e^{-2\pi i\alpha n}f) \neq 0$. There is a more elegant and illuminating definition which is also longer; it is discussed, for example, in [1]. In the continuum case, the 1 is not included; it is a reflection of the periodicity of the lattice. There are definitions around which differ by factors of $\pi, 2\pi$, and 2 from the one we give here, and the fact that the gap labeling theorem we give and the one in Johnson-Moser [20] differ by a factor of 2 is resolved by differing definitions of the frequency module. Our definition is such that, if f is a periodic function of period L, its frequency module is $\{n/L \mid n\in\mathbb{Z}\}$.

If f is quasiperiodic, i.e. if

$$f(n) = F(2\pi\alpha_1 n,\ldots,2\pi\alpha_\nu n)$$

for a continuous function, F, on the $\nu$-dimensional torus, then the frequency module is always contained in the set $\{\sum_{j=0}^{\nu} n_j\alpha_j \mid n_j\in\mathbb{Z};\alpha_0=1\}$ and will equal that set if F has enough non-zero Fourier coefficients. f is quasiperiodic if and only if its frequency module is finitely generated, and it is limit periodic (i.e. a uniform limit of periodic functions) if and only if its frequency module has the property that any two elements in it have a common divisor in it.

The basic gap labeling theorem is:

<u>Theorem 9.1</u> Let H be an almost periodic one-dimensional Jacobi matrix. Then the value of the ids in any gap in the spectrum of H lies in the

frequency module.

Of course, the value also always lies in [0,1]. This phenomenon was first found by Claro and Wannier [44], and first rigorously proven in continuum models by Johnson-Moser [20], and then for discrete models by Bellissard, Lima and Testard [4]. Johnson-Moser use a homotopy argument which was carried over to the discrete case by Delyon-Souillard [15]. Bellissard, Lima and Testard use some C*-algebra techniques, and their argument has been extended to the higher dimensional case by them [3].

The relevance of the gap labeling theorem to regularity of the ids comes from the following meta-theorem:

Meta-theorem 9.2 A "generic" almost periodic one-dimensional Jacobi matrix has gaps in its spectrum where the ids takes each possible allowed value (i.e. all numbers in the frequency module which lie in (0,1)).

This result has been proven in the limit periodic case [9.33,1] and (in a weakened form) for the case where $V(n) = \lambda \cos(2\pi\alpha n+\theta)$ [5] for suitable notions of generic. If the almost periodic function is not strictly periodic, then the frequency module is dense in $\mathbb{R}$. Gaps in the spectrum are open sets on which k is constant, so on which k has a zero derivative. If every allowed value occurs in a gap, then the spectrum is a Cantor set (nowhere dense, but not necessarily of zero measure), and k is a Cantor function, which means it cannot be $C^1$ in the neighborhood of any point of the spectrum.



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References

1. Avron, J. and Simon, B., Almost periodic Schrödinger operators, I. Limit periodic potentials. Comm. Math. Phys. 82 (1982), 101-120

2. Avron, J. and Simon, B., Almost periodic Schrödinger operators, II. The integrated density of states, Duke Math. J. 50 (1983), 369-391

3. Bellissard, J., Lima, R. and Testard, D., Almost periodic Schrödinger operators, ZIF preprint

4. Bellissard, J., Lima, R. and Testard, D., A metal-insulator transition for the almost Mathieu model, Comm. Math. Phys. 88 (1983), 207-234

5. Bellissard, J. and Simon, B., Cantor spectrum for the almost Mathieu equation, J. Func. Anal. 48 (1982), 408-419

6. Benderskii, M. and Pastur, L., On the spectrum of the one-dimensional Schrödinger equation with a random potential, Math. USSR Sb. 11 (1970), 245

7. Campanino, M. and Klein, A., A supersymmetric transfer matrix and differentiability of the density of states in the one-dimensional Anderson model, Comm. Math. Phys., to appear

8. Carmona, R., Lectures on random Schrödinger operators, to appear in Proc. 14th St. Flour Probability Summer School (1985), Springer

9. Chulaevsky, V., On perturbations of a Schrödinger operator with periodic potential, Russian Math. Surveys 36 (1981), 143

10. Constantinescu, F., Fröhlich, J. and Spencer, T., Analyticity of the density of states and replica method for random Schrödinger operators on a lattice, J. Stat. Phys. 34 (1984), 571-596

11. Craig, W., Pure point spectrum for discrete almost periodic Schrödinger operators, Comm. Math. Phys. 88 (1983), 113-131

12. Craig, W. and Simon, B., Subharmonicity of the Lyaponov index,

Duke Math. J. 50 (1983), 551-560

13. Craig, W. and Simon, B., Log Hölder continuity of the integrated density of states for stochastic Jacobi matrices, Comm. Math. Phys. 90 (1983), 207-218

14. Cycon, H., Froese, R., Kirsch, W. and Simon, B., *Topics in the theory of Schrödinger operators*, to appear

15. Delyon, F. and Souillard, B., The rotation number for finite difference operators and its properties, Comm. Math. Phys. 89 (1983), 415

16. Delyon, F. and Souillard, B., Remark on the continuity of the density of states of ergodic finite difference operators, Comm. Math. Phys. 94 (1984), 289

17. Endrullis, M. and Englisch, H., Special energies and special frequencies, preprint

18. Grempel, D., Fishman, S. and Prange, R., Localization in an incommensurate potential: An exactly solvable model, Phys. Rev. Lett. 49 (1982), 833

19. Halperin, B., Properties of a particle in a one-dimensional random potential, Adv. Chem. Phys. 31 (1967), 123-177

20. Johnson, R. and Moser, J., The rotation number for almost periodic potentials, Comm. Math. Phys. 84 (1982), 403

21. Kirsch, W. and Martinelli, F., Large deviations and Lifshitz singularity of the integrated density of states of random Hamiltonians, Comm. Math. Phys. 89 (1983), 27-40

22. Kirsch, W. and Simon, B., Lifshitz tails for periodic plus random potentials, J. Stat. Phys., to appear

23. Kotani, S., Ljaponov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators, Stochastic analysis (1984), K. Ito (ed.), North Holland, pp. 225-248

24. Kunz, H. and Souillard, B., Sur le spectre des operateurs aux differences finies aleatoires, Comm. Math. Phys. 78 (1980), 201-246

25. LePage, E., Empirical distribution of the eigenvalues of a Jacobi matrix, Probability measures on groups, VII, Springer Lecture Notes Series 1064, 1983, pp. 309-367



26. LePage, E., in prep.

27. Lifschitz, I., Energy spectrum structure and quantum states of disordered condensed systems, Sov. Phys. Usp. 7 (1965), 549

28. Lloyd, P., Exactly solvable model of electronic states in a three-dimensional disordered Hamiltonian: Non-existence of localized states, J. Phys. C2 (1969), 1717-1725

29. Luck, J. and Nieuwenhuizen, T., Singular behavior of the density of states and the Lyaponov coefficient in binary random harmonic chains, J. Stat. Phys., to appear

30. March, P. and Snitzman, A., Some connections between excursion theory and the discrete random Schrödinger equation, with applications to analyticity and smoothness properties of the density of states in one dimension, Courant preprint

31. Mezincescu, G., Bounds on the integrated density of electronic states for disordered Hamiltonians, Phys. Rev., to appear

32. Mezincescu, G., Internal Lifschitz singularities of disordered finite-difference Schrödinger operators, Comm. Math. Phys., to appear

33. Moser, J., An example of a Schrödinger equation with an almost periodic potential and nowhere dense spectrum, Comm. Math. Helv. 56 (1981), 198

34. Pastur, L., Spectral properties of disordered systems in one-body approximation, Comm. Math. Phys. 75 (1980), 179

35. Poschel, J., Examples of discrete Schrödinger operators with pure point spectrum, Comm. Math. Phys. 88 (1983), 447-463

36. Simon, B., Kotani theory for one-dimensional stochastic Jacobi matrices, Comm. Math. Phys. 89 (1983), 227

37. Simon, B., The equality of the density of states in a wide class of tight binding Lorentzian models, Phys. Rev. B27 (1983), 3859-3860

38. Simon, B., Almost periodic Schrödinger operators, IV. The Maryland model, Ann. Phys. 159 (1985), 157-183

39. Simon, B., Lifschitz tails for the Anderson model, J. Stat. Phys. 38 (1985), 65-76

40. Simon, B., Internal Lifschitz singularities in the Anderson model, in prep.

41. Simon, B. and Taylor, M., Harmonic analysis on SL(2,R) and smoothness of the density of states in the one dimensional Anderson model, Comm. Math. Phys., to appear

42. Simon, B. and Wolff, T., Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians, Comm. Pure Appl. Math., to appear

43. Spencer, T., The Schrödinger equation with a random potential - a mathematical review, to appear in Proc. 1984 Les Houches Summer School

44. Wannier, G. and Claro, F., Magnetic subband structure of electrons in hexagonal lattices, Phys. Rev. B19 (1979), 6068

45. Wegner, F., Bounds on the density of states in disordered systems, Z. Phys. B44 (1981), 9-15.