Hypercontractivity: A Bibliographic Review

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§1. Overview and Self-Adjointness

For the past twenty years, a set of ideas known as hypercontractivity has played a continuing role in analysis with ramifications in quantum field theory, self-adjointness of Schrödinger operators, best constants in classical inequalities and bounds on semigroup kernels. A significant role was played by a paper of Raphael Hoegh-Krohn and one of us [98] which codified previous work and coined the term "hypercontractive." Our goal here is to give a brief historical review and a rather complete bibliography.

In looking at the history, one must bear in mind that for several papers there were lengthy delays between submission and publication, roughly two years for Simon and Hoegh-Krohn [98] and Gross [48].

The theory of hypercontractive semigroups was introduced in a fundamental paper of Nelson [74], who also discovered the simplest and most basic example.

Definition Let (Ω,μ_0) be a probability measure space. $H_0 \geq 0$ is a self-adjoint operator on $L^2(\Omega,\mu)$. We say that e^{-tH_0} $(t \geq 0)$ is a hypercontractive semigroup if and only if:

(a) e^{-tH_0} is a contraction on $L^2(\Omega,\mu_0)$ for all $t>0$.

(b) For some T, e^{-TH_0} is a bounded map from $L^2(\Omega,\mu_0)$ to $L^4(\Omega,\mu_0)$.

General principles (interpolation and duality) imply that e^{-tH_0} is then a contraction from any L^p to itself and bounded from any L^p to any $L^q(1<p,q<\infty)$ if t is sufficiently large (depending on p,q).

Example (Nelson[74]) Let $H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 - \frac{1}{2}$, the harmonic oscillator on $L^2(\mathbb{R}, dx)$. Let $\Omega_0 = (\pi)^{-1/4} \exp(-\frac{1}{2}x^2)$ be the unique positive unit vector with $H\Omega_0 = 0$ and consider

$H_0 = \Omega_0^{-1}\ H\Omega_0 = \frac{1}{2}\left(-\frac{d}{dx^2} + 2x\frac{d}{dx}\right)$. Then e^{-tH_0} is hypercontractive on $L^2(\mathbb{R}, \Omega_0^2\ dx)$.

Nelson applied this result to deduce semiboundedness of certain cutoff quantum field Hamiltonians. Glimm [46] made an important observation: Suppose that e^{-tH_0} is hypercontractive. If $H_0 1 = 0$, zero is a simple eigenvalue and for some $m > 0$: $\sigma(H_0) \subset \{0\} \cup [m, \infty)$, then e^{-TH_0} is actually a contraction from L^2 to L^4 for T sufficiently large. This allows one to extend Nelson's example to the second quantization of any strictly positive operator (see [98]).

Nelson implemented his semiboundedness proof by making extensive use of path integrals. An alternative prooi of semiboundedness was given by P. Federbush [44] based on differentiating the already established hypercontractive inequality $\|e^{-tH}f\|_{p(t)} \leq \|f\|_2$ with respect to t at $t = 0$. This yielded the first log Sobolev inequalities, providing a precursor to the work we will discuss in §2.

Segal [90,91,92] studied an abstract version of the theory, and in particular showed that essential self-adjointness followed from the same L^p properties that Nelson used. Simon - Hoegh-Krohn [98] codified and extended this work, and, in particular show that :

<u>Theorem 1</u>: Suppose that e^{-tH_0} is a hypercontractive semigroup on $L^2(\Omega,\mu)$. Let V be a function on Ω (and also the associated multiplication operator). Suppose that $e^{-V} \in \bigcap_{p<\infty} L^p$ and $V\in L^p$ for some $p>2$. Then $H_0 + V$ is bounded from below and essentially self-adjoint on $D(H_0) \cap D(V)$. The same result is true if $V \geq 0$ and $V \in L^2$.

At the time of this work, self-adjointness of the spatially cut off quantum field Hamiltonian was important in the construction of infinite volume asymptotics of the dynamics. It was first proven for $(\phi^4)_2$ field theories by Glimm-Jaffe [47] using different methods and then by Rosen [83] and Segal [91] for general $P(\phi)_2$ theories. Segal used variants of the above theorem. Hoegh-Krohn [54] applied it to the $:\exp(\alpha\phi):$ interaction.

A second example of some historical significance is in the Simon-Hoegh-Krohn paper [98]. Let $V \geq 0$ be in $L^2(\mathbb{R}^d, e^{-x^2}dx)$. Then, by the above theorem and a small additional argument, $-\Delta + x^2 + V(x)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$. This is of interest because prior to this, all such theorems had required V to be locally L^p with $p \geq d/2$ if $d \geq 4$. Simon [96] showed how to get rid of the x^2 and prove $-\Delta + V$ essentially self adjoint on C_0^∞ if $V \geq 0$ in $L^2(\mathbb{R}^d, e^{-ax^2}dx)$ some $a>0$. Motivated by this, in a celebrated work, Kato [62], using different methods, showed that $V \in L^2_{loc}(\mathbb{R}^d)$ suffices if $V \geq 0$.

It is interesting that the self-adjointness result is now mainly of historical interest. Kato's work has replaced any application to Schrödinger operators. And, because of the Euclidean revolution in quantum field theory, self-adjointness became irrelevant, although the L^p estimates associated to hypercontractivity are still significant; see Guerra et al. [52,53].

§2 Logarithmic Sobolev Inequalities and hypercontactivity

Some aspects of these concepts are most easily understood in finite dimensions. Let $d\mu(x) = (2\pi)^{-n/2} \exp[-\|x\|^2/2]dx$ denote Gauss measure on $\mathbb{R}^n$. The inequality

$$\int_{\mathbb{R}^n} |f(x)|^2 \ell n|f(x)| d\mu(x) \leq c \int_{\mathbb{R}^n} |\nabla f(x)|^2 d\mu(x) + \|f\|^2_{L^2(\mu)} \ell n\|f\|_{L^2(\mu)} \tag{2.1}$$

is the prototype of logarithmic Sobolev inequalities. If one defines an operator N on $L^2(\mathbb{R}^n, \mu)$ by $(Nf,g)_{L^2(\mu)} = \int_{\mathbb{R}^n} \nabla f(x) \cdot \nabla \bar{g}(x) d\mu(x)$ then (2.1) reads

$$\int_{\mathbb{R}^n} |f(x)|^2 \ell n|f(x)| d\mu(x) \leq c(Nf,f)_{L^2(\mu)} + \|f\|^2_{L^2(\mu)} \ell n\|f\|_{L^2(\mu)}. \tag{2.2}$$

In P. Federbush's proof [44] of semiboundedness of H_0+V, he first differentiated the hypercontractivity inequality for e^{-tH_0} at $t=0$ to obtain the inequality (2.2) for H_0 (with $\mathbb{R}^n$ replaced by an infinite dimensional space and N replaced by H_0). He then showed that the inequality (2.2) was itself sufficient to prove semiboundedness. L. Gross [48] later showed that, conversely, one could recover hypercontractivity of e^{-tN} from (2.1) so that hypercontractivity and logarithmic Sobolev inequalities were actually equivalent for Dirichlet form operators (such as N). The techniques in Gross' argument for the direction "log. Sobolev for H_0 implies bounds on e^{-tH_0}" has yielded tremendous advances recently in the understanding of heat kernels. This will be discussed in the next section. A direct proof of (2.1) with the best constant $c = 1$ was also given by Gross [48] using a central limit theorem argument applied to a logarithmic Sobolev-like inequality for an operator on L^2 of a two point measure space. The "two point inequality" was an outgrowth of his earlier work [49] on hypercontractivity for Fermions but was also anticipated by Bonami [14] in his work on harmonic analysis on finite groups.

There are now many proofs of these two types of inequalities for Gauss measure. Some prove hypercontractivity directly [9, 10, 19, 46, 74, 75, 76, 77, 91] while others prove the logarithmic Sobolev inequality directly [2, 5, 13, 43, 48, 84]. The most elementary direct proof of hypercontractivity with best constants is that of E. Nelson [76] while the most elementary and simplest direct proof of the logarithmic Sobolev inequality with best constants ($c = 1$) is is that of O. Rothaus [84].

Both kinds of inequalities were developed in various directions in the 1970's. Inequality (2.2) can be interpreted as saying that $(N + 1)^{-1/2}$ is a bounded operator from $L^2(\mu)$ to the Orlicz space $L^2 \ell n L$. G. Feissner showed more generally that $(N + 1)^{-k/2}$ is a bounded operator from $L^p(\mu)$ to $L^p \ell n^k L$. See also [8]. Furthermore one may ask whether, given a measure ν on $\mathbb{R}^n$ with a reasonable density, its Dirichlet form operator satisfies a logarithmic Sobolev inequality. There is a procedure by which Dirichlet form operators arise naturally in quantum mechanics and quantum field theory; if V is a suitable real valued function on $\mathbb{R}^n$ then the operator $H := -\Delta + V$ is a self-adjoint operator with a unique lowest eigenvector ψ of unit norm which may be taken strictly positive. If $d\nu(x) = \psi(x)^2 dx$ and $\lambda = \inf \text{spectrum } H$, then the unitary operator $U : f \longrightarrow f/\psi$ from $L^2(\mathbb{R}^n, dx)$ to $L^2(\mathbb{R}^n, d\nu)$ converts $(H - \lambda)$ into an operator $\hat{H} := U(H - \lambda)U^{-1}$ on $L^2(\mathbb{R}^n, d\nu)$ which turns out to be a Dirichlet operator for ν [40]. Hence, by Gross' theorem, hypercontractivity and the logarithmic Sobolev inequality are equivalent for $\hat{H}$. Conditions on V which assure that both hold were obtained by J. P. Eckmann [40], R. Carmona [25], J. Rosen [82], J. Hooton [58]. Moreover J. Rosen [82] showed that if the density of ν decreases very quickly at ∞, then for $1 < p < \infty$, $\|e^{-t\hat{H}}\|_{L^2 \to L^p} < \infty$ for all $t > 0$. This is strictly stronger than hypercontractivity and was called supercontractivity. For a review of studies of $\hat{H}$, see B. Simon [94]. An even stronger notion, ultracontractivity, will be discussed in §3.

An application of hypercontractive ideas in yet another direction was made by W.

Beckner [10] who used extensions of the above-mentioned two point inequality to get the exact bounds in the Hausdorff-Young inequality. The study of e^{-zN}, for complex z, from the point of view of hypercontractivity has recently been completed in a definitive manner by J. Epperson [41]. One may ask whether the Laplace-Beltrami operator on a manifold other than $\mathbb{R}^n$ generates a logarithmic Sobolev inequality. This has been addressed in a number of works [32, 33, 34, 35, 87, 89] with startlingly complete results for S^n in [106]. Finally we mention that applications of logarithmic Sobolev inequalities in infinite dimensions to statistical mechanics were made by Holley and Stroock [55, 56, 57]. The bibliography contains many more works which touch on hypercontractivity or logarithmic Sobolev inequalities in one way or another and not described here or below.

§3 Ultracontractivity

It was shown in [28] that if $H = -\frac{d^2}{dx^2} + V$ on $L^2(a, b)$ where $V \in L^1$ and $-\infty < a < b < \infty$, then the semigroup $e^{-t\hat{H}}$ is bounded from $L^2(d\nu)$ to $L^\infty(d\nu)$ for all $t > 0$. This property is called ultracontractivity and is equivalent to $e^{-t\hat{H}}$ having a pointwise bounded integral kernel $K(t, x, y)$ for all $t > 0$.

A more general investigation of such inequalities from the point of view of logarithmic Sobolev inequalities was initiated by Davies and Simon in [38]. If (Ω, μ) is a measure space then there is a very close relationship between bounds of the form

$$\| e^{-tH} \|_{L^2 \to L^\infty} \leq c(t) < \infty \tag{3.1}$$

for all $t > 0$, and

$$\int_\Omega |f|^2 \ln |f| \, d\mu \leq \epsilon \, \| H^{1/2} f \|_2^2 + \beta(\epsilon) \, \| f \|_2^2 + \| f \|_2^2 \log \| f \|_2 \tag{3.2}$$

for all $f \in \mathrm{Dom}\,(H^{1/2})$ and all $\epsilon > 0$. In particular one has an equivalence between (3.1) and (3.2) in the case where

$$c(t) = c\,t^{-N/4} \tag{3.3}$$

and

$$\beta(\epsilon) = a - \frac{N}{4} \ln \epsilon.$$

Although theorems of this type are not applicable to the harmonic oscillator, they are important for some other Schrödinger operators $H = -\Delta + V$ on $L^2(\mathbb{R}^n)$, For example if $\lambda > 0$ and

$$V(x) = (a^2 + |x|^2)^\lambda$$

then $e^{-t\hat{H}}$ is ultracontractive if and only if $\lambda > 1$, with constants $c(t)$ which diverge as $t \to 0$ much more rapidly than (3.3). The harmonic oscillator therefore stands at the borderline between ultracontractive behavior and the absence of any $L^2 \to L^p$ smoothing properties.

It was subsequently shown by Varopoulos [102] and Fabes and Stroock [42] that less singular problems of this type, for which (3.3) is valid, could be handled by the use of ordinary Sobolev inequalities or what were called Nash inequalities. We remark that the paper of Nash [73] involves "entropy inequalities" which antedate the introduction of logarithmic Sobolev inequalities by a decade.

The paper [38] has spawned a substantial literature concerning pointwise bounds on the heat kernels of second order elliptic operators in divergence form; see[34] for a comprehensive survey. One can easily handle uniformly elliptic operators with measurable coefficients on $\mathbb{R}^n$ and on regions in $\mathbb{R}^n$ subject to Dirichlet or Neumann boundary conditions. The use of logarithmic Sobolev inequalities also allows one to obtain pointwise bounds on the heat kernels of many singular elliptic operators of second order [32,78].

When one turns to the study of the Laplace-Beltrami operator on a Riemannian manifold, many different techniques can be used to obtain pointwise bounds on the heat kernel [34]. Some [26, 71, 36] make little or no use of logarithmic Sobolev inequalities, while others [32, 33, 35] depend essentially upon such a use.

Suppose one has a uniform bound on the heat kernel of the form (3.1), or equivalently

$$0 \leq K(t, x, y) \leq a(t) < \infty.$$

By proving a logarithmic Sobolev inequality for the non-self-adjoint operator $\varphi\, e^{-Ht}\, \varphi^{-1}$, where φ is a suitable weight, it is often possible to show that

$$0 \leq K(t, x, y) \leq c_\delta(t) \exp\left[-d(x, y)^2/(4 + \delta)\, t\right] \tag{3.4}$$

for any $\delta > 0$, where $d(x, y)$ is a Riemannian metric constructed directly from the operator

[30, 33, 35]. See also [31, 42, 21, 34], and [33] for an analogous result for certain second order hypoelliptic operators. By comparison with earlier literature [3, 81] the advantage of (3.4) is that one has obtained the sharp constant 4 in the exponential.

The possibility of obtaining sharp constants is a recurring feature of the use of logarithmic Sobolev inequalities, and demonstrates their deep significance. Many developments of the above applications are being currently investigated, and one particularly looks forward to the proof of lower bounds on heat kernels which are of comparable accuracy to the upper bounds mentioned above.

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