Topics in Functional Analysis

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INTRODUCTION

The subject of "functional analysis" spans an extremely large amount of mathematics from results as "soft" as the characterization of some category of locally convex spaces to results as "hard" as some of the detailed estimates in the theory of singular integral transformations. In these lectures, we focus on the area of greatest importance in mathematical physics: the study of self-adjoint operators and algebras on a separable Hilbert space.

Most of what we discuss is sufficiently elementary to be included in "a standard text", but much of it is not in any of the standard texts which tend to be written with a bias towards partial differential equations or towards general operator theory in general sorts of spaces.

Some of the choices of proofs and orderings of theorems were arrived at in discussion with Mike Reed. He and I have written a functional analysis text with a bias towards mathematical physics (Reed and Simon, Vol. I, 1972; In press, Vols II, III). When it appears, it will be a suitable reference for the reader who wishes to delve further into the mysteries which will unfold. I hope that no reader is offended by the above "plug".

In Section 1, we will discuss self-adjoint operators and in particular prove a variety of self-adjointness criteria of use in mathematical physics. In Section 2, we discuss the various decompositions of the spectra of self-adjoint operators and summarize what is known about the spectrum of some dynamical operators arising in physical situations. Section 3 deals with semi-bounded quadratic forms and their relation to self-adjoint operators. Sections 4, 5, 6, 7 deal with subjects in the theory of operator algebras: the Gel'fand theory of Banach algebras, the GNS construction, some simple facts about von Neumann algebras and some theory of the CAR and CCR.

Throughout, all our Hilbert spaces will be separable unless otherwise indicated. Many of the results extend to non-separable spaces, but we cannot be bothered with such obscurities. We use $\langle .,. \rangle$, for ordered pair and $(.,.)$ for inner product. Our inner product is linear in the *second* factor. If A and B are subsets of a set $X, A\backslash B = \{x \in A \mid x \notin B\}$. A "subspace" of Hilbert space need not be closed.

We have decided to emphasize the main ideas rather than the technical details. Some of our proofs thereby tend to have a surreal aspect.

1. OPERATOR THEORY

Since we suppose the reader has some familiarity with the theory of unbounded operators, we will review the basic definitions without commentary.

Definition. An **operator,** A, is a linear transformation from a subspace, $D(A)$, of a Hilbert space, $\mathscr{H}$, into $\mathscr{H}$.

$D(A)$, the **domain** of A, may not be closed, but we will suppose it dense unless otherwise indicated.

Definition. The **graph,** $\Gamma(A)$, of an operator, A, is the subset of $\mathscr{H} \times \mathscr{H}$ given by

$$\Gamma(A) = \{\langle \phi, A\phi \rangle \mid \phi \in D(A)\}.$$

We express many operator-theoretic definitions in terms of graphs. It is useful to translate the notions back to primitive non-graph language.

Definition. We call B an **extension** of A and write $A \subset B$ if $\Gamma(A) \subset \Gamma(B)$. $\mathscr{H} \times \mathscr{H}$ is a Hilbert space when given the inner product

$$(\langle \phi, \psi \rangle, \langle \eta, \chi \rangle) = (\phi, \eta) + (\psi, \chi).$$

Definition. A is called a **closed operator** if $\Gamma(A)$ is closed. A is **closable** if it has closed extensions.

A is closable if and only if $\overline{\Gamma(A)}$ is the graph of an operator, i.e. if $\langle 0, \psi \rangle \in \overline{\Gamma(A)}$ implies $\psi = 0$. In that case, we define the **closure** of A, written $\bar{A}$, by $\Gamma(\bar{A}) = \overline{\Gamma(A)}$.

Definition. Let A be a (densely defined) operator. $D(A^*)$ is the set of vectors, ψ, in $\mathscr{H}$ for which there is a C so that

$$|(\psi, A\phi)| \leqslant C\|\phi\|$$

for all $\phi \in D(A)$. If $\psi \in D(A^*)$, there exists (by the Reisz lemma) a unique vector $A^*\psi \in \mathscr{H}$ with $(A^*\psi, \phi) = (\psi, A\phi)$ for all $\phi \in D(A)$. The (not necessarily densely defined) operator A^* is called the **adjoint** of A.

THEOREM 1.1. *A is closable if and only if A^* is densely defined. In that case $\bar{A} = A^{**}$.*

Proof. Let $V: \mathscr{H} \times \mathscr{H} \to \mathscr{H} \times \mathscr{H}$ by $V\langle\psi, \phi\rangle = \langle -\phi, \psi\rangle$. The key idea of the proof is that $\Gamma(A^*) = V\Gamma(A)^{\perp}$. Since V is unitary, $^{\perp}$ and V commute. If A^* is densely defined, it follows that $\Gamma(A^{**}) = [-\Gamma(A)]^{\perp\perp} = \overline{\Gamma(A)}$. That $\overline{\Gamma(A)}$ is not the graph of an operator if $D(A^*)$ is not dense is easy (it turns out that $D(A^*)^{\perp} = \{\psi \mid \langle 0, \psi\rangle \in \Gamma(A)\}$). ∎

Notice that this proof implies that A^* is always closed.

Definition. An operator A is called

(i) **symmetric** if $A \subset A^*$,

(ii) **self-adjoint** if $A = A^*$,

(iii) **essentially self-adjoint** if $A^* = A^{**}$.

Remarks 1. Since A^* is closed, $A^* = A^{***}$ so A is essentially self-adjoint if and only if $\bar{A}$ is self-adjoint.

2. If A is self-adjoint and B is symmetric with $A \subset B$, then $B = A$ by the following abstract nonsense: $A = A^*, B \subset B^*, A \subset B$. The last implies $B^* \subset A^* = A$. Thus $B = A$.

3. By Remark 2, if A is essentially self-adjoint, it has exactly one self-adjoint extension. We will later see the converse of this is true, i.e. an operator with exactly one self-adjoint extension is essentially self-adjoint.

4. The distinction between self-adjoint and symmetric is best illustrated by classical boundary value problems. The reader unfamiliar with this example is referred to Wightman's Cargèse lectures (Wightman, 1965) for the basic notion and to any standard text: e.g. Coddington and Levinson (1955) for the full theory.

5. In quantum mechanics, self-adjoint operators arise in two distinct ways. First, observables must be self-adjoint. And dynamics is associated with a self-adjoint operator by the following theorem which we state without proof:

THEOREM 1.2 (Stone's theorem). *Let $U(t)$ be a one parameter strongly continuous unitary group, i.e.*

(1) *$U(t)$, a unitary operator is given for each $t \in \mathbf{R}$,*

(2) *$U(t+s) = U(t)U(s)$,*

(3) *$t \to U(t)$ is strongly continuous.*

Then H defined on

$$D(H) = \left\{ \psi \,\middle|\, \lim_{t\to 0} \frac{U(t)-1}{t} \psi \text{ exists} \right\} \text{ by } H\psi = i \lim_{t\to 0} \frac{U(t)-1}{t} \psi$$

is self-adjoint and $U(t) = e^{-iHt}$. H is called the **infinitesimal generator** *of $U(t)$.*

The definition of e^{-iHt} requires the spectral theorem which we discuss in Section 2. Thus $\psi(t) = U(t)\psi$ solves Schrödinger's equation, $i\dot{\psi} = H\psi$. Typically, H is given as a formal expression on some reasonable domain and the first goal of a mathematical physicist is to establish the self-adjointness of H (or the essentially self-adjointness which means that H determines a self-adjoint operator uniquely). Our goal in the rest of this section is to discuss a variety of self-adjointness criteria. Most are based on the following fundamental criterion:

THEOREM 1.3 (Fundamental criterion). *An operator A is essentially self-adjoint if and only if A is symmetric and $A^*\psi = \pm i\psi$ only has the solutions $\psi = 0$.*

Proof. (1) Since $A^* = (\bar{A})^*$, it is enough to show that a closed symmetric operator is self-adjoint if and only if $A^*\psi = \pm i\psi$ has no solutions.

(2) If $A = A^*$ and $A^*\psi = +i\psi$, then

$$\begin{aligned}\|\psi\|^2 &= i\langle i\psi, \psi\rangle = i\langle A\psi, \psi\rangle = i\langle \psi, A\psi\rangle \\ &= i\langle \psi, i\psi\rangle = -\|\psi\|^2\end{aligned}$$

so $\psi = 0$. Thus self-adjointness implies that $A^*\psi = \pm i\psi$ has no solutions.

(3) Conversely let $A^*\psi = \pm i\psi$ have no solutions and let A be a closed

and symmetric. First notice that $\operatorname{Ker}(A^* \mp i) = \operatorname{Ran}(A \pm i)^{\perp}$ so the basic assumption implies $A \pm i$ have dense ranges.

(4) $\|(A+i)\psi\|^2 = \|A\psi\|^2 + \|\psi\|^2$ since A is symmetric. It is a simple exercise to show that this equality and the fact that A is closed implies that $\operatorname{Ran}(A+i)$ is closed. Thus (3) and (4) imply $\operatorname{Ran}(A+i) = \mathscr{H}$ if A is closed symmetric operator with $\operatorname{Ker}(A^* - i) = \{0\}$.

(5) Now let $\psi \in D(A^*)$. Since $\operatorname{Ran}(A+i) = \mathscr{H}$, we can find $\eta \in D(A)$ with $(A^* + i)\psi = (A+i)\eta$. Thus $\psi - \eta \in \operatorname{Ker}(A^* + i) = \{0\}$ so $\psi = \eta \in D(A)$. Thus $D(A^*) \subset D(A)$. Since $A \subset A^*$, $A = A^*$.∎

We emphasize point (3) of the proof. If A is symmetric (and not necessary closed) $\operatorname{Ker}(A^* \pm i) = \{0\}$ if and only if $\operatorname{Ran}(A \mp i)$ are dense. Before our first application of Theorem 1.3, we need a definition.

Definition. $M \subset D(A)$ is called a **core** for A if M is a subspace and $\overline{A \restriction M} = \bar{A}$. In particular, if A is self-adjoint, M is a core for A if and only if $A \restriction M$ is essentially self-adjoint.

THEOREM 1.4 (due to Nelson (1959)). *Let $U(t)$ be a strongly continuous one-parameter unitary group with infinitesimal generator H. Let M be a dense subspace of $\mathscr{H}$ such that*

(i) *$U(t)M \subset M$ for all t,*

(ii) *$M \subset D(H)$, i.e. $\lim_{t \to 0} \dfrac{U(t)\psi - \psi}{t}$ exists for all $\psi \in M$.*

Then M is a core for H.

Proof. Let $B = H \restriction M$. We need only prove that $B^*\psi = \pm i\psi$ implies $\psi = 0$. So suppose $B^*\psi = i\psi$. Let $\phi \in M$ and let $f(t) = (\psi, U(t)\phi)$. Then (using (i), (ii))

$$\begin{aligned} \dot{f}(t) &= (\psi, -iH\,U(t)\,\phi) = -i(B^*\psi, U(t)\,\phi) \\ &= -(\psi, U(t)\phi) = -f(t) \end{aligned}$$

Thus $f(t) = e^{-t}(\psi, \phi)$. But $|f(t)| \leqslant \|\phi\|\,\|\psi\|$. To avoid a contradiction as $t \to -\infty$, we must have $(\psi, \phi) = 0$, i.e. $\psi \in M^{\perp} = \{0\}$. Similarly, $B^*\psi = -i\psi$ has no solutions. ∎

Applications

1. In the proof of Theorem 1.4, we did not use the self-adjointness of H to prove that B was essentially self-adjoint. In fact, the method of proving

Theorem 1.4 can be used to prove Stone's theorem. One takes for M the Gårding domain for $U(t)$, i.e. finite linear combinations of $\int f(t)\, U(t)\psi dt$ where ψ is an arbitrary vector in $\mathscr{H}$ and f an arbitrary element of C_0^∞, the C^∞ functions of compact support. For details, see Reed and Simon (In press, Vol. I).

2. (Hunziker (1968a) Theorem 1.4 can be used to establish the essential self-adjointness of certain *Liouville operators* of classical mechanics. Let Γ be the phase space a classical mechanical system with Hamiltonian, H. The Liouville operator is the operator, L, given by:

$$Lf = \sum_i \left(\frac{\partial H}{\partial p_i} \frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial f}{\partial p_i} \right).$$

We wish to prove that iL is self-adjoint on suitable domains in $\mathscr{H} = L^2(\Gamma, d^n p\, d^n q)$ in some cases. Let $\Gamma = \mathbf{R}^6$ and $H = p^2 + V$ where $V \in C_0^\infty$. The theory of ordinary differential equations assures us the existence of global solutions of Hamilton's equations: $p(t), q(t)$ solving

$$p(0) = p_0; \quad q(0) = q_0; \quad \dot{p} = -\frac{\partial H}{\partial q}; \quad \dot{q} = \frac{\partial H}{\partial p}.$$

Define

$$\omega_t : \mathbf{R}^6 \to \mathbf{R}^6 \quad \text{by} \quad \omega_t(p_0, q_0) = (p(t), q(t)) \quad \text{and} \quad U(t): \mathscr{H} \to \mathscr{H}$$

by

$$(U(t)f)(p,q) = f(\omega_t(p,q)).$$

By Liouville's theorem, $U(t)$ is unitary and formally

$$\lim_{t \to 0} \frac{U(t) - 1}{t} \psi = L\psi.$$

Let $M = C_0^\infty(\mathbf{R}^6)$. Then $U(t)M \subset M$ and if $\psi \in M$,

$$\frac{U(t) - 1}{t} \psi$$

has a limit equal to $L\psi$ as $t \to 0$. It follows that iL is essentially self-adjoint on $M = C_0^\infty(\mathbf{R}^6)$. A similar argument works for smooth potentials which are bounded at ∞ and in the n-body case.

We need one last general result before turning to Nelson's and Kato's criteria.

THEOREM 1.5. *Let A be a symmetric operator and let $d_\pm = \dim \operatorname{Ker}(A^* \pm i)$. Then A has self-adjoint extensions if and only if $d_+ = d_-$.*

Proof. First suppose that $d_+ = d_-$. Find a unitary operator

$$U : \operatorname{Ker}(A^* + i) \to \operatorname{Ker}(A^* - i).$$

Let

$$D(B) = \{\phi + \psi + U\psi \mid \phi \in D(A), \psi \in \operatorname{Ker}(A^* + i)\}$$

and define

$$B(\phi + \psi + U\psi) = A\phi - i\psi + iU\psi.$$

It is not hard to prove that B is well-defined and essentially self-adjoint.

If $d_+ \neq d_-$, supgose without loss that $d_+ < d_-$. Also suppose that A is closed. If B is any symmetric extension of A, one can show that

$$d = \dim\big(D(B)/D(A)\big) \leqslant d_+ \quad \text{and} \quad \dim\big(\operatorname{Ker}(B^* - i)\big) = d_- - d > 0$$

so B is not self-adjoint. ∎

Remarks. 1. As a consequence of the proof we see that if $d_+ = d_- \neq 0$, A has several self-adjoint extensions (pick distinct unitaries U!). It turns out that all the self-adjoint extensions come from unitaries in this manner.

2. Remark 1 and the theorem tell us if $d_+ \neq 0$ or $d_- \neq 0$ then A has none or more than 1 self-adjoint extension. By Theorem 1.3, we see that A is essentially self-adjoint if it only has exactly one self-adjoint extension.

3. An important corollary of this theorem is:

THEOREM 1.6 (von Neumann). *A map $C : \mathscr{H} \to \mathscr{H}$ is called a* **complex conjugation** *if it is conjugate linear, isometric and involutive ($C^2 = 1$). If A is a symmetric operator and there is a complex conjugation so that $CA = AC$, then A has self-adjoint extensions.*

Proof. $A^*\psi = i\psi$ if and only if $A^*(C\psi) = -i(C\psi)$. Thus C is a *bijection* of $\operatorname{Ker}(A^* - i)$ and $\operatorname{Ker}(A^* + i)$. Thus $d_+ = d_-$. ∎

Application

Let $V \in L^2(\mathbf{R}^n)$, real-valued; then $-\Delta + V$ defined on C_0^∞ has self-adjoint extensions. In the case $n = 3$, we will improve this considerably.

To state and prove Nelson's criterion, we will need two special kinds of vectors.

Definition. ψ is called an **analytic vector for** A, a given operator, if $\psi \in D(A^n)$ for all n and

$$\sum_{n=0}^{\infty} \frac{\|A^n \psi\|}{n!} t^n < \infty \quad \text{for some} \quad t > 0.$$

Definition. Let A be a symmetric operator. Suppose that

$$\psi \in C_0^{\infty}(A) \equiv \cap D^n(A).$$

Let

$$D_\psi = \left\{ \sum_{n=0}^{N} a_n A^n \psi \right\} \quad \text{and let} \quad \mathscr{H}_\psi = \overline{D_\psi}.$$

Define the operator A_ψ on $\mathscr{H}_\psi$ by $A_\psi = A \restriction D_\psi$. ψ is called a **vector of uniqueness** if A_ψ is essentially self-adjoint.

We are heading towards a theorem of Nelson which tells us that a symmetric operator with a dense set of analytic vectors is essentially self-adjoint (Nelson, 1959). Our proof is patterned after one given by Nussbaum (1965).

NUSSBAUM'S LEMMA. *Let A be a symmetric operator with a dense set of vectors of uniqueness. Then A is essentially self-adjoint.*

Proof. We need only show that Ran $(A \pm i)$ are dense. Given $\phi \in \mathscr{H}$, find ψ, a vector of uniqueness, with $\|\phi - \psi\| < \varepsilon/2$. Since $\psi \in \mathscr{H}_\psi$ and A_ψ is essentially self-adjoint, find $\eta \in \mathscr{H}_\psi$ with $\|(A_\psi + i)\eta - \psi\| < \varepsilon/2$. Then $\|(A + i)\eta - \phi\| < \varepsilon$. Since ε was arbitrary, Ran $(A + i)$ is dense. Similarly, Ran $(A - i)$ is dense. ∎

THEOREM 1.7 (Nelson's Theorem). *Let A be a symmetric operator. If D(A) contains a dense set of analytic vectors for A, then A is essentially self-adjoint.*

Proof. By Nussbaum's lemma, it is enough to show that every analytic vector ψ is a vector of uniqueness. Define $C: \mathscr{H}_\psi \to \mathscr{H}_\psi$ by

$$C\left(\sum_{n=0}^{N} a_n A^n \psi \right) = \sum_{n=0}^{N} \bar{a}_n A^n \psi.$$

It is not hard to see that C is well-defined and a complex conjugation. Moreover, $CA_\psi = A_\psi C$. Thus A_ψ has self-adjoint extensions. Let B be any self-adjoint extension of A_ψ. By using the spectral theorem (see Section 2) one can show that if

$$\sum_{n=0}^{\infty} t^n \frac{\|A^n \psi\|}{n!} < \infty \quad \text{for} \quad t < 0 \quad \text{then} \quad e^{itB}\psi = \sum_{n=0}^{\infty} (it)^n \frac{A^n \psi}{n!}.$$

Thus $e^{itB}\psi$ is independent of which self-adjoint extension of A_ψ is taken. Similarly any $\phi \in D_\psi$ has $e^{itB}\phi$ determined. Thus if $\tilde{B}$ is another self-adjoint extension, $e^{itB} = e^{it\tilde{B}}$ on D_ψ and thus on $\mathscr{H}_\psi$. It follows (by Stone's theorem) that $B = \tilde{B}$. ∎

Remarks. 1. Nelson's proof while not as slick is more transparent and is not particularly complicated.

2. There is a close connection between vectors of uniqueness and the uniqueness aspect of the classical problem of moments. Using this connection, Nussbaum has found a simple extension of Nelson's theorem which we discuss below.

Application

Nelson's theorem is the simplest way of showing that field operators in Fock space are essentially self-adjoint. Nelson's theorem has been extended in a variety of directions.

Nussbaum (1965) has used the theory of the moment problem to prove a slightly stronger theorem: A vector, ψ, is called *quasianalytic* if and only if $\psi \in C^\infty(A)$ and $\sum_{n=0}^{\infty} \|A^n\psi\|^{-1/n} = \infty$. Every analytic vector is quasi-analytic but the converse is not true. Nussbaum proved that A is essentially self-adjoint if $D(A)$ contains a dense set of quasi-analytic vectors for A.

A second generalization employs positivity to obtain a stronger theorem. Call $\psi \in C^\infty(A)$, **semi-analytic** if

$$\sum_{n=0}^{\infty} \frac{\|A^n \psi\|}{(2n!)} t^n < \infty \quad \text{for some} \quad t > 0.$$

Then:

THEOREM 1.8 (Nussbaum). *Let A be a symmetric operator and suppose there is a fixed constant c so that $(\psi, A\psi) \geqslant -c\|\psi\|^2$ for all $\psi \in D(A)$ (A is then called* **semi-bounded**). *If $D(A)$ contains a dense set of semi-analytic vectors, then A is essentially self-adjoint.*

Proof. See Nussbaum (1969), Masson and McClary (1972) or Simon (1971d) or Chernoff (In press, a). ∎

In the various discussions of Theorem 1.8, two other useful criteria are presented:

THEOREM 1.9 (Krein, 1947; Simon, 1971d). *If A is semi-bounded and has a unique semi-bounded self-adjoint extension, then A is essentially self-adjoint.*

THEOREM 1.10 (Nussbaum, 1969). *Let A be a symmetric operator and define $D(A^2) = \{\psi \in D(A) \mid A\psi \in D(A)\}$ and A^2 on $D(A^2)$ by $A^2\psi = A(A\psi)$. If A^2 is essentially self-adjoint, then A is essentially self-adjoint.*

Proof. Let $\psi \in D((A^*)^2)$. Then a simple exercise shows that $\psi \in D(A^{2*})$ and $(A^2)^*\psi = (A^*)^2\psi$, i.e. $(A^*)^2 \subset (A^2)^*$. Thus if $A^*\psi = \pm i\psi$ has solutions, then $(A^2)^*\psi = -\psi$ has solutions. But A^2 is positive so $\overline{A^2}$ is positive. If A^2 is essentially self-adjoint, $(A^2)^* = \overline{A^2}$ so $\langle\psi, (A^2)^*\psi\rangle = -\|\psi\|^2$ is impossible. ∎

Application of Theorem 1.8

Let

$$H = -\frac{d^2}{dx^2} + x^2 + x^4 \quad \text{on} \quad L^2(\mathbf{R}).$$

Any finite sum of Hermite functions is a semi-analytic vector, so H is essentially self-adjoint on $\mathscr{S}$, the Schwartz space.

As a final general criterion, we consider the "perturbation" result of Rellich (1939) and Kato (1951a).

THEOREM 1.11 (The Kato–Rellich theorem). *Let H_0 be a self-adjoint operator. Let V be symmetric obeying:*

(i) $D(V) \supset D(H_0)$,

(ii) *For some $a < 1, b > 0$, and all $\psi \in D(H_0)$.*

$$\|V\psi\| \leqslant a\|H_0\psi\| + b\|\psi\|. \tag{1.1}$$

Then $H_0 + V$ defined on $D(H_0) \cap D(V) = D(H_0)$ is self-adjoint and any core for H_0 is a core for $H_0 + V$. If H_0 is semi-bounded, so is $H_0 + V$.

Proof. Since (by the equality $\|(H_0 \pm i\alpha)\psi\|^2 = \|H_0\psi\|^2 + \alpha^2\|\psi\|^2$ for α real), $\|(H_0 \pm i\alpha)^{-1}\| \leqslant \alpha^{-1}$ and $\|H_0(H_0 \pm i\alpha)^{-1}\| \leqslant 1$ we see that

$$\|V(H_0 \pm i\alpha)^{-1}\| \leqslant a + b\alpha^{-1}.$$

Taking α large, we can be sure that $(1 + V(H_0 \pm i\alpha))^{-1}$ exists and is given by a geometric series. In particular, $1 + V(H_0 \pm i\alpha)^{-1}$ has $\mathscr{H}$ as range. But then

$$\{(H_0 + V + i\alpha)\psi \mid \psi \in D(H_0)\}$$

$$= \{(1 + V(H_0 \pm i\alpha)^{-1})\phi \mid \phi = (H_0 \pm i\alpha)\psi; \psi \in D(H_0)|$$

$$= \mathrm{Ran}\,(1 + V(H_0 \pm i\alpha)^{-1})$$

is all of $\mathscr{H}$. By the fundamental criterion, $H_0 + V$ is self-adjoint. The rest of the theorem follows in a similar vein. ∎

Remarks. 1. It is fairly easy to see that an inequality of form (1.1) holds for all $\psi \in D(H_0)$ if and only if for some $a' < 1$, $b' > 0$,

$$\|V\psi\|^2 \leqslant a' \|(H_0\psi)\|^2 + b' \|\psi\|^2. \tag{1.2}$$

2. Condition (1.1) is also equivalent to the statement that $\|V(H_0 + i\alpha)^{-1}\| < 1$ for some α.

3. It is enough to establish (1.1) or (1.2) on a core of H_0.

Application (Kato)

Let $H_0 = -\Delta$ on $L^2(\mathbf{R}^3)$.

$$\left[-\Delta = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right].$$

This differential operator is essentially self-adjoint on $\mathscr{S}(\mathbf{R}^3)$. Alternately, one can explicitly define $D(-\Delta)$ by using the Fourier transform

$$\hat{f}(k) = (2\pi)^{-3/2} \int e^{-ik.x} f(x)\, dx.$$

$f \in D(-\Delta)$ if and only if $f \in L^2$ and $k^2 \hat{f}(k) \in L^2(\mathbf{R}^3)$. Using the Fourier transform, it can be proven that

$$((H_0 + k^2)^{-1}\psi)(x) = \frac{1}{4\pi} \int \frac{e^{-k|x-y|}}{|x-y|} \psi(y)\, dy.$$

Let $W \in L^2(\mathbf{R}^3)$. Then, $W(H_0 + k^2)^{-1}$ is a Hilbert-Schmidt operator with Schmidt norm $ck^{-1/2}$. Thus for k large, $\|W(H_0 + k^2)^{-1}\| < \frac{1}{2}$. If $P \in L^\infty$, $\|P(H_0 + k^2)^{-1}\| \leqslant ck^{-2} < \frac{1}{2}$ if k is large. We conclude:

THEOREM 1.12 (Kato, 1951a). *Let* $V \in (L^2 + L^\infty)(\mathbf{R}^3)$. *Then* $-\Delta + V$ *is self-adjoint on* $D(-\Delta)$ *and any core of* $-\Delta$ *is a core for* $-\Delta + V$.

Example: $V = cr^{-1}$ (Coulomb potential). V is not in L^2 or L^∞ but the small r part of V is in L^2 and the large r part is in L^∞ so $V \in L^2 + L^\infty$.

Now let $H_0 = \sum_{i=1}^n (2\mu_i)^{-1} \Delta_i$ in $L^2(\mathbf{R}^{3n})$ and let V_{ij} be multiplication by a function V_{ij} of $r_{ij} = \mathbf{r}_i - \mathbf{r}_j$. First notice that for any $\phi \in \mathscr{S}(\mathbf{R}^{3n})$

$$\|V_{ij}\phi\|^2 \leqslant a\,\|-\Delta_{ij}\phi\|^2 + b\,\|\phi\|^2 \tag{1.3}$$

where Δ_{ij} is the Laplacian in the variable r_{ij}. For we have just proven (1.3) if $\|.\|^2$ is the integral over r_{ij} when $r_1, \ldots, \hat{r}_i, \ldots, \hat{r}_j, \ldots r_n$ are fixed. If we then integrate over the other variables, we get (1.3). Next notice that $-\Delta_{ij}$ in k-space (after Fourier transform) is multiplication by $(k_i - k_j)^2$ while H_0 is multiplication by $\sum_{i=1}^n (2\mu_i)^{-1} k_i^2$. Thus $\|-\Delta_{ij}\phi\|_2 \leqslant c\,\|H_0\phi\|^2$ for any $\phi \in \mathscr{S}(\mathbf{R}^{3n})$ with c independent of ϕ. Since a in (1.3) can be made arbitrarily small (by making b large), we conclude by using Remarks 1 and 3 above that:

THEOREM 1.13 (Kato, 1951a). *Let* $V_{ij} \in (L^2 + L^\infty)(\mathbf{R}^3)$ *and let* $V_{ij}(r_{ij})$ *stand for multiplication by* $V_{ij}(r_i - r_j)$ *on* $L^2(\mathbf{R}^{3n})$. *Then* $-\sum_{i=1}^n (2\mu_i)^{-1} \Delta_i + \sum_{i,j} V_{ij}(r_{ij})$ *is self-adjoint on* $D(-\Delta)$.

Remarks. 1. We have seen some multiplication operators on $L^2(\mathbf{R}^3)$ obey an inequality of form (1.1) with $H_0 = -\Delta$. One can ask precisely which V obey such an inequality. The answer is simple:

PROPOSITION. *Let* V *be a multiplication operator on* $L^2(\mathbf{R}^3)$. *Then the following are equivalent*:

(a) V *obeys* (1.1) *with* $H_0 = -\Delta$ *for some* a *and* b.

(b) V *obeys* (1.1) *with* $H_0 = -\Delta$ *for any* $a > 0$ *if* b *is chosen suitably.*

(c) V *is uniformly locally* L^2, *i.e.* $\int_S |V(x)|^2\, dx < c$ *for any sphere* S *of radius* 1.

2. One can ask what happens on $\mathbf{R}^n$. For $n = 1$ or 2 the proposition in Remark 1 holds. For $n > 3$, the situation is not as simple. However, one can give L^p sufficient conditions. For $n \geqslant 4$:

PROPOSITION (Nelson, 1964b). *Let* $n \geqslant 4$ *and let* $V \in L^p(\mathbf{R}^n) + L^\infty(\mathbf{R}^n)$ *with* $p > n/2$. *Then for any* $a > 0$, *there is a* b *so that* (1.1) *holds.*

PROPOSITION (Simon, In press, a). *Let* $V \in L^2(\mathbf{R}^n) + L^\infty(\mathbf{R}^n)$ *with* $V \geqslant 0$. *Then* $-\Delta + V$ *is essentially self-adjoint on* $C_0^\infty(\mathbf{R}^n)$.

This last result does not use smallness techniques directly*. For $n \geqslant 5$, smallness techniques extend slightly further than indicated in Nelson's proposition:

PROPOSITION (which follows from results of Strichartz, 1967). *Let $n \geqslant 5$. Let μ be Lebesgue measure. Let V be a measurable function on* **R**. *If $t^{n/2}\,\mu\{x\}\,|\,|V(x)| > t\} \to 0$ as $t \to \infty$, in particular if $V \in L^n|^2 + L^\infty$, then* (1.1) *holds for any $a > 0$ (with b dependent on a).*

The Kato–Rellich theorem has a variety of generalizations:

THEOREM 1.14. *Let H_0 be self-adjoint. Let V be a symmetric operator so that*

(i) $D(V) \supset D(H_0)$,

(ii) *For some $b > 0$ and all $\psi \in D(H_0)$:*

$$\|V\psi\| \leqslant \|H_0\psi\| + b\,\|\psi\|. \tag{1.4}$$

Then $H_0 + V$ is essentially self-adjoint on any core for H_0.

Proof. See Wust(1971). We note that the self-adjointness on $D(H_0)$ part of Theorem 1.11 does not hold in general, for example, take $V = -H_0$. ■

Application (Konrady 1972)

In the spatially cutoff $(\phi^4)_2$ field theory one is interested in an operator $H_0 + V$ where one has the following structure:

(i) There is an operator, N (the number of operator) which commutes with H_0 and $0 \leqslant N \leqslant c\,H_0$.

(ii) For d, e, suitable, $\|V\psi\|^2 \leqslant \frac{1}{2} d\,\|N^2\psi\|^2 + e\,\|\psi\|^2$.

(iii) For suitable, f, $\pm\,[N, [N, V]] \leqslant \frac{1}{2} N^3 + f$.

(iv) For suitable, g, $\frac{1}{2} N \leqslant H_0 + V + g$.

All estimates hold on $C^\infty(H_0)$. Let us prove that $C^\infty(H_0)$ is a core for $H_0 + V$. By (i) and (ii), V is a small perturbation of $H_0 + dN^2$ in the sense of Theorem 1.12, so $C^\infty(H_0)$ is a core for $H_0 + dN^2 + V$. If we can establish that $(dN^2)^2 \leqslant (H_0 + dN^2 + V)^2 + h^2$ for some constant h, then we can apply Theorem 1.14 to $H_0 + dN^2 + V - dN^2$. Thus we must show that

$$(H_0 + V)^2 + d[N^2(H_0 + V) + N^2(H_0 + V)] \equiv A + (H_0 + V)^2$$

is bounded from below. But

$$\begin{aligned} A = [N, [N, H_0 + V]] &+ 2N(H_0 + V)N \\ &\geqslant -\tfrac{1}{2}N^3 - f + 2N(H_0 + V)N \qquad \text{(by iii)} \\ &\geqslant \tfrac{1}{2}N^3 - 2gN^2 - f \qquad \text{(by iv)} \\ &\geqslant -h \qquad \text{for some } h. \end{aligned}$$

Since $(H_0 + V)^2 \geqslant 0$, we have the requisite estimate.

* Added in proof. T. Kato (Berkeley preprint) has improved the last proposition to require $V \in (L^2)_{\mathrm{loc}}$; $V \geqslant 0$.

Let us briefly refer to other methods of proving self-adjointness which are useful in the study of Hamiltonian operators of quantum physics:

(1) *Differential Equation Techniques.* See Stummel (1956) or Ikebe and Kato (1962). The most interesting operators treatable by these methods are operators $H = -\Delta + V$ where V does not go to zero at ∞. For example:

THEOREM 1.15* (Carlemann, 1934; Stummel, 1956; Jaffe, 1965). *If V is a C^∞ function on $\mathbf{R}^n$ which is bounded below, $-\Delta + V$ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^n)$.*

THEOREM 1.16 (Stummel, 1956; Ikebe and Kato, 1962). *Let $V = V_1 + V_2$. Where V_1 is a continuous function, $|V_1(x) - V_1(y)| < C|x - y|$ if $|x - y| > 1$ (So V_1 grows at worst linearly) and let V_2 be a sum if a bounded function and a function in L^2. Then $-\Delta + V$ (on $\mathbf{R}^3$) is essentially self-adjoint on $C_0^\infty(\mathbf{R}^3)$.*

(2) *Hypercontractive Semigroups.* This is a technique which can be used to prove essential self-adjointness of some spatially cut-off two-dimensional Bose field theories. The basic hypotheses and some of the estimates are due to Nelson (1966). The self-adjointness proof in the concrete situation is due to Rosen (1970). Its proof has been abstracted and simplified by Segal (1970) and further discussed by Høegh–Krohn and Simon (1972). An application is found in Simon (In press a).

(3) *Limit Methods.* Certain kinds of limits of self-adjoint operators are self-adjoint. The oldest theorems are due to Trotter (1958). Extensions can be found in Glimm and Jaffe (1969, 1972). In Glimm and Jaffe (1972), the application to two-dimensional Yukawa field theories are discussed.

2. SPECTRAL PROPERTIES OF DYNAMICAL GENERATORS

There has been intensive study of "spectral" properties of the Hamiltonians in quantum mechanical systems and to a lesser extent of Liouville operators. We will first describe what we mean by "spectral" properties and then describe some of what is known about the spectral problems for Hamiltonians and Liouville operators.

The *spectrum* of a self-adjoint operator H is the set, $\sigma(H)$, of $\lambda \in \mathbf{C}$ for which $(H - \lambda)^{-1}$ does not exist. In the proof of the fundamental criterion for self-adjointness, we essentially showed that $\pm i \notin \sigma(H)$ if H is self-adjoint. In a similar way, one sees that $\sigma(H) \subset \mathbf{R}$.

* See footnote, added in proof, page 29.

In Section 4, we will briefly sketch the proof of the following basic theorem for self-adjoint operators:

THEOREM 2.1 (Spectral Theorem). *Let A be a self-adjoint operator on a separable Hilbert space, $\mathscr{H}$. Then, there exist measures $\{\mu_n\}_{n=1}^N$ (N may be finite or infinite) and $U : \mathscr{H} \to \oplus_{n=1}^N L^2(\mathbf{R}, d\mu_n)$, an isometric isomorphism so that:*

(a) *$\psi \in D(A)$ if and only if*

$$\sum_{n=1}^{N} \int |x|^2 \, |(U\psi)_n(x)|^2 \, d\mu_n < \infty$$

(b) *if $\psi \in D(A)$ then $(UA\psi)_n(x) = x\, U\psi_n(x)$.*

Here we write a typical

$$f \in \bigoplus_{n=1}^{N} L^2(\mathbf{R}, d\mu_n)$$

as $\langle f_1(x), \ldots, f_n(x), \ldots\rangle$ with $f_n \in L^2(\mathbf{R}, d\mu_n)$.

The relation of the "spectrum" to the spectral theorem is given by:

THEOREM 2.2. *We say $\lambda \notin \operatorname{Supp}\{\mu_n\}$ if and only if for some $\varepsilon > 0$, $\mu_n(\lambda - \varepsilon, \lambda + \varepsilon) = 0$ for all n. Then $\sigma(H) = \operatorname{Supp}\{\mu_n\}$.*

Proof. If $\lambda \notin \operatorname{Supp}\{\mu_n\}$, suppose $\mu_n(\lambda - \varepsilon, \lambda + \varepsilon) = 0$. Then multiplication by $(x - \lambda)^{-1}$ on $\oplus_{n=1}^N L^2$ is a bounded operator. $U^{-1}(x-\lambda)^{-1}\,U$ is precisely $(H-\lambda)^{-1}$. Thus $\lambda \notin \sigma(H)$. Conversely, suppose $\lambda \notin \sigma(H)$ so $\|(H-\lambda)^{-1}\| = \varepsilon^{-1}$ for some ε. We claim that $\mu_n(\lambda - \varepsilon, \lambda + \varepsilon) = 0$ for all n. For suppose not. Then we can find $f \neq 0$ in $\oplus_{n=1}^N L^2$ with $\|f\| = 1$ and $\|(x-\lambda)^{-1} f\| < \varepsilon$. But then letting $\psi = U^{-1} f$,

$$\varepsilon > \|(H-\lambda)\psi\| \geqslant \|(H-\lambda)^{-1}\|^{-1}\, \|(H-\lambda)^{-1}(H-\lambda)\psi\| = \varepsilon$$

This contradiction proves that $\lambda \notin \operatorname{Supp}\{\mu_n\}$. ∎

The problem with $\sigma(H)$ is that it is a crumby invariant of H, i.e. it does not distinguish rather different operators. For example let A_1 be multiplied by x on $L^2([0,1], dx)$. $\sigma(A_1) = [0,1]$. A_1 has no eigenvectors.

Let A_2 be given on l_2 by $A_2 \langle a_1, a_2, \ldots \rangle = \langle q_1 a_1 \, q_2 a_2, \ldots \rangle$ where $\{q_n\}_{n=1}^{\infty}$ is a counting of the rationals in $[0, 1]$. Then $\sigma(A_2) = [0, 1]$. A_2 has a complete set of eigenvectors.

To get better invariants for A we first distinguish certain invariant subspaces for A.

Definition. Let H be a self-adjoint operator on $\mathscr{H}$ and let U and $\{\mu_n\}_{n=1}^{N}$ be the elements of a spectral representation as in Theorem 2.1. The measure μ_ψ on $\mathbf{R}$ given by

$$\int f(x)\, d\mu_\psi(x) = \sum_{n=1}^{N} \int f(x)\, |(U\psi)_n(x)|^2\, d\mu_n(x)$$

is called the **spectral measure** for ψ.

While it is not completely obvious, it can be shown that μ_ψ is independent of the precise choice of U and the μ_n. We will see this in Section 4.

Definition. $\psi \in \mathscr{H}_{p.p.}$ (the **pure point space**) if and only if μ_ψ is a pure point measure (recall that ν is a pure point measure if $\nu(A) = \sum_A \nu(\{x\})$). $\psi \in \mathscr{H}_{p.p.}$ if and only if ψ is a linear combination of eigenvectors. $\psi \in \mathscr{H}_{a.c.}$ (**the absolutely continuous space**) if and only if μ_ψ is absolutely continuous relative to Lebesgue measure (recall that ν is absolutely continuous relative to Lebesgue measure if and only if $\nu(A) = \int_A f(x)\, dx$ for some locally L^1 function f). $\psi \in \mathscr{H}_{sing}$ (the **singular** or **continuous singular space**) if and only if ν is singular relative to Lebesgue measure and continuous (recall that ν is singular relative to Lebesgue measure if $\nu(\mathbf{R}\backslash A) = 0$ for a set A of Lebesgue measure 0 and ν is continuous, if $\nu(\{x\}) = 0$ for all x).

THEOREM 2.3. *$\mathscr{H} = \mathscr{H}_{p.p.} \oplus \mathscr{H}_{a.c.} \oplus \mathscr{H}_{sing}$. Each subspace is an invariant subspace for A.*

Proof. This is essentially a consequence of the Lebesgue decomposition theorem, that any measure is uniquely a sum $\mu_{p.p.} + \mu_{a.c.} + \mu_{sing}$. That the spaces are invariant is a consequence of the fact that ν pure point implies $\tilde{\nu}$ given by $d\tilde{\nu} = x d\nu$ is pure point and similarly for the other cases. ∎

Now, one defines

Definition. $\sigma_{a.c.}(H) = \sigma(H \restriction \mathscr{H}_{a.c.})$; $\sigma_{sing}(H) = \sigma(H \restriction \mathscr{H}_{sing})$ $\sigma_{p.p.}(H) = \{\lambda | \lambda$ is an eigenvalue of $H\}$.

$\sigma_{p.p.}(H)$ is not quite $\sigma(H \restriction \mathscr{H}_{p.p.})$ rather $\overline{\sigma_{p.p.}(H)} = \sigma(H \restriction \mathscr{H}_{p.p.})$. Essentially all $\lambda \in \sigma_{sing}$ or $\sigma_{a.c.}$ are equally important but the eigenvalues in

$\sigma(H \restriction \mathscr{H}_{p.p.})$ are special so we single them out. Theorem 2.3 immediately implies $\sigma = \sigma_{a.c.} \cup \sigma_{sing} \cup \overline{\sigma_{p.p.}}$. But $\sigma_{a.c.}$, σ_{sing} and $\overline{\sigma_{p.p.}}$ need not be disjoint. We then have a perhaps overlapping "decomposition" of σ into three pieces. As we will see this decomposition has physical meaning in many cases.

There is another different decomposition of σ into two *disjoint* subsets whose union is σ.

Definition. $\lambda \in \sigma(H)$ is said to be discrete if and only if

(i) λ is an isolated point of $\sigma(H)$, i.e.

$$(\lambda - \varepsilon, \lambda + \varepsilon) \cap \sigma(H) = \{\lambda\} \qquad \text{for some } \varepsilon > 0.$$

(ii) λ is an eigenvalue of finite multiplicity, i.e.

$$\dim \{\psi \mid H\psi = \lambda \psi\} < \infty.$$

The **discrete spectrum,** $\sigma_{disc}(H) = \{\lambda \in \sigma(H) \mid \lambda$ is discrete$\}$. $\sigma_{ess}(H)$ the **essential spectrum** is defined by $\sigma_{ess}(H) = \sigma(H) \backslash \sigma_{disc}(H)$.

In this definition, σ_{ess} seems rather artificial. It is not. First, it has an intrinsic definition:

THEOREM 2.4. $\lambda \in \sigma_{ess}(H)$ *if and only if one* **or** *more of the following hold*:

(i) $\lambda \in \sigma_{cont}(H) \equiv \sigma_{a.c.}(H) \cup \sigma_{sing}(H)$,

(ii) λ *is a limit point of* $\sigma_{p.p.}(H)$,

(iii) $\dim \{\psi \mid H\psi = \lambda\psi\} = \infty$.

In addition, σ_{ess} has an invariance property:

THEOREM 2.5. *Let* H_0, H_1 *be self-adjoint operators and suppose that* $(H_0 + i)^{-1} - (H_1 + i)^{-1}$ *is compact. Then* $\sigma_{ess}(H_0) = \sigma_{ess}(H_1)$.

For proofs of Theorems 2.4 and 2.5 (the first is easy, the second is deep) see Reed and Simon (In press, a).

Examples

(1) If $H_0 - H_1$ is compact (in which case $D(H_0) = D(H_1)$ by Kato's theorem), then

$$(H_0 + i)^{-1} - (H_1 + i)^{-1} = (H_0 + i)^{-1}(H_1 - H_0)(H_1 + i)^{-1}$$

is compact.

(2) Consider $A = -d^2/dx^2 + V(x)$ on $L^2([0, \infty), dx)$ with $V(x)$ a nice function. Suppose $D(A) = C_0^\infty(0, \infty)$, the smooth functions with support strictly in $(0, \infty)$. A is not essentially self-adjoint but any two self-adjoint extensions A_0, A_1 have the property that $(A_0 + i)^{-1} - (A_1 + i)^{-1}$ has rank 1 so $\sigma_{ess}(A_0) = \sigma_{ess}(A_1)$. The essential spectrum is thus the spectrum independent of boundary conditions.

σ_{ess} is always closed; σ_{disc} may not be closed.

Given a physical Hamiltonian, the spectral questions one usually asks are:

(1) What is $\sigma_{ess}(H)$?

(2) Is $\sigma_{disc}(H)$ finite or infinite?

(3) What is $\sigma_{a.c.}(H)$?

(4) As we shall see, vectors $\psi \in \mathscr{H}_{sing}$ have a "non-physical" behaviour so we hope $\mathscr{H}_{sing} = \{0\}$, i.e. that $\sigma_{sing} = \varnothing$; so one asks: Is $\sigma_{sing} = \varnothing$?

(5) How do σ_{disc} and $\sigma_{p.p.}$ differ? Put differently: Can there be eigenvalues embedded in the continuum?

Kato (1967b) reviews the answers to some of these questions in cases A and B.

A. Two Body Quantum Systems

The Hamiltonian (after centre of mass motion has been removed) is just $-\Delta + V$ on $L^2(\mathbf{R}^3)$ where V is a multiplication operator on $\mathbf{R}^3$ which we suppose is in $L^2 + L^\infty$.

(1) In physical two-body systems, $V \to 0$ at infinity at least in the weak sense that $V \in L^2 + (L^\infty)_\varepsilon = \{f \in L^2 + L^\infty \mid (\forall\, \varepsilon) f = f_{1,\varepsilon} + f_{2,\varepsilon}$ with $f_{1,\varepsilon} \in L^2$ and $\|f_{2,\varepsilon}\|_\infty < \varepsilon\}$ (Consider $V = r^{-1}$ again). We have already seen that $V(H_0 + i)^{-1}$ is a Hilbert–Schmidt integral operator if $V \in L^2$. If $V \in L^2 + (L^\infty)_\varepsilon$, $V(H_0 + i)^{-1}$ is a norm limit of Hilbert–Schmidt operators and so is compact. We conclude that for any $V \in L^2 + (L^\infty)_\varepsilon$, $(H_0 + i)^{-1} - (H + i)^{-1} = (H + i)^{-1}[V(H_0 + i)^{-1}]$ is compact. By Theorem 2.5, $\sigma_{ess}(H) = \sigma_{ess}(H_0) = [0, \infty]$ so:

THEOREM 2.6 (Courant and Hilbert, 1953; Rejto, 1966). *If* $V \in L^2 + (L^\infty)_\varepsilon(\mathbf{R}^3)$, *and* $H = H_0 + V = -\Delta + V$, *then* $\sigma_{ess}(H) = [0, \infty)$.

(2) This problem has been studied by several authors: Birman (1961), Faddeev (1957), Schwinger (1961). The basic result is:

THEOREM 2.7 (Courant and Hilbert, 1953). *Let* $V \in L^2 + (L^\infty)_\varepsilon$ *and suppose for some* $R > 0$ *and some* $\varepsilon > 0, V(\mathbf{r}) < -r^{-2+\varepsilon}$ *for* $r = |\mathbf{r}| > R$. *Then* $\sigma_{disc}(H)$ *is an infinite set (with limit point* 0). *Suppose for some* $R > 0$ *and some* $\varepsilon > 0\, |V(\mathbf{r})| > -r^{-2-\varepsilon}$ *for* $r > R$. *Then* $\sigma_{disc}(H)$ *is finite.*

Proof. See Simon (1970). ∎

Birman, Faddeev and Schwinger have refinements in case V does not have simple power behavior at infinity or is in the r^{-2} border line.

(3) As Hepp will explain, the problem of locating $\sigma_{a.c.}$ turns out to be related to scattering theory. Using methods of time-dependent scattering theory, Hack (1958) and Kuroda (1959) have proven

THEOREM 2.8 *Let* $V \in L^2 + L^p$; $2 < p < 3$. *Then* $\sigma_{a.c.}(H) = [0, \infty)$.

$$(1 + |r|)^{-\alpha} \in L^2 + L^p \quad \textit{for some} \quad 2 < p < 3 \quad \textit{so long as} \quad \alpha > 1.$$

Thus Theorem 2.8 breaks down precisely at Coulomb fall-off where it is known that scattering theory in an unmodified form does not hold. Potentials with r^{-1} fall-off have been shown to have $\sigma_{a.c.}(H) = [0, \infty)$ by Dollard (1964) and with $r^{-\alpha}$ fall-off $\alpha > 0$, by Buslaev and Matveev (1970). These authors use a modified scattering theory. For additional discussion, see Lavine (1969) or Amrein *et al.* (1970).

(4) The basic idea behind the proof of Theorem 2.9 is that states which describe "asymptotically free" particles must be in $\mathscr{H}_{a.c.}$. "Bound States" are in $\mathscr{H}_{p.p.}$. Physically we expect only bound states and scattering states to occur so we hope that $\mathscr{H}_{sing} = \{0\}$. This has been proven for a variety of situations by a variety of authors: for example, see Agmon (In press), Ikebe (1960), Lavine (1972), Rejto (1966), Simon (1971a), Weidmann (1967a), Kato (1968), Aguilar and Combes (1971). Typical is the following result which was the first one of its genre:

THEOREM 2.9 (Ikebe). *Let* $V \in L^2 + (L^\infty)_\varepsilon$ *and suppose*

(a) *V is Hölder continuous outside a finite set,*

(b) $r^{2+\varepsilon} V(\mathbf{r}) \to 0$ *as* $r \to \infty$ *for some* $\varepsilon > 0$.

Then $-\Delta + V$ *has no singular continuous spectrum.*

The other results either remove smoothness conditions or allow slower fall-off at infinity.

(5) The occurance of an eigenvalue at $E = 0$ is not very exceptional. For square well potentials of the right strength they occur. But eigenvalues at $E > 0$ are "unphysical", for "a particle of positive energy should be able to reach $r = \infty$ and so be unbound". Nevertheless, there are explicit examples (von Neumann and Wigner (1929), Weidmann (1967a)) of potentials with positive energy bound states. These potentials fall off slowly (as r^{-1}) and have rapid oscillations at infinity. The earliest result assuring no positive energy bound states is due to Kato (1959) (who is responsible for earliest results in many aspects for the subject). An extension of Kato's result is:

THEOREM 2.10 (Agmon, (1970; Simon, 1969). *Suppose* $V \in L^2 + (L^\infty)_\varepsilon$, and that

(1) *V obeys some regularity conditions* ($V \in C^2$ *outside a finite set is more than enough*).

$$V = V_1 + V_2$$

where

(2) $r\,V_1(\mathbf{r}) \to 0$ *as* $r \to \infty$

(3) $V_2(\mathbf{r})$ *and* $r\dfrac{\partial V_2}{\partial r}(\mathbf{r}) \to 0$ *as* $r \to \infty$.

Then $-\Delta + V$ *has no eigenvalues in* $[0, \infty)$.

In the above theorem, V_1 is a potential which dies out somewhat rapidly. V_2 need not die out rapidly but it cannot "wiggle" too much.

B. *N*-Body Quantum Mechanics

The basic operator is

$$A = \sum_{i=1}^{n} -\frac{\Delta_i}{2\mu_i} + \sum_{i<j} V_{ij}(\mathbf{r}_i - \mathbf{r}_j)$$

in $L^2(\mathbf{R}^{3n})$. One first removes the centre of mass motion; that is one writes $\mathbf{R}^{3n} = \mathbf{R}^3 \times \mathbf{R}^{3n-3}$ where the first factor is relative to the coordinate

$\mathbf{r} = (\Sigma\mu_i)^{-1}\,\Sigma\mu_i\mathbf{r}_i$. Then $L^2(\mathbf{R}^{3n}) = L^2(\mathbf{R}^3) \otimes L^2(\mathbf{R}^{3n-1})$ and $A = A_0 \otimes 1 + 1 \otimes H$ where $A_0 = -\Delta_{\mathbf{r}}/2M$ $(M = \Sigma\mu_i)$. The exact form of H depends on the coordinates we use (but up to unitary operators which change coordinates, H is uniquely determined). If we take coordinates $\boldsymbol{\eta}_i = \mathbf{r}_i - \mathbf{r}_n$ $(i = 1, \ldots, n-1)$ in $\mathbf{R}^{3n-1}$ and write $m_i = (\mu_i^{-1} + \mu_n^{-1})^{-1}$, then

$$H = -\sum_{i=1}^{n-1} (2m_i)^{-1}\,\Delta_{\boldsymbol{\eta}_i} + \sum_{\substack{i,j=1\\ i<j}}^{n-1} (\mu_n)^{-1}\,\nabla_{\boldsymbol{\eta}_i}\cdot\nabla_{\boldsymbol{\eta}_j}$$

$$+ \sum_{\substack{i<j\\ i,j=1}}^{n-1} V_{ij}(\boldsymbol{\eta}_i - \boldsymbol{\eta}_j) + \sum_{i=1}^{n-1} V_{in}(\boldsymbol{\eta}_i).$$

The problem with H is that, $V = \sum_{i<j} V_{ij} + \sum_i V_{in}$ does not go to 0 at infinity but remains large in tubes where

$$\sum_{i=1}^{n} |\boldsymbol{\eta}_i|^2 \to \infty$$

while some η_i or some $\eta_i - \eta_j$ remains bounded. Given a subset, D, of $\{1, \ldots, n\}$ with m elements A_D is the operator on $L^2\,(\mathbf{R}^{3m})$ obtained by only taking the sums over i, and i and j in D. H_D is the operator obtained from A_D by removing the center of mass for D.

(1) One has the beautiful theorem of Hunziker (1964a):

THEOREM 2.11. *Suppose each* $V_{ij} \in L^2 + (L^\infty)_\varepsilon$. *Then* $\sigma_{ess}(H) = [\Sigma, \infty)$ *where*

$$\Sigma = \inf_{\substack{D_1, D_2 \subset \{1,\ldots,n\}\\ D_1 \cap D_2 = \varnothing}} \sigma(H_{D_1} + H_{D_2}).$$

This has a simple physical interpretation: States with energy less than Σ cannot "decay" into two bound clusters D_1, D_2 and so must be bound. Hunziker's theorem is further discussed in Hepp's lectures.

(2) That this is a hard problem is perhaps best shown by a recent conjecture of Effimov (1970) which *at first sight* seems contrary to intuition but which may be true. The reader interested in the problem may consult Kato (1951b) (who had the first result!), Simon (1970), Uchiyama (1966, 1970), Zhislin (1960), Sigalov and Zhislin (1965). A typical result is:

THEOREM 2.12. (Zhislin). *Atoms have σ_{disc} infinite. Explicitly if*

$$V_{ij}(\eta_i - \eta_j) = \frac{1}{|\eta_i - \eta_j|} \quad \text{for} \quad i, j \leqslant n - 1$$

and

$$V_{in}(\eta_i) = \frac{-(n-1)}{|\eta_i|},$$

then $\sigma_{disc}(H)$ is infinite.

Proof. See Zhislin (1960), Simon (1970) or Uchiyama (1966). ■

(3) One can use scattering theory to prove:

THEOREM 2.13. *If $V_{ij} \in L^2 + L^p$ for $2 < p < 3$, then*

$$\sigma_{a.c.}(H) = [\Sigma, \infty) = \sigma_{ess}(H).$$

Proof. See Hack (1958) or Hunziker (1968b). ■

(4) That $\sigma_{ess} = \varnothing$ for some 3-body systems has been proven by Faddeev (1963). For some n-body system with repulsive forces, $\sigma_{ess} = \varnothing$ is a result of Hepp (1969a) and Lavine (1971a). In all these cases (except Faddeev's three body results), the Hamiltonians have $\Sigma = \inf \sigma_{ess}(H) = 0$ and are what are known as 1-channel Hamiltonians because no subsystem has bound states†. The jump to a small but interesting class of general n-body systems has been made by Balslev and Combes (1971):

THEOREM 2.14 *Let $V_{ij}(\mathbf{r})$ be functions of $|\mathbf{r}| = r$ which have analytic continuations to the sector $\{r \mid |\arg r| < \alpha\}$ for some $\alpha > 0$ so that the continuation goes to zero as $r \to \infty$ (in the sector) and with $V_{ij} \in L^2 + L^\infty$. Then $\sigma_{sing}(H) \equiv \phi$.*

Proof. See Balslev and Combes (1971) and Simon (In press, c). ■

This class of potentials include the Coulomb and Yukawa potentials. The class of potentials introduced by Combes (unpublished) for which Theorem 2.14 holds is larger than we have indicated—it is defined by certain abstract conditions and includes certain non-central of non-local or momentum dependent potentials. We have just described the class in the central, local case for simplicity of presentation.

† Another 1-channel result (weak potentials) has been got by Iono and O'Carroll (1972).

(5) It is not so pathological for bound states to occur in $[\Sigma, \infty]$. Very often they occur for reasons of symmetry. Explicitly, it can happen that for reasons of symmetry $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ where H leaves both $\mathscr{H}_1$ and $\mathscr{H}_2$ invariant but $\Sigma_2 = \inf \sigma_{ess}(H \upharpoonright X_2) > \Sigma_1 = \Sigma$. Bound states occurring in $\mathscr{H}_2$ below Σ_2 seems to be embedded in the continuum $[\Sigma, \infty]$. For example, if $H = -\Delta_1 - \Delta_2 - (2/r_1) - (2/r_2) + (1/|r_1 - r_2|)$ (Helium atom Hamiltonian), and $\mathscr{H}_{1(2)}$ is the set of states of natural (unnatural) parity, then $\Sigma_1 = -1; \Sigma_2 = -\frac{1}{4}$. On $\mathscr{H}_2$, H has an infinite number of bound states in $[\Sigma_1, \Sigma_2)$. (See Sigalov and Zhislin, 1965; Balslev, In press).

For proofs of the absence of bound states in (a, ∞) where $a = 0$ or $a > 0$, see Weidmann (1967a), Agmon (1969) and Alberverio (In press, a).

C. Spatially Cutoff $P(\phi)_2$ Hamiltonians

In his lectures, Glimm will discuss the model $H = H_0 + \int g(x) :P(\phi(x)): dx$ where H_0 is the free boson Hamiltonian of mass $m_0 > 0$ in two-dimensional space time and where $g \geqslant 0, g \in L^1 \cap L^2$. $P(X)$ must be a polynomial which is bounded below when X is real.

(1) THEOREM 2.15. *Let* $E_0 = \inf \sigma(H)$, *Then* $\sigma_{ess}(H) = [m_0 + E_0, \infty)$.

Proof. That $\sigma(H) \cap (-\infty, m_0 + E_0)$ is purely discrete was proven by Glimm and Jaffe (1970b) by approximating H with operators which have discrete spectrum in $(-\infty, m_0 + E_0)$. (See also Høegh-Krohn and Simon, 1972). That $[m_0 + E_0, \infty) \subset \sigma(H)$ has been proven by Høegh-Krohn (1971) and Kato and Migubayashi (1971). ■

(2) Very little is known about spectral properties in general. But the $P(X) = X^2$ model is "exactly soluble" and its spectral properties have been examined by Rosen (In 1972a). *If* this model is a guide, one expects that σ_{disc} is finite, at least when g has compact support.

(3) Using ideas from scattering theory, Høegh-Krohn (1971) and Kato and Migubayashi (1971) have proven:

THEOREM 2.16. $\sigma_{a.c.}(H) = [m_0 + E_0, \infty)$ *where* $E_0 = \inf \sigma(H)$.

(4) No results are known except in the ϕ^2 model. In that case it is known that $\sigma_{sing}(H) = \phi$.

(5) No real results but it is known that in general models, there can be eigenvalues in the continuum below $E_0 + 2m$ essentially for symmetry reasons as in the Helium atom (see Simon, In press, b)—the symmetry here is $(-1)^N$ where N is the number operator).

D. Liouville Operators

There are actually several Liouville operators:

(a) *Constant energy surface operators.* In phase space there are $6n-1$ dimensional energy surfaces, Ω_E, with a natural Liouville measure, dL. The dynamics on $L^2(\Omega_E, dL)$ induces a Liouville operator, L_E. Sinai's work in ergodicity for the hard sphere gas implies:

THEOREM 2.17. (Sinai, In press). *The Liouville operator, L_E, in the case of a hard sphere gas in a spherical box, has $\sigma(L_E) = \sigma_{ess}(L_E) = \sigma_{a.c.}(L_E) = (-\infty, \infty)$; $\sigma_{sing}(L_E) = \sigma_{disc}(L_E) = \varnothing$; $\sigma_{p.p.}(L_E) = \{0\}$. 0 is a simple eigenvalue; the absolutely continuous spectrum is of infinite multiplicity.*

(b) *Full operator in a box.* Very little is known about the spectrum of L, the operator discussed after Theorem 1.4. In the case of non-interacting particles in a box, one has:

THEOREM 2.18. *Under certain conditions (non-interacting particles in a spherical box with smooth wall forces)*

$$\sigma(L) = \sigma_{ess}(L) = \sigma_{a.c.}(L) = (-\infty, \infty);$$

$$\sigma_{disc}(L) = \sigma_{sing}(L) = \varnothing; \quad \sigma_{p.p.}(L) = \{0\}.$$

For the exact conditions and a proof, see Prosser (1969).

(c) *Full operator in all space.* Again, a limited amount is known. However the scattering theory of Hunziker (1968a) gives some information:

THEOREM 2.19. *Let L be the Liouville operator for an n-particle system (with center of mass removed) with two-body forces which are* C^∞ *and bounded by* $r^{-2-\varepsilon}$ *at infinity. Then* $\sigma(L) = \sigma_{ess}(L) = \sigma_{a.c.}(L) = (-\infty, \infty)$.

3. QUADRATIC FORMS

A striking result in the theory of bounded operators is the one to one correspondence between bounded self-adjoint operators and bounded symmetric quadratic forms, where:

Definition. A **quadratic form,** a, is a map $a: Q(a) \times Q(a) \to \mathbf{C}$, where $Q(a)$ is a dense subspace of $\mathscr{H}$, so that $a(\psi, .)$ is linear and $a(., \phi)$ is conjugate linear. a is called a **bounded** quadratic form if and only if $Q(a) = \mathscr{H}$ and $|a(\psi, \phi)| \leqslant C \|\psi\| \|\phi\|$.

It is a trivial consequence of the Riesz lemma that:

THEOREM 3.1. *There is a one-one correspondence between bounded quadratic forms and bounded operators given by* $a \leftrightarrow A$ *with* $a(\psi, \phi) = (\psi, A\phi)$.

We wish to discuss unbounded forms and their relation to unbounded operators. We restrict ourselves to the case of symmetric forms. For "sectorial forms" see Kato (1966b) or Reed and Simon (In press, Vol. I):

Definition. A quadratic form, a, is called **symmetric** if and only if $a(\psi, \phi) = \overline{a(\phi, \psi)}$ for all $\phi, \psi \in Q(a)$. a is called **bounded below** or **semibounded** if there is a C with $a(\phi, \phi) > -C \|\phi\| \|\phi\|$. If $C = 0$ can be taken, a is called **positive.** If a is a semibounded form, introduce the norm $\|\cdot\|_{+1}$ or $\|\cdot\|_{+1,a}$ by $\|\psi\|_{+1,a} = a(\psi, \psi) + (C+1) \|\psi\|^2$. If $Q(a)$ is a complete space in $\|.\|_{+1}$, we say a is **closed.** If a has a closed extension, we say a is **closable.**

Example. Let $Q(a) = C_0^\infty(\mathbf{R})$ the smooth functions of compact support. Define $a(\psi, \phi) = \overline{\psi(0)}\, \phi(0)$ for $\phi, \psi \in Q(a)$. Let ϕ_n be a sequence of functions in $Q(a)$ with $\phi_n(0) = 1$ and $\phi_n \to 0$ in $L^2(\mathbf{R}) = \mathscr{H}$. Then (a) $\|\phi_n - \phi_m\|_{+1} = \|\phi_n - \phi_m\| \to 0$ as $n, m \to \infty$. (b) $\|\phi_n\|_{+1} \to 1$ as $n \to \infty$. Suppose a has a closed extension to a form $\tilde{a}$ on $Q(\tilde{a})$. Then there is a $\phi \in Q(\tilde{a})$ with $\|\phi_n - \phi\|_{+1} \to 0$. In particular, $\|\phi\|_{+1} = \lim_{n \to \infty} \|\phi_n\|_{+1} = 1$. But since $\|.\| \leqslant \|\ \|_{+1}$, $\phi_n \to \phi$ in L_2. We conclude $\phi = 0$. This contradiction shows that this particular a is not closable.

We thus see: *A positive symmetric form may not be closable.* In this respect forms are not as nice as operators. On the other hand, recall an operator could be closed and symmetric but not self-adjoint. This is not the case for forms. First we define when a form is "self-adjoint".

Definition. Let A be a semibounded self-adjoint operator. Passing to a spectral representation let

$$Q(A) = \{\psi \sum_n \int (|x| + 1) \,|\, (U\psi)_n(x) \,|_2 \, d\mu_n(x) < \infty \} = D(|A|^{\frac{1}{2}})$$

and define the **form of A,** a, by

$$a(\psi, \phi) = \sum_n \int x \, \overline{(U_n\psi)_n(x)} \, (U\phi)_n(x) \, d\mu_n(x).$$

It is not hard to prove that a is closed and is the smallest closed extension of the form $(\psi, A\phi)$ defined on $D(A)$.

THEOREM 3.2. *Let a be a closed, semibounded, symmetric quadratic form. Then a is the form of a unique self-adjoint operator.*

Proof. Denote the Hilbert space, $Q(a)$, with norm $\|.\|_{+1}$ by $\mathscr{H}_{+1}$. Denote its dual by $\mathscr{H}_{-1}$ (supressing the natural map $\mathscr{H}_{+1} \to \mathscr{H}_{-1}$). Map i: $\mathscr{H}_{+1} \to \mathscr{H}$ by identification and let j: $\mathscr{H} \to \mathscr{H}_{-1}$ be the adjoint of i. Explicitly, given $\psi \in \mathscr{H}$ define $j(\psi)$ by

$$j(\psi)\,(\eta) = \psi(i\eta). \tag{3.1}$$

We suppress the maps i and j and use the standard $(.,.)$ for action of a vector on the dual. (3.1) then becomes

$$(\psi, \eta) = (\psi, \eta). \tag{3.2}$$

The left-hand side of (3.2) denotes the action of $\psi \in \mathscr{H}_{-1}$ (really $j(\psi)$) on $\eta \in \mathscr{H}_{+1}$ and the right-hand side the inner product in of ψ and η (really ψ and $i(\eta)$). Now map $\hat{B}$: $\mathscr{H}_{+1} \to \mathscr{H}_{-1}$ by

$$(\hat{B}\phi, \psi) = a(\phi, \psi) + (c + 1)\,(\phi, \psi) = (\phi, \psi)_{+1}.$$

Notice that on the one hand

$$\|\hat{B}\,\phi\|_{-1} \geqslant \frac{(\hat{B}\phi, \phi)}{\|\phi\|_{+1}} = \frac{\|\phi\|_{+1}^2}{\|\phi\|_{+1}} = \|\phi\|_{+1}.$$

while on the other hand $(\hat{B}\phi, \psi) \leqslant \|\phi\|_{+1} \|\psi\|_{+1}$ for all ψ so

$$\|\hat{B}\phi\|_{-1} = \|\phi\|_{+1}.$$

We conclude that $\hat{B}$ is injective, and Ran $\hat{B}$ is complete, hence closed. If Ran $\hat{B} \neq \mathscr{H}_{-1}$ we can find $0 \neq \eta \in \mathscr{H}^*_{-1} = \mathscr{H}_{+1}$ with $(\hat{B}\phi, \eta) = 0$ for all $\phi \in \mathscr{H}_{+1}$. Letting $\phi = \eta$ we see that $\|\eta\|_{+1} = 0$ so $\eta = 0$. This shows $\hat{B}$ is onto. Thus $\hat{B}$ is invertable. Let $D(A) = \hat{B}^{-1}[\mathscr{H}] = \{\psi \in \mathscr{H}_{+1} \mid \hat{B}_{+1} \in \mathscr{H}\}$ (really $\{\psi \in \operatorname{ran} i \mid \hat{B}\psi \in \operatorname{ran} j\}$). $\hat{B}^{-1} \restriction \mathscr{H}$ is an everywhere defined map on $\mathscr{H}$ which is injective and symmetric, for

$$\begin{aligned}
(\psi, \hat{B}^{-1}\phi) &= (\hat{B}^{-1}\psi, \hat{B}(\hat{B}^{-1}\phi)) \\
&= a(\hat{B}^{-1}\psi, \hat{B}^{-1}\phi) + (c+1)(\hat{B}^{-1}\psi, \hat{B}^{-1}\phi) \\
&= \overline{a(\hat{B}^{-1}\phi, \hat{B}^{-1}\psi) + (c+1)(\hat{B}^{-1}\phi, \hat{B}^{-1}\psi)} \\
&= \overline{(\phi, \hat{B}^{-1}\psi)}.
\end{aligned}$$

Thus $\hat{B}^{-1} \restriction \mathscr{H}$ is self-adjoint and (by the spectral theorem and the injectivity of $\hat{B}^{-1} \restriction \mathscr{H}$) its inverse $B: D(A) \to \mathscr{H}$ is self-adjoint.

Finally, let A: $D(A) \to \mathscr{H}$ be given by $A = B - c - 1$. A is self-adjoint, and it is easy to see that a is the form of A. To prove uniqueness, one shows that any other possible operator C with a the form of C has $C \subset A$ so $C = A$. ∎

One thus needs criteria for forms to be closed (or closable). One simple one is:

THEOREM 3.3. *Let a and b be closed, semibounded, symmetric quadratic forms with $Q(a) \cap Q(b)$ dense. Then $a + b$ defined on $Q(a) \cap Q(b)$ is closed.*

Proof. Without loss of generality, suppose that $a > 1, b > 1$ so $\|\psi\|_{+1,a+b} = a(\psi, \psi) + b(\psi, \psi)$. If $\psi_n \in Q(a) \cap Q(b)$ and ψ_n is $\|\cdot\|_{+1,a+b}$-Cauchy it is Cauchy in $\| \ \|_{+1,a}, \| \ \|_{+1,b}$ and $\| \ \|$, since $\| \ \| \leqslant \| \ \|_{+1,a} \leqslant \| \ \|_{+1,a+b}$ and $\| \ \|_{+1,b} \leqslant \| \ \|_{+1,a+b}$. Thus, there is $\psi \in Q(a) \cap Q(b)$ so that $\|\psi_n - \psi\|_{+1,a \text{ or } b} \to 0$. Then $\|\psi_n - \psi\|_{+1,a+b} \to 0$ so $a + b$ is closed. ∎

WARNING. It can happen that A and B are self-adjoint operators with $A > 0, B > 0$ and $A + B$ defined on $D(A) \cap D(B)$ essentially self-adjoint but so that the form associated to $A + B$ is not $a + b$.

Example. Let V be a function on $\mathbf{R}^n$ such that $\int_S |V(x)|\, d^n x < \infty$ for any bounded open set, S, with $\bar{S}$ disjoint from a closed set Γ of measure 0 (so V is locally L^1 away from Γ where it can have singularities) and let $V \geqslant 0$. Let $H_0 = -\Delta$ on $L^2(\mathbf{R}^n)$. Then $-\Delta$ and V are self-adjoint, positive operators and $Q(H_0) \cap Q(V)$ is dense since $C_0^\infty(\mathbf{R}^n \backslash \Gamma)$ is dense and contained in $Q(H_0) \cap Q(V)$. Thus, as a *form sum* we can define a self-adjoint operator which we naturally associate with $-\Delta + V$. In a certain sense the natural meaning in quantum mechanics for $C = A + B$ where A, B, C are observables is that $(\psi, C\psi) = (\psi, A\psi) + (\psi, B\psi)$ for all ψ so a physical case can be made for taking a form sum.

There is also a perturbation theory result similar in appearance (although not in proof) to the Kato–Rellich theorem:

THEOREM 3.4 (KLMN Theorem). *Let H_0 be a positive self-adjoint operator and let V be a symmetric quadratic form so that*

(i) $Q(H_0) \subset Q(V)$,

(ii) *For some $a < 1, b > 0$ and all $\psi \in Q(H_0)$*

$$|V(\psi, \psi)| \leqslant a(\psi, H_0 \psi) + b(\psi, \psi). \tag{3.3}$$

Then $H_0 + V$ defined as a sum of forms on $Q(H_0)$ is closed and thus the form of a unique self-adjoint operator, H.

Proof. Since H_0 is positive and (3.3) holds

$$H(\psi, \psi) = (\psi, H_0\psi) + V(\psi, \psi) \geqslant (1 - a)(\psi, H_0\psi) - b(\psi, \psi)$$

$$\geqslant -b(\psi, \psi)$$

so H is semibounded. Moreover, letting $\| \ \|_{+1,h}$ and $\| \ \|_{+1,h_0}$ be the form norms associated with H and H_0, it is easy to see that (3.3) implies $C_1 \|\cdot\|_{+1,h_0} \leqslant \|\cdot\|_{+1,h} \leqslant C_2 \|\cdot\|_{+1,h_0}$. Since $Q(H_0)$ is complete in $\| \ \|_{+1,h_0}$, it is complete in $\|\cdot\|_{+1,h}$. Thus H is a closed form. ∎

Application

Let R be the set of functions on $\mathbf{R}^3$ with

$$\int \frac{|V(x)|\,|V(y)|}{|x - y|^2}\, d^3x d^3y < \infty.$$

Let $V \in R + L^\infty$ and let $H_0 = -\Delta$. Then V obeys a condition of form (3.3) for any $a > 0$ (b is a-dependent of course; the bound is proven as we proved Theorem 1.13, using the fact that if $W \in R$, $|W|^{1/2}(H_0 + E)^{-1}|W|^{1/2}$ is Hilbert–Schmidt). Thus:

THEOREM 3.5 (Simon, 1971b). *Let $V \in R + L^\infty$. Then $H_0 + V$ defined as a sum of forms on $Q(H_0)$ is the form of a unique self-adjoint operator, H.*

The "Rollnik class" described in Theorem 3.5 contains the Kato class, i.e. $L^2 + L^\infty \subset R + L^\infty$. Moreover, it allows worse finite singularities; $r^{-\alpha} \in L^2 + L^\infty$ only if $0 < \alpha < \frac{3}{2}$ while it is in $R + L^\infty$ if $0 < \alpha < 2$. We also note that there are $V \in R$ with $D(V) \cap D(H_0) = \{0\}$ so the form sum is essential. The quantum mechanics for systems with potentials $V \in R + L^\infty$ is developed in Simon (1971a).

Remarks. (1) It is hard to describe precisely which multiplication operators, V, on $\mathbf{R}^3$ obey an estimate of type (3.3) with $H_0 = -\Delta$. It is certainly larger than $R + L^\infty$: for example $r^{-2}(1 + |ln\, r|)^{-\alpha} \in R + L^\infty$ only if $\alpha > \frac{1}{2}$, but obeys an estimate of type 3.3 with a arbitrarily small if $\alpha > 0$.

(2) On $\mathbf{R}^n, n = 1, 2$ one can develop a theory analogous to the Rollnik class theory. In $\mathbf{R}^1$, V obeys an estimate of form 3.3 if and only if V is uniformly locally L^1.

(3) By the results we discussed after Theorem 1.14, in $\mathbf{R}^n (n \geqslant 4)$, $r^{-\alpha}$ is a small **operator** perturbation of $-\Delta$ if $\alpha < 2$ (for $r^{-\alpha} \in L^{\frac{1}{2}n+\varepsilon}$ in that case). On the other hand if $\alpha > 2$, $-\Delta - r^{-\alpha}$ is not bounded below on $C_0{}^\infty$, so $r^{-\alpha}$ is not even a small form perturbation of $-\Delta$ if $\alpha > 2$. The $\frac{3}{2} < \alpha < 2$ case filled by the Rollnik class in $\mathbf{R}^3$ does not exist in $\mathbf{R}^n$ if $n \geqslant 4$!

(4) It is a general result that if $H_0 \geqslant 0$ and $\|V\psi\|^2 \leqslant a\,\|H_0\psi\|^2 + b\,\|\psi\|^2$, then $(\psi, V\psi) \leqslant a(\psi, H_0\psi) + (\psi, \psi)$.

Example. To see that V in Theorem 3.4 need not be an operator, it is an amusing exercise to prove that on $L^2(\mathbf{R})$, for any $a > 0$, there is a b with $|\phi(0)|^2 \leqslant a\,\|d/dx\,\phi\|^2 + b\,\|\phi\|^2$ for all $\phi \in D(d/dx)$. Thus $(-d^2/dx^2) + \delta(x)$ can be defined. It is instructive to find which piecewise C^∞ functions are in $D((-d^2/dx^2) + \delta(x))$ and explicitly see the cancellations involved in the form sum.

As a final general result:

THEOREM 3.6. *Let A be a symmetric operator with $(\psi, A\psi) \geqslant 0$ for all $\psi \in D(A)$ (We say A is positive). Then the form a with $Q(a) = D(A)$ and $a(\psi, \phi) = (\psi, A\phi)$ is closable.*

Proof. Let $\|\psi\|_{+1,a}^2 = a(\psi, \psi) + \|\psi\|^2$ as usual. Let $\mathscr{H}_{+1}$ be the abstract completion of $D(A)$ with $\| \quad \|_{+1,a}$. Let $i = \mathscr{H}_{+1} \to \mathscr{H}$ by extending the identity map $i = D(A) \to \mathscr{H}$ which is continuous from $\|.\|_{+1}$ to $\| \quad \|$. If we show that i is injective, we can define $\bar{a}$ on $i\ (\mathscr{H}_{+1})$ by $\bar{a}(\psi, \phi) = (\psi, \phi)_{+1} - (\psi, \phi)$. $\bar{a}$ will be a closed form extending a. Suppose $\psi \in \mathscr{H}_{+1}$ and $i(\psi) = 0$. Then there are $\psi_n \xrightarrow{\mathscr{H}_{+1}} \psi$ with $\psi_n \to 0$ in $\mathscr{H}$. But then for any $\phi \in D(A)$ $(\phi, \psi_n)_{+1} = (\phi, A\psi_n) + (\phi, \psi_n) = (A\phi + \phi, \psi_n) \to 0$. Since $D(A)$ is dense in $\mathscr{H}_{+1}$, $(\phi, \psi)_{+1} = 0$ for all $\phi \in \mathscr{H}_{+1}$. This implies $\psi = 0$. We conclude that i is injective. ■

As a corollary we have:

THEOREM 3.7 (Friedrichs). *Let A be a positive symmetric operator. Then A has a positive self-adjoint extension A_F.*

Proof. That the operator A_F associated with the form $\bar{a}$ of Theorem 3.6 extends A is left to the reader (see the proof of Theorem 3.2) ■

Remarks. (1) A_F, the Friedrichs extension, is canonically associated with A. It has the property

$$\inf_{\psi \in D(A)} (\psi, A\psi) = \inf_{\psi \in D(A_F)} (\psi, A_F\psi).$$

There may however be other self-adjoint extensions of A with this property.

(2) Historically, the Friedrichs extension was found by Friedrichs (1934) and Stone (1932) without using forms although some proofs, e.g. Freudenthal (1936) were disguised form theoretic results. That forms are so nice, (Theorem 3.2) was realized by Kato (1955), Lax and Milgram (1954) and Lions (1961). The $\mathscr{H}_{+1} \to \mathscr{H}_{-1}$ language to prove Theorem 3.2 was introduced by Nelson (1964a). It is for these authors, we use the term KLMN theorem.

As a final application of form methods, let us sketch how they can be used if one wants to carefully treat quantum statistical mechanics in boxes of arbitrary shape. For additional details, see the technical appendix to Lebowitz and Lieb (In press) or see Robinson (1972). Ω will stand for an arbitrary bounded open region of $\mathbf{R}^{3n}$. Usually it will be of the form $\{\langle \mathbf{x}_1, \ldots, \mathbf{x}_n \rangle \mid \mathbf{x}_j \in \Omega_0\}$ for some set $\Omega_0 \subset \mathbf{R}^3$. To take care of hard cores, one need only remove $\{\mathbf{x}_1, \ldots, \mathbf{x}_n \mid \mathbf{x} \in \Omega_0\ |\mathbf{x}_i - \mathbf{x}_j| \leqslant a$ for some $i, j\}$ from Ω. We won't worry about statistics or spins. They both can be added with trivial modifications:

A. The Free Hamiltonian

Let $C_0^\infty(\Omega)$ be the C^∞ functions with support compact in Ω. $-\Delta$ defined on $C_0^\infty(\Omega)$ is a positive symmetric operator. Its Friedrichs extension on $L^2(\Omega)$ will be denoted by H_0^Ω; it is just the "zero boundary condition" extension of $-\Delta$. Notice if we want hard cores, we just remove the core region from Ω and H_0^Ω has the hard cores built in. H_0^Ω has one very important property:

THEOREM 3.8. *Let $\Omega \subset \Omega'$. Let $L^2(\Omega)$ be mapped into $L^2(\Omega')$ in the natural way. If $\phi \in Q(H_0^\Omega)$ it is in $Q(H_0^{\Omega'})$.*

Proof. $\phi \in Q(H_0^\Omega)$ means that there are $\phi_n \in C_0^\infty(\Omega)$ with $\phi_n \to \phi$ in $L^2(\Omega)$ and such that $\nabla \phi_n$ is $L^2(\Omega)$-Cauchy. Since $C_0^\infty(\Omega) \subset C_0^\infty(\Omega')$, any $\phi \in Q(H_0^\Omega)$ is in $Q(H_0^{\Omega'})$. ∎

Notice this theorem would *not* be true if Q were replaced by D; for a function in Ω which is C^∞ and vanishes on $\partial\Omega$ is in $D(H_0^\Omega)$ if Ω is a nice region. If $\nabla\psi$ does not vanish on $\partial\Omega$, and Ω' contains $\overline{\Omega}$, ψ will not be in $D(H_0^{\Omega'})$. For comparison of operators, form domains are enough:

THEOREM 3.9 (Weyl's min–max principle—form version) *Let A be a self-adjoint operator which is bounded from below. Let*

$$\mu_n(A) = \max_{\phi_1,\ldots,\phi_{n-1}} \left(\min_{\substack{\psi \in Q(A) \\ \psi \in [\phi_1,\ldots,\phi_{n-1}]^\perp \\ \|\psi\| = 1}} (\psi, A\psi) \right).$$

Then either (a) μ_n is the nth *eigenvalue from the bottom of $\sigma(A)$ (counting multiplicity) or*

(b) $\mu_n = \inf \sigma_{ess}(A)$.

In particular, $\sigma_{ess}(A) = \varnothing$ if and only if $\lim_{n\to\infty} \mu_n(A) = \infty$. *$e^{-\beta A}$ is trace class, if and only if $\Sigma \exp[-\beta\mu_n(A)] < \infty$ and in that case* $\operatorname{tr}[\exp(-\beta A)] = \sum_{n=1}^\infty \exp[-\beta\mu_n(A)]$.

We then have:

THEOREM 3.10. *H_0^Ω has purely discrete spectrum and* $\exp(-\beta H_0^\Omega)$ *is trace class. If $\Omega \subset \Omega'$, then*

$$Z_\Omega^0(\beta) = \operatorname{tr}[\exp(-\beta H_0^\Omega)] \leqslant Z_{\Omega'}^0(\beta).$$

Proof. By Theorems 3.8 and 3.9 $\mu_n(H_0^\Omega) \geqslant \mu_n(H_0^{\Omega'})$ if $\Omega \subset \Omega'$. By explicit computation, $\Sigma \exp[-\beta\mu_n(H_0^\Omega) < \infty$ if Ω is a cube. The theorem then follows easily. ∎

B. Interacting Hamiltonians

Now let V_{ij} be a function on $\mathbf{R}^3$ which is a Rollnik potential. Then, as we have seen, for any $a > 0$ $(\psi, V_{ij}\psi) \leqslant a(\psi, -\Delta_{\mathbf{R}^3}\psi) + b(\psi, \psi)$ for suitable n independent of ψ. Now view V_{ij} as a function of $\mathbf{r}_i - \mathbf{r}_j$ on $\mathbf{R}^{3n}$. As in our proof of Theorem 1.14, $(\psi, V_{ij}\psi) \leqslant a(\psi, -\Delta_{\mathbf{R}^{3n}}\psi) + b'(\psi, \psi)$ for $\psi \in Q(-\Delta_{\mathbf{R}^{n3}})$. But, by Theorem 3.8 (boundedness of Ω played no role until Theorem 3.10), any $\psi \in Q(H_0{}^\Omega)$ is in $Q(-\Delta_{\mathbf{R}^{3n}})$ and so obeys $(\psi, V_{ij}\psi) \leqslant a(\psi, H_0{}^\Omega\psi) + b'(\psi, \psi)$. Applying Theorem 3.4, we conclude:

THEOREM 3.11. *Let* $\{V_{ij}\}_{i,j=1}^n$ *be functions in* $(R + L^\infty)(\mathbf{R}^3)$. *Let* $\Omega \subset \mathbf{R}^{3n}$ *open. Then* $H_0{}^\Omega + \sum_{ij} V_{ij}$ *defined as a sum of forms on* $Q(H_0{}^\Omega)$ *is the form of a self-adjoint operator,* H^Ω.

We also have:

THEOREM 3.12. *Let* Ω *be a bounded region of* $\mathbf{R}^{3n}$ *and let* H^Ω *be as in Theorem* 3.11. *Then* $e^{-\beta H^\Omega}$ *is trace class for any* $\beta > 0$. *If* $\Omega \subset \Omega'$:

$$Z_\Omega(\beta) = \operatorname{tr}[(\exp - \beta H_\Omega)] \leqslant Z_{\Omega'}(\beta).$$

Proof. By the basic Rollnik estimate, $\mu_n(H^\Omega) \geqslant (1-a)\,\mu_n(H_0{}^\Omega) + b$. The discreteness of $\sigma(H^\Omega)$ and existence of Z_Ω follow from the analogous facts for $H_0{}^\Omega$. The inequality on the partition functions comes from the inequality $\mu_n(H^\Omega) \geqslant \mu_n(H^{\Omega'})$ proven as in Theorem 3.10. ∎

4. THE GEL'FAND THEORY OF COMMUTATIVE BANACH ALGEBRAS

The material we discussed in this section is of a more standard nature than that of Sections 1.3. A beautiful pedagogic discussion can be found in the little monograph by Gel'fand *et al.* (1964).

Definition. A **Banach algebra** (with identity), $\mathbf{B}$, is a complex Banach space together with

(i) An associative, distributive multiplication with identity, 1,

(ii) $\|ab\| \leqslant \|a\|\,\|b\|$,

(iii) $\|1\| = 1$.

Our first goal is to construct a natural map from $\mathbf{B}$ to continuous functions on a compact Hausdorff space when $\mathbf{B}$ is commutative. We first need a technical lemma whose content is familiar from operator theory on a Banach space X (with $\mathbf{B} = L(X)$, the bounded operators on X):

LEMMA 4.1 (a) *The family of invertable elements, I, of* **B** *is open. Inverse is continuous on I.*

(b) *Maximal proper (i.e. not equal to* **B**) *two-sided ideals are closed.*

(c) *If $x \in \mathbf{B}$, the* **spectrum** *of x, $\sigma(x) = \{\lambda \in \mathbf{C} \mid x - \lambda 1$ is non-invertible$\}$ is a compact subset of* **C**.

(d) *For any $x \in \mathbf{B}, \sigma(x) \neq \varnothing$.*

(e) *Let* spr (x), *the spectral radius of x, be defined by* spr $(x) = \sup_{\lambda \in \sigma(x)} |\lambda|$ *is given by (Gel'fand spectral radius formula).*

$$\operatorname{spr}(x) = \lim_{n \to \infty} \|x^n\|^{1/n}.$$

Remark. (e) asserts $\lim_{n \to \infty} \|x^n\|^{1/n}$ exists.

Proof. (a) Let $A \in I$ and let $\|B\| \leqslant \|A^{-1}\|^{-1}$. Then the geometric series $A^{-1} - A^{-1}BA^{-1} + A^{-1}(BA^{-1})^2 - \ldots (-1)^n A^{-1}(BA^{-1})^n + \ldots$ converges and yields a two-sided inverse for $A + B$. This proves that I is open and if $x_n, x \in I$ and $x_n \to x$, then $x_n^{-1} \to x^{-1}$.

(b) If I is a maximal ideal, (which is proper) $1 \notin I$ so $1 \notin \bar{I}$ by (a). Since $\bar{I}$ is also an ideal and is proper since $1 \notin \bar{I}, I = \bar{I}$ by maximality.

(c) $\sigma(x)$ is bounded, for if $\lambda \geqslant \|x\|$, then $1 + \lambda^{-1}x + (\lambda^{-1}x)^2 + \ldots + (\lambda^{-1}x)^n + \ldots$ converges to C_λ and $(x - \lambda)^{-1} = (-\lambda)^{-1}(1 - \lambda^{-1}x) = (-\lambda)^{-1} C_\lambda$. $\sigma(x)$ is closed because I is open, the map $M : \lambda \to x - \lambda$ is continuous and $\mathbf{C} \backslash \sigma(x) = M^{-1}(I)$.

(d) Consider the function $f(\lambda) = (1 - \lambda x)^{-1} = \lambda^{-1}(\lambda^{-1} - x)^{-1}$. f is a vector valued analytic function in the entire plane if $\sigma(x) = \varnothing$ and in addition $\lim_{\lambda \to \infty} \|f(\lambda)\| = 0$ since $\lim_{\lambda \to \infty} (\lambda^{-1} - x)^{-1} \to (-x)^{-1}$. By the vector-valued Liouiville theorem, $f(\lambda) = 0$.

(e) We first note that $\lim_{n \to \infty} \|x^n\|^{1/n}$ exists and equals $\inf_n \|x^n\|^{1/n}$. For let $l(m) = \log \|x^m\|$. Then $l(m + n) \leqslant l(m) + l(n)$. Fix n and write $k = an + q$. Where a and q are integers and $0 \leqslant q < n - 1$. Then $l(k) \leqslant al(n) + l(q)$. As a result

$$\overline{\lim} \frac{l(k)}{k} \leqslant \frac{l(n)}{n} \text{ so } \overline{\lim} \frac{l(k)}{k} \leqslant \inf \left[\frac{l(k)}{k}\right].$$

This proves the limit exists and is equal to the inf.

Next we note that Hadarmard's theorem implies the radius of convergence of $1 + \mu x + \mu^2 x^2 \ldots$ is $(\lim \|x^n\|^{1/n})^{-1}$. It is also the inverse of the $\sup_{\lambda \in \sigma(x)} |\lambda|$ because the nearest singularity of $(1 - \mu x)^{-1}$ to $\mu = 0$ determines the radius of convergence. ■

THEOREM 4.2. (a) (Gel'fand–Mazur theorem). *Every Banach field (in fact, every Banach division ring) is canonically isomorphic to* $\mathbf{C}$.

(b) *Let* $\mathbf{B}$ *be commutative. There is a canonical bijection between maximal ideals,* $I \subset \mathbf{B}$ *and continuous non-zero homomorphisms,* l, *from* $\mathbf{B}$ *to* $\mathbf{C}$, *given by* $I = \ker l$.

Remark. By using (b) of Lemma 4.1, one can show that any homomorphism of $\mathbf{B}$ into C is continuous.

Proof. (a) Let $x \in \mathbf{B}$. Then $\sigma(x) \neq \varnothing$ so $x - \lambda$ is not invertible for some λ. The only non-invertible element in a division ring is 0 so $x = \lambda 1$ for some λ.

(b) If I is a maximal ideal, it is closed by Lemma 3.1 so $\mathbf{B}/I$ is a Banach space. Since I is an ideal, $\mathbf{B}/I$ is an algebra and simple manipulations prove it is a Banach algebra. Finally since I is maximal, $\mathbf{B}/I$ is a field; hence $\mathbf{B}/I \simeq \mathbf{C}$, The natural homomorphism π: $\mathbf{B} \to \mathbf{B}/I$ defines a (unique) homomorphism from $\mathbf{B}$ to $\mathbf{C}$ with I. ∎

Remark. Homomorphisms from $\mathbf{B}$ to $\mathbf{C}$ are sometimes called **multiplicative linear functionals.**

Definition. The set of maximal proper ideals of a commutative Banach algebra, $\mathbf{B}$, (equivalently the set of multiplicative linear functionals) is called the **spectrum** of $\mathbf{B}$; $\sigma(\mathbf{B})$.

THEOREM 4.3 (Gel'fand). *Let* $\mathbf{B}$ *be a commutative Banach algebra with spectrum* $\sigma(\mathbf{B})$. *Then*:

(a) *Viewing* $\sigma(\mathbf{B})$ *as a set of multiplicative linear functionals,* $\sigma(\mathbf{B})$ *is a weak *-closed subset of the unit ball of* $\mathbf{B}$.

(b) $\sigma(\mathbf{B})$ *with the induced topology (induced from the weak *-topology on* $\mathbf{B}^*$*) is a compact Hausdorff space.*

(c) *Given* $x \in \mathbf{B}$ *and* $l \in \sigma(\mathbf{B})$, *define* $\hat{x}(l) = l(x)$. *The function* $\hat{x}$ *on* $\sigma(\mathbf{B})$ *has the property* $\operatorname{ran} \hat{x} = \sigma(x)$.

(d) $\wedge : \mathbf{B} \to C(\sigma(\mathbf{B}))$ *is a homorphism and* $\|\hat{x}\|_\infty \leqslant \|x\|$.

Proof. (a) Let $l \in \sigma(\mathbf{B})$ correspond to the maximal ideal, I. Let $x \in \mathbf{B}$. Then $l(x - l(x)\,1) = 0$ so $x - l(x) \in I$. Thus $x - l(x)$ is not invertible, i.e. $l(x) \in \sigma(x)$. As a result of the spectral radius formula, $|l(x)| \leqslant \|x\|$ so $\|l\|_{\mathbf{B}^*} \leqslant 1$. To see that $\sigma(\mathbf{B}) \subset \mathbf{B}^*$ is weak *-closed, let $l_\alpha \to l$ in $\mathbf{B}^*$ with $l_\alpha \in \sigma(\mathbf{B})$. For any $x, y \in X$, $l_\alpha(xy) = l_\alpha(x)\, l_\alpha(y)$ and $l_\alpha(z) \to l(z)$ for $z = x, y$ or xy. Thus $l(xy) = l(x)\, l(y)$ so $l \in \sigma\,(\mathbf{B})$.

(b) This is just the Banach–Alaoglu theorem which says the unit ball of $\mathbf{B}^*$ is weak $*$-compact.

(c) Our argument in (a) shows that $\operatorname{ran}\hat{x} \subset \sigma(x)$. So suppose $\lambda \in \sigma(x)$. Then $x - \lambda$ is not invertible so $(x-\lambda)\mathbf{B} \equiv \{(x-\lambda)y \mid y \in \mathbf{B}\}$ is a proper ideal in $\mathbf{B}$. By a Zorn's lemma argument, $(x-\lambda)\mathbf{B}$ is contained in a maximal proper ideal which corresponds to some $l \in \sigma(\mathbf{B})$. Then $\hat{x}(l) = l(x) = \lambda$ so $\lambda \in \operatorname{ran}\hat{x}$.

(d) That $\wedge$ is a homomorphism is obvious. Finally $\|\hat{x}\|_\infty = \operatorname{spr}(x) = \lim \|x^n\|^{1/n} \leqslant \|x\|$. ∎

Remark. The weak $*$-topology on $\sigma(\mathbf{B})$ is called the **Gel'fand topology** and $\hat{x}$ is called the **Gel'fand transform.**

Why is $\sigma(\mathbf{B})$ called the spectrum of $\mathbf{B}$? A partial answer is:

THEOREM 4.4. *Let* $\mathbf{B}$ *be a commutative Banach algebra. Suppose B is generated by* $x \in \mathbf{B}$, *i.e.* $\{\sum_{n=0}^{N} a_n x^n\}$ *is dense in* $\mathbf{B}$. *Then the map* $\hat{x}\colon \sigma(\mathbf{B}) \to \sigma(x) \subset \mathbf{C}$ *is a homomorphism of* $\sigma(\mathbf{B})$ *and* $\sigma(x)$.

Proof. $\hat{x}$ is continuous and $\sigma(\mathbf{B}), \sigma(x)$ are compact and Hausdorff so it is enough to prove that $\hat{x}$ is bijective. By (c) of the last theorem, $\hat{x}$ is subjective so we need only prove it injective. Suppose $\hat{x}(l_1) = \hat{x}(l_2)$. Since l_1 and l_2 are multiplicative, $l_1(\sum_{n=0}^{N} a_n x^n) = l_2(\sum_{n=0}^{N} a_n x^n)$. Since l_1 and l_2 are continuous and x generates $\mathbf{B}, l_1 = l_2$. ∎

Remark. More generally, if $x_1, \ldots, x_k$ generate $\mathbf{B}$, $\sigma(\mathbf{B})$ is homomorphic to the joint spectrum of $x_1, \ldots, x_k$, a subset of $\mathbf{C}^k$ under $\hat{x}_1 \otimes \ldots \otimes \hat{x}_k$.

As a final element of the abstract theory we can precisely describe for which $\mathbf{B}$, $\wedge$ is a isometric isomorphism of $\mathbf{B}$ and $C(\sigma(\mathbf{B}))$.

Definition. A **Banach $*$-algebra** is a Banach algebra, $\mathbf{B}$, with a map $*$ obeying

(i) $*$ is conjugate linear,

(ii) $x^{**} = x$ for all $x \in \mathbf{B}$,

(iii) $(xy)^* = y^* x^*$,

(iv) $\|x^*\| = \|x\|$.

A **C*-algebra** or **abstract C*-algebra** (also called B^*-algebra) is a Banach $*$-algebra obeying the additional property

(v) $\|a^* a\| = \|a\|^2$.

Remarks. (1) (v) implies (iv).

(2) If $\hat{f}^*(l) = \overline{\hat{f}(l)}$, we say ^ is a *-isomorphism.

THEOREM 4.5 (Commutative Gel'fand Naimark theorem). *^ is an isometric *-isomorphism of* **B** *onto* $C(\sigma(\mathbf{B}))$ *if and only if* **B** *is an abstract* C^**-algebra.*

Proof. (1) If ^ is isometric and *-isomorphism, the C^* property follows from $\|f\|_\infty^2 = \| \, |f|^2 \, \|_\infty$.

(2) Let **B** be a C^*-algebra. Let h be a hermitian element of **B**, i.e. $h = h^*$. Then $\hat{h}$ is real, for let $u_t = e^{ith}$ with $t \in \mathbf{R}$ defined by

$$e^{ith} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} h^n.$$

Then $u_t^* = u_{-t}$ and $u_t^* u_t = u_0 = 1$. Thus $\|u_t\| = \|u_{-t}\| = 1$. But for $l \in \sigma(B)$, $l(u_t) = e^{itl(h)}$ and $l(u_{-t}) = e^{-itl(h)}$. Since $|l(u_t)|$ and $|l(u_{-t})|$ are not greater than 1 we conclude $|l(u_t)| = 1$, i.e. $l(h)$ is real.

(3) If $x \in \mathbf{B}, x = y + iz$ with y and z hermitian

$$[y = \tfrac{1}{2}x + \tfrac{1}{2}x^*; \quad z = \frac{i}{2}(x^* - x)]$$

so

$$\hat{x}^*(l) = l(x^*) = \overline{l(x)} = \overline{\hat{x}(l)}.$$

Thus ^ is a *-homomorphism.

(4) Let h be hermitian. Then $\|h\|^2 = \|h^2\|$ so $\|h\|^{2^n} = \|h^{2^n}\|$. Thus $\|\hat{h}\|_\infty = \operatorname{spr}(h) = \lim_{h \to \infty} \|h^{2^n}\|^{1/2^n} = \|h\|$.

(5) For $x \in \mathbf{B}$, $\|\hat{x}\|_\infty^2 = \|\bar{\hat{x}}\hat{x}\|_\infty = \|\widehat{x^*x}\|_\infty = \|x^*x\| = \|x\|^2$ so ^ is isometric.

(6) By 5, ran ^ is complete, thus closed and ^ is injective.

(7) ^ is closed, a subalgebra of $C(\sigma(\mathbf{B}))$ and is conjugate symmetric by 3. Moreover, $1 = \hat{1} \in \operatorname{ran}$ ^ and if $l_1 \neq l_2$ in $\sigma(\mathbf{B})$, $\hat{x}(l_1) \neq \hat{x}(l_2)$ for some $\hat{x}$ so ran ^ separates points. By the Stone–Weierstrass theorem, $\hat{x}$ is surjective. ∎

Theorems 4.4 and 4.5 have one elementary synthesis:

THEOREM 4.6. *Let* **B** *be a commutative* C^**-algebra. Suppose* $x \in \mathbf{B}$, *so that* x *and* x^* *generate* **B**; *then* $\sigma(\mathbf{B}) \to \sigma(x)$ *is a homomorphism.*

Before discussing the application to the spectral theorem we need one more abstract theorem. First note if $x \in \mathbf{B}$, by enlarging $\mathbf{B}$ we can sometimes shrink $\sigma(x)$:

Example. Let $\mathbf{B}$ be the algebra of functions analytic in $\{z \mid |z| < 1\}$ continuous on $\{z \mid |z| \leqslant 1\}$ with $\|f\| = \sup_{|z|=1} |f(z)|$. Let z be the obvious element of $\mathbf{B}$. Then $\sigma_{\mathbf{B}}(z) = \{\lambda \mid |\lambda| \leqslant 1\}$ (We use $\sigma_{\mathbf{B}}$ to emphasize this is the spectrum with respect to $\mathbf{B}$). Let $\mathbf{C} = C(\{z \mid |z| = 1\})$ with $\|.\|_\infty$-norm. Then $\mathbf{B} \subset \mathbf{C}$ and $\sigma_{\mathbf{C}}(z) = \{\lambda \mid |\lambda| = 1\}$.

This shrinkage cannot happen for elements of C^*-algebras:

THEOREM 4.7 (Permanence of spectrum). *Let $x \in B \subset C$ where B and C are C^*-algebra (not necessarily commutative). Then $\sigma_B(x) = \sigma_C(x)$.*

Proof. We note that this theorem is proven if we can show that whenever $x - \lambda$ has an inverse in $\mathbf{C}$, this inverse is in the C^*-algebra generated by x and x^*. Equivalently, we need only show if $a \in \mathbf{C}$ and a is invertible, then a^{-1} is in the C^*-algebra generated by a and a^*. First suppose $a = h$ is hermitean and h^{-1} exists. Since $\sigma(h) \subset \mathbf{R}$ we can obtain h^{-1} by analytically continuing $(h - \lambda)^{-1}$ along the imaginary axis from $\lambda = 2i\,\|h\| \equiv \lambda_0$. $(h - \lambda_0)^{-1} = \lambda_0{}^{-1}(1 + \lambda_0{}^{-1}h + \lambda_0{}^{-2}h^2 + \ldots)$ is in the C^*-algebra generated by h and at each stage in the continuation all the Taylor series coefficients are in the C^*-algebra generated by h. Thus h^{-1} is in the C^*-algebra generated by h. Now let a be arbitrary. If a^{-1} exists then a^*a inverse (equal to $a^{-1}\,(a^{-1})^*$) exists. Since a^*a is hermitian, $(a^*a)^{-1}$ is in the C^*-algebra generated by a^* and a .Since $a^{-1} = (a^*a)^{-1}a^*$, a^{-1} is in the C^*-algebra generated by a and a^*. ∎

Application

Let us apply the Gel'fand theory to sketch a proof of the spectral theorem. First, let A be a bounded self-adjoint operator on a Hilbert space, $\mathscr{H}$. Let $L(\mathscr{H})$ be the bounded operators on $\mathscr{H}$ and let $\mathbf{B}$ be the subalgebra generated by A. $\mathbf{B}$ is abelian and both $\mathbf{B}$ and $L(\mathscr{H})$ are C^*-algebras since $\|A^*A\| = \|A\|^2$ for operators on a Hilbert space. Thus $\sigma_{\mathbf{B}}(A) = \sigma(A)$. Then $\wedge : \mathbf{B} \to C(\sigma(A))$ is an isometric isomorphism of $\mathbf{B}$ onto $C(\sigma(A))$. Let $\phi : f \to f(A)$ be its inverse. Suppose temporarily that $\mathbf{B}$ has a cyclic vector, i.e. $\{B\psi \mid B \in \mathbf{B}\}$ is dense in $\mathscr{H}$ for some ψ. Let μ_ψ be defined on $C(\sigma(A))$ by $\mu_\psi(f) = (\psi, f(A)\psi)$. μ_ψ is a positive linear functional and thus there is a measure $d\mu_\psi$ with $\mu_\psi(f) = \int_{\sigma(A)} f d\mu$. Map $U: C(\sigma(A)) \to$ by $Uf = f(A)\psi$. Then

$$\|Uf\|^2 = \langle\psi, f^*(A)f(A)\psi\rangle = \int d\mu f^*f = \|f\|^2_{L^2(\sigma, d\mu_\psi)}.$$

By the cyclic assumption, U extends to a unitary map of $L^2(\sigma(A), d\mu_\psi)$ onto $\mathscr{H}$. A simple computation shows that $(U^{-1}AUf)(\lambda) = \lambda f(\lambda)$. More generally, if $\mathbf{B}$ does not have a cyclic vector, we can find (using Zorn's lemma), orthogonal subspaces $\{\mathscr{H}_\alpha\}_{\alpha \in I}$ in $\mathscr{H}$ with $\oplus_\alpha \mathscr{H}_\alpha = \mathscr{H}$ and vectors $\psi_\alpha \in \mathscr{H}_\alpha$ with $\{B\psi_\alpha \mid B \in \mathbf{B}\} = \mathscr{H}_\alpha$. Then we construct $U: \oplus_\alpha L^2(\sigma(A), d\mu_\alpha) \to \mathscr{H}$ with $U^{-1}AU =$ multiplication by λ.

In the same way we can realize any bounded normal operator as a multiplication operator (by a non-real-valued function). Finally, if A is self-adjoint but unbounded, $(A + i)^{-1}$ exists and is a bounded normal operator. By realizing it as a multiplication operator, we can realize A as a multiplication operator.

5. THE GNS CONSTRUCTION

The GNS (for Gel'fand, Naimark and Segal) construction which we will describe can be viewed as a non-commutative version of the Reisz–Markov theorem which describes positive linear functionals on $C(X)$ where X is a compact Hausdorff space. Since we now know (Theorem 4.5) that any commutative C^*-algebra, $\mathbf{B}$, is isometrically isomorphic to $C(\sigma(\mathbf{B}))$ we can:

1st Translation of Reisz–Markov

Given a linear functional ϕ on $\mathbf{B}$, a commutative C^*-algebra, obeying $\phi(x^*x) \geqslant 0$ for all x, there is a measure μ on $\sigma(\mathbf{B})$ with $\phi(x) = \int_{\sigma(B)} \hat{x}(l)\, d\mu(l)$.

We want to somehow eliminate $\sigma(\mathbf{B})$ from this picture since there is not a really nice analogue of $\sigma(\mathbf{B})$ when $\mathbf{B}$ is not abelian. A fairly standard way of "eliminating" a measure space is to transform conclusions into statements about the Hilbert space L^2 $(M, d\mu)$. In the above case, letting $\mathscr{H} = L^2$ $(M, d\mu)$ we have for each $x \in \mathbf{B}$, a bounded operator, $\pi(x)$ on $\mathscr{H}$ given by $(\pi(x)f)(l) = \hat{x}(l)f(l)$. Moreover, since $C(\sigma(\mathbf{B}))$ is dense in L^2 $(\sigma(\mathbf{B}), d\mu)$ the vector $\Omega_0 = 1$ on $\sigma(\mathbf{B})$ has the property that $\{\pi(x)\Omega_0 \mid x \in \mathbf{B}\} = C(\sigma(\mathbf{B}))$ is dense in L^2. Or Ω_0 is cyclic for $\pi(\mathbf{B})$ where:

Definition. $\Omega_0 \in \mathscr{H}$ is called cyclic for $\mathfrak{A}$, an algebra of bounded operators on $\mathscr{H}$, if $\{A\Omega_0 \mid A \in \mathfrak{A}\}$ is dense in $\mathscr{H}$.

Thus:

2nd Translation of Reisz–Markov

Given a linear functional ϕ on $\mathbf{B}$, a commutative C^*-algebra, obeying $\phi(x^*x) \geqslant 0$, there is a representation, π, of $\mathbf{B}$ as operators on a Hilbert space, (i.e. π is a *-homomorphism of $\mathbf{B}$ into $L(\mathscr{H})$) and a vector $\Omega_0 \in \mathscr{H}$ which is cyclic for $\pi(B)$ and so that

$$(\Omega_0, \pi(x)\Omega_0) = \phi(x).$$

This is the form of the Reisz–Markov theorem that has a non-commutative analogue. First, we define:

Definition. Let B be a C^*-algebra. A linear functional, ϕ, from $\mathbf{B}$ to $\mathbf{C}$ is called *positive* if $\phi(x^*x) \geqslant$ for all $x \in \mathbf{B}$.

There are other candidates for *positivity* and we discuss some of them below.

THEOREM 5.1 (The GNS Construction). *Let ϕ be a positive linear functional on a C^*-algebra $\mathbf{B}$. Then there is a Hilbert space, $\mathscr{H}$, a representation $\pi: \mathbf{B} \to L(\mathscr{H})$ and a cyclic vector $\Omega_0 \in \mathscr{H}$ for $\pi\,[\mathbf{B}]$ so that $\phi(x) = (\Omega_0, \pi(x)\Omega_0)$. Moreover, the triple $\langle \mathscr{H}, \pi, \Omega_0 \rangle$ is uniquely associated to ϕ in the sense that if $\langle \mathscr{H}^{(1)}, \pi^{(1)}, \Omega_0{}^{(1)} \rangle$ is another triple with $\pi^{(1)}$ a representation, $\Omega_0{}^{(1)}$ cyclic and $\phi(x) = (\Omega_0{}^{(1)}, \pi^{(1)}(x)\,\Omega_0{}^{(1)})$ then there is a unitary map $u: \mathscr{H}^{(1)} \to \mathscr{H}$ so that $u\Omega_0{}^{(1)} = \Omega_0$ and $u\pi^{(1)}(x)\,u^{-1} = \pi(x)$.*

Proof. We must first find $\mathscr{H}$. There is only one object around from which to build $\mathscr{H}$, namely $\mathbf{B}$! And there is only one candidate for an inner product. Define $[.,.]$ on $\mathbf{B} \times \mathbf{B}$ by $[x, y] = \phi(x^*y)$. $[.,.]$ has all the properties of inner product but the definiteness condition. Letting $\|x\|_\phi = \phi(x^*x)$ we conclude from the Cauchy–Schwartz inequality that

$$|\phi(x^*y)| \leqslant \sqrt{\phi(x^*x)}\,\sqrt{\phi(y^*y)}. \tag{5.1}$$

We first show that ϕ is continuous. (Notice we haven't assumed it continuous). First let h be hermitian; and let $\mathbf{B}_h \subset \mathbf{B}$ be the C^*-algebra generated h. $\mathbf{B}_h$ is abelian so $\mathbf{B}_h \simeq C(\sigma(\mathbf{B}_h))$. ϕ thus acts as a positive linear functional on $C(\sigma(\mathbf{B}_h))$ by $\hat{\phi}(x) = \phi(^{\wedge\,-1}\,x)$. Since $\|\hat{h}\| \pm h \geqslant 0$, $\hat{\phi}(\|\hat{h}\| \pm \hat{h}) \geqslant 0$ or $|\phi(h)| \leqslant \phi(1)\,\|h\|$. For general $y \in \mathbf{B}$, $\phi(y^*y) \leqslant \phi(1)\,\|y^*y\|$ since y^*y is hermitian, so by (5.1)

$$|\phi(y)|^2 \leqslant (\phi(1))^2\,\|y^*y\| = (\phi(1)\,\|y\|\,)^2.$$

Now let $\mathfrak{I}_\phi = \{x \in \mathbf{B} \mid \|x\|_\phi = 0\}$. $\mathfrak{I}_\phi$ is clearly a subspace. In fact it is a left ideal. For applying (5.1), if $x \in \mathfrak{I}_\phi$:

$$\|y\,x\|_\phi = (\phi(x^*y^*y\,x))^2 \leqslant \phi(y^*y\,x\,x^*y^*y)\,\phi(x^*x) = 0.$$

Thus $\|yx\|_\phi = 0$, i.e. $yx \in \mathfrak{I}_\phi$.

Now we note that $\mathbf{B}/\mathfrak{I}_\phi$ is an inner product space so it has a completion $\mathscr{H}$. We define $[x] \in \mathscr{H}$, to be the class of $\mathbf{B}/\mathfrak{I}_\phi$ containing x. We define $\pi(y)[x]$ by $\pi(y)[x] = [yx]$. Since $\mathfrak{I}_\phi$ is a left ideal, $\pi(y)[x]$ is well defined. Moreover:

$$\begin{aligned}\|[yx]\|^4 = \|yx\|_\phi{}^4 &= \phi(x^*y^*yx)^2 \\ &\leqslant \phi(yy^*x^*xy^*y)\,\phi(x^*x) \leqslant \phi(1)\,\|[x]\|^2\,\|x\|^2\,\|y\|^4\end{aligned}$$

and

$$\begin{aligned}\|[yx]\|^{16} &\leqslant \phi(yy^*x^*xy^*y)^4\,\phi(x^*x)^4 \\ &\leqslant \phi(x^*xyy^*yy^*x^*x)^2\,\|y\|^8\,\|[x]\|^8 \\ &\leqslant \phi(xyy^*yy^*x^*xx^*xyy^*yy^*x^*)\,\|y\|^8\,\|[x]\|^{10} \\ &\leqslant \phi(1)\,\|[x]\|^{10}\,\|x\|^6\,\|y\|^{16}\end{aligned}$$

By repeating this argument:

$$\|[xy]\| \leqslant \|y\|\,\|[x]\|,$$

so $\pi(y)$ extends to a bounded linear transformation of $\mathscr{H}$ into $\mathscr{H}$. π is thus a representation of $\mathbf{B}$. Next, we define $\Omega_0 = [1]$. Then $\pi(x)\Omega_0 = [x]$ so $\pi[\mathbf{B}]\Omega_0 = \mathbf{B}/\mathfrak{I}_\phi$ is dense, i.e. Ω_0 is cyclic and

$$\begin{aligned}(\Omega_0, \pi[x]\Omega_0) &= ([1], [x]) \\ &= [1, x]_\phi = \phi(x).\end{aligned}$$

This completes the proof of existence. Uniqueness is left to the reader. ∎

Applications and Refinements

(1) *Non-Commutative Gel'fand–Naimark Theorem*

Suppose, $\mathbf{B}$ is an arbitrary C^*-algebra. We have essentially defined a positive element to be an $x \in \mathbf{B}$ of the form $x = y^*y$. A detailed and deep analysis of C^*-algebras shows that:

THEOREM 5.2. *Let* $\mathbf{B}$ *be a* C^**-algebra. Then* x *is of the form* y^*y *if and only if* x *is Hermitian and* $\sigma(x) \subset [0, \infty)$. *The set of all positive elements in* $\mathbf{B}$ *is a convex cone.*

Proof. See Dixmier (1964) or Reed and Simon (In press, Vol. III). ∎

An application of the Hahn–Banach theorem shows that for any $0 \neq x \in \mathbf{B}$, we can find ϕ, a positive linear functional, with $\phi(x) \neq 0$. We thus conclude:

THEOREM 5.3 (Non-Commutative Gel'fand–Naimark theorem). *Any C^*-algebra is isometrically isomorphic to an algebra of operators on a Hilbert space.*

Proof. Let Φ be a family of positive linear functions separating points. Let $\mathscr{H} = \bigoplus_{\phi \in \Phi} \mathscr{H}_\phi$ and $\pi\colon \mathbf{B} \to L(\mathscr{H})$ by $[\pi(x)\psi]_\phi = \pi_\phi(x)\psi_\phi$ where $\langle \mathscr{H}_\psi, \pi_\psi, \Omega_\phi \rangle$ is the representation associated to ϕ by the GNS Construction. π is clearly an isomorphism so we need only prove it is isometric. By our construction in Theorem 5.1, $\|\pi(x)\| \leqslant \|x\|$. Since $\|\pi(x)^*\pi(x)\| = \|\pi(x^*x)\|$ and $\|x^*x\| = \|x\|^2$, $\|\pi(x)\|^2 = \|\pi^*(x)\pi(x)\|$, we need only prove $\|\pi(h)\| = \|h\|$ for h hermitian. If h is hermitian, let $\mathbf{B}_h$ be the C^*-algebra generated by h and $\pi_h \equiv \pi[\mathbf{B}_h]$. π_h and $\mathbf{B}_h$ are algebraically isomorphic so they have the same maximal ideals. Since $\|h\|_{\mathbf{B}_h} = \sup_{l \in \sigma(\mathbf{B}_h)} |l(h)|$ we conclude $\|h\|_{\mathbf{B}_h} = \|\pi(h)\|_{\mathscr{H}}$. This completes the proof. ∎

(2) *Pure States and Irreducible Representations*

A positive linear functional ϕ on $\mathbf{B}$, a C^*-algebra is called a *state* if $\phi(1) = 1$. If $\langle \mathscr{H}_\phi, \pi_\phi, \Omega_\phi \rangle$ is the triple associated to ϕ via the GNS construction, $\|\Omega_\phi\|^2 = \phi(1) = 1$. The states form a compact convex set when given the weak $*$-topology—this set is a *cap* for the cone of positive linear functionals, i.e. each half line from 0 interests the states in precisely one point. We want to ask when the representation π_ϕ is irreducible; it is unclear that it is ever irreducible at this point! We recall the basic definition and Schur's lemma:

Definition. A representation π of $\mathbf{B}$ is called irreducible if the only subspaces of $\mathscr{H}$ left invariant by all the $\pi(A)$ are $\{0\}$ and $\mathscr{H}$.

SCHUR'S LEMMA. *π is irreducible if and only if $\pi[\mathbf{B}]' \equiv \{C \mid C\pi(A) = \pi(A)C$ for all $A \in \mathbf{B}\} = \{\lambda 1\}$.*

Proof. We first note that since $\pi[\mathbf{B}]^* \equiv \{\pi(A)^* \mid A \in \mathbf{B}\} = \{\pi(A^*) \mid A \in \mathbf{B}\} = \pi[\mathbf{B}]$, $V^\perp$ is invariant if V is invariant. If V and $V^\perp$ are invariant, the orthogonal projection onto V commutes with all $\pi(A)$. Thus, if π is not irreducible, then there is a projection $P \neq 0, 1 \in \pi[\mathbf{B}]'$ so $\pi[\mathbf{B}]' \neq \{\lambda 1\}$. On the other hand, if $C \neq \lambda 1 \in \pi[\mathbf{B}]'$, we can find C self-adjoint in $\pi[\mathbf{B}]'$ with $C \neq \lambda 1$. But then C has non-trivial spectral projections P which are in $\pi[\mathbf{B}]'$. P is then a non-trivial invariant subspace. ∎

To determine which states lead to irreducible representations, we note:

THEOREM 5.4. *Let ϕ be a state and let $\langle \mathcal{H}_\phi, \pi_\phi, \Omega_\phi \rangle$ be the corresponding GNS triple. Suppose ψ is a positive linear functional with $\psi \leqslant \phi$, i.e. $\phi - \psi$ is positive. Then there exists a unique $T \in L(\mathcal{H}_\phi)$ with $\psi(B^*A) = (\pi(B)\,\Omega_\phi, T\pi(A)\,\Omega_\phi)$ and this $T \in \pi_\phi[\mathbf{B}]'$ and obeys $0 \leqslant T \leqslant 1$. Conversely, if $T \in \pi_\phi[\mathbf{B}]'$ and $0 \leqslant T \leqslant 1$, then $\psi(A) = (\Omega_\phi, TA\,\Omega_\phi)$ is a positive linear functional with $\psi \leqslant \phi$.*

Proof. Let ψ be given. First notice that

$$|\psi(B^*A)|^2 \leqslant \psi(B^*B)\,\psi(A^*A) \leqslant \phi(B^*B)\,\phi(A^*A) = \|[\mathbf{B}]\|\,\|[A]\|.$$

Thus, ψ lifts to a bounded sesquilinear function l on $\mathcal{H}_\phi$ with $l([B],[A]) = \psi(B^*A)$. By the Reisz lemma, there is a unique $0 \leqslant T \leqslant 1$ with $l([B],[A]) = ([B], T[A])$ or $\psi(B^*A) = (\pi[B]\Omega_\phi, T\pi[A]\Omega_\phi)$. Moreover

$$[[B], (T\pi(A) - \pi(A)T)\,[C]]_\phi = \psi(B^*AC) - \psi((A^*B)^*C) = 0$$

so $T \in \pi_\phi[B]'$. The inverse is trivial. ■

To identify the states which yield irreducible representations under the GNS construction, we must do a little geometry:

Definition. An **extreme point** of a convex set C is a point $x \in C$ for which $x = \frac{1}{2}y + \frac{1}{2}z$ with $y, z \in C$ implies $y = z = x$. An extreme point of the convex set of states on a C^*-algebra is called a **pure state**.

The Krein–Milman theorem assures us that there are lots of pure states, enough so that the weak* -closure of the convex combinations of the pure states is all states. We also need a simple geometric lemma:

LEMMA. *A state ϕ is a pure state if and only if ψ_1, ψ_2 positive functionals and $\psi_1 + \psi_2 = \phi$ implies $\psi_1 = \lambda_1\,\phi$* and *$\psi_2 = \lambda_2\,\phi$.*

Proof. If ϕ is not pure, then $\phi = \frac{1}{2}\phi_1 + \frac{1}{2}\phi_2$ for $\phi_1 \neq \phi \neq \phi_2$. So letting $\psi_i = \frac{1}{2}\phi_i$, we see that $\phi = \psi_1 + \psi_2$ with $\psi_1 \neq \lambda_i\,\phi$. Now suppose ϕ is pure. We first note that if $0 < \lambda < 1$ and $\phi = \lambda\,\phi_1 + (1-\lambda)\phi_2$ where ϕ_1, ϕ_2 are states then $\phi_1 = \phi = \phi_2$. (Interior points of line segments are midpoints of some smaller line segment). Suppose $\phi = \psi_1 + \psi_2$ with ψ_1, ψ_2 positive functionals. If $\psi_i(1) = 0$, the conclusion is trivial so suppose $\psi_i(1) \neq 0$ for $i = 1, 2$. Let $\phi_i = \psi_i(1)^{-1}\,\psi_i$. Then ϕ_i is a state and $\phi = \lambda\phi_1 + (1-\lambda)\phi_2$ where $\lambda = \psi_1(1)$ and $(1-\lambda) = \phi(1) - \psi_1(1) = \psi_2(1)$. Thus $\phi_1 = \phi_2 = \psi_i = \lambda_i\,\phi$. ■

We can now prove:

THEOREM 5.5. *Let ϕ be a state on a C^*-algebra and let $(\mathscr{H}_\phi, \pi_\phi, \Omega_\phi)$ be the associated GNS triple. Then π_ϕ is irreducible if and only if ϕ is pure.*

Proof. π_ϕ is irreducible if and only if $\pi_\phi[\mathbf{B}]' = \{\lambda 1\}$. But by Theorem 5.4, $\pi_\phi[\mathbf{B}]' = \{\lambda 1\}$ if and only if $0 \leqslant \psi \leqslant \phi$ implies $\psi = \lambda\phi$. But $0 \leqslant \psi \leqslant \phi$ if and only if $\phi = \psi + (\phi - \psi)$ and $\psi \geqslant 0$, $\phi - \psi \geqslant 0$ so we conclude that π_ϕ is irreducible if and only if $\phi = \psi_1 + \psi_2$; with $\psi_1, \psi_2 \geqslant 0$ implies $\psi_1 = \lambda\phi$ (and thus $\psi_2 = (1 - \lambda)\phi$). By the lemma, this happens if and only if ϕ is pure. ∎

(3) *Group Covariance*

In applications, one often has more structure than a C^*-algebra, $\mathbf{B}$. One has a set of automorphisms $\beta\colon \mathbf{B} \to \mathbf{B}$ in addition. In fact, one often has a topological group, G, and a representation of G by automorphisms of $\mathbf{B}$, i.e. $\beta_g \beta_h = \beta_{gh}$ which is continuous in the weak sense that $g \to \beta_g(A)$ is weakly continuous, i.e. $l(\beta_g(A))$ is continuous in g for any fixed l and A†. One has the following nice result:

THEOREM. *Let $\mathbf{B}$ be a C^*-algebra, G a topological group continuously represented by automorphisms, β_g, of $\mathbf{B}$. Suppose ϕ is a state of $\mathbf{B}$ left invariant by the β_g in the sense that $\phi(\beta_g(A)) = \phi(A)$ for all $g \in G$ and $A \in \mathbf{B}$. Then there is a strongly continuous unitary representation of G on $\mathscr{H}_\phi$ with $U(g)\,\Omega_\phi = \Omega_\phi$ and $U(g)\,\pi(A)\,U(g)^{-1} = \pi(\beta_g(A))$. The $U(g)$ with these properties are uniquely determined.*

Proof. We first notice that $\beta_g\colon \mathfrak{I}_\phi \to \mathfrak{I}_\phi$ for if $\phi(x^*x) = 0$, then $\phi(\beta_g(x)^*\,\beta_g(x)) = \phi(\beta_g(x^*x)) = \phi(x^*x) = 0$. Thus $[\beta_g(x)]$ is only dependent on $[x]$. Moreover, by the above computation $\|[\beta_g x]\| = \|[x]\|$ so by continuity $[x] \to [\beta_g x]$ extends to an isometry U_g on $\mathscr{H}_\phi$. Since $U_g^{-1} = U_{g^{-1}}$, U_g is surjective, i.e. U_g is unitary. Moreover U_g is weakly continuous for $([y], U_g[x]) = \phi(y^*\beta_g(x)) = l_y(\beta_g(x))$ with $l_y(.) \equiv \phi(y^*.)$. Strong continuity follows from weak continuity by abstract nonsense. Moreover,

$$U_g\,\Omega_\phi = U_g[1] = [\beta_g(1)] = [1] = \Omega_\phi$$

and

$$\begin{aligned} U_g\,\pi(x)\,U_g^{-1}\,[y] &= U_g\,\pi(x)\,[\beta_g^{-1}\,y] = U_g[x\,\beta_g^{-1}\,y] \\ &= [\beta_g(x)\beta_g^{-1}\,y] = [\beta_g(x)\,y) = \pi(\beta_g(x))\,[y]. \end{aligned}$$

† Editor's note: Sometimes, $g \mapsto l(\beta_g(A))$ is continuous in g only for a distinguished subclass of states l, the "regular" states.

By continuity,

$$U_g \pi(x) U_g^{-1} = \pi(\beta_g(x)).$$

The proof of uniqueness is left to the reader. ■

Remark. 1. If π_ϕ is not irreducible, there will be many V_g with $V_g \pi(x) V_g^{-1} = \pi(\beta_g x)$; for let $B \in \pi[\mathbf{B}]'$ and $V_g = B U_g B^{-1}$. It is the condition $U_g \Omega_\phi = \Omega_\phi$ that fixes U_g.

2. It is a useful exercise to prove that $[\{U_g\} \cup \{\pi(\mathbf{B})\}]' = \{0\}$ if and only if ϕ is an extreme invariant state.

(4). *Non-Commutative Ergodic Theory*

Let T be a continuous map of a compact set X into itself. Let $\mathbf{B}$ be $C(X)$ and let β_T: $\mathbf{B} \to \mathbf{B}$ by $(\beta_T f)$ $(x) = f(Tx)$. Then μ is an invariant measure for T if and only if $\phi \circ \beta_T = \phi$ where $\phi(f) = \int f(x)\, d\mu$. The extreme points of $\{\phi \mid \phi \circ \beta_T = \phi\}$ are precisely the ergodic measures of T. More generally, this definition can be extended: if $\mathbf{B}$ is a C^*-algebra and a group C acts by automorphism on $\mathbf{B}$, then extreme invariant states are called ergodic states and the GNS construction plays a role in the development of this ergodic theory. Some of its nicest features are described in Ruelle's book. (Ruelle, 1969a).

(5) *Change of Hilbert Space*

In passing to the infinite volume limit in quantum field theory or in discussing "spontaneously broken symmetry", the phenomena of change of Hilbert space occurs. This is described in the field theory case in Glimm and Jaffe (1970a) and in the broken symmetry case in Wightman (1969a). Both applications are also discussed in Reed and Simon (In press, Vol. III).

(6) *Quantum Statistical Mechanics in Infinite Volume*

As will be discussed in Hugenholtz's lectures (see also Ruelle (1969a), quantum statistical "states" in infinite volume are naturally associated with states on a certain C^*-algebra. However, the GNS construction does *not* produce physically reasonable objects (although it is sometimes a useful technical device for studying the states): at temperature $T \neq 0$, a statistical equilibrium state is not "physically" a vector state but should be a density matrix or a more general object—(a constant family of density matrices if we look at finite volumes). To see how physically absurd the GNS triple $(\mathscr{H}_\phi, \pi_\phi, \Omega_\phi)$ is we note that the energy is not bounded below on $\mathscr{H}_\psi$.

6. VON NEUMANN ALGEBRAS: AN INTRODUCTION

We can only scratch the surface of the theory of von Neumann algebras in this short discussion. We will try to explain why the ultraweak topology is not mysterious but is quite natural. We will not have a chance to discuss the general theory. There is some discussion in Reed and Simon (In press, Vol. III) and Kadison (1958) and a great deal in Dixmier (1957) and Schwartz (1968). A nice presentation of type theory may be found in Lanford (1972).

Definition. An algebra, $\mathfrak{A}$, of operators on $\mathscr{H}$, a Hilbert space, is called a von Neumann algebra if $1 \in \mathfrak{A}$, $\mathfrak{A}^* \equiv \{A^* \mid A \in \mathfrak{A}\} = \mathfrak{A}$ and $\mathfrak{A}$ is closed in the weak vector topology.

Remark. The condition $1 \in \mathfrak{A}$ is often dropped.

We are first heading towards proving that a von Neumann algebra, $\mathfrak{A}$, is in a natural way the Banach space of continuous linear functionals on $W_{\mathfrak{A}}$, the normed linear space of weakly continuous functionals on $\mathfrak{A}$.

We first recall some elementary facts and definitions from duality theory:

Definition. Let X and Z be vector spaces and suppose there is a map m: $Z \times X$ into $\mathbf{C}$ which is bilinear. We write $m(z, x) = (z, x)$. Let Y be a subset of Z and let $[Y]$ be the algebraic closure of Y, i.e.

$$[Y] = \left\{ \sum_{i=1}^{n} \alpha_i y_i \mid y_1, .., y_n \in Y; \alpha_1, .., \alpha_n \in \mathbf{C}; n = 1, 2, \dots \right\}.$$

Then the $\sigma(X, Y)$-topology is the weakest topology on X making $(y, .)$ continuous for each $y \in Y$. We say Y separates points of X if for all $x \in X, x \neq 0, \exists\, y \in Y$ with $(y, x) \neq 0$. We say X separates points in Y if for $\forall\, y \in Y, \exists\, x \in X$ with $(y, x) \neq 0$.

THEOREM 6.1. (a) *If Y separates points of X, then the $\sigma(X, Y)$ topology is Hausdorff. Henceforth, we suppose Y separates points of X.*

(b) $\sigma(X, Y) = \sigma(X, [Y])$.

(c) *If Y is a subspace and X separates points of Y, then $X^*_{\sigma(X,Y)}$ the set of $\sigma(X, Y)$ continuous linear functionals on X is Y in the sense that every $l \in X^*_{\sigma(X,Y)}$ is of the form $l(x) = (y, x)$ for a unique $y \in Y$.*

(d) *More generally if Y is a subspace and $k_X(Y) = \{y \in Y \mid (y, x) = 0$ for all $x \in X\}$, then $X^*_{\sigma(X,Y)} = Y/k_X(Y)$ in that every $l \in X_\sigma{}^*$ is of the form $l(x) = (y, x)$ for a $y \in Y$ unique precisely up to elements of $k_X(Y)$.*

(e) *Suppose X separates points of Z and Z separates points of X. Given $Y \subset X$ a subspace define $Y^0 \subset Z$ by $Y^0 = \{z \in Z \mid (z, y) = 0$ for all $y \in Y\}$. Similarly if $M \subset Z$ we define $M^0 \subset X$. If $Y \subset X$, a subspace, Y^{00}, the double annihilator of Y relative to the (X, Z) pairing is the $\sigma(X, Z)$ closure of Y.*

Proof. (a) and (b) are simple and (c) is a special case of (d). Since each $y \in Y$ defines a $\sigma(X, Y)$-continuous functional and there is uniqueness up to $k_X(Y)$ trivially, we only need show that every $l \in X_\sigma{}^*$ is of the form $l(x) = (y, x)$. Since $l \in X_\sigma{}^*, l^{-1}(\{\lambda \mid |\lambda| \leqslant 1\})$ is a $\sigma(X, Y)$ neighbourhood of 0 and so contains a set of the form $N = \{x \mid |(y_1, x)| < \varepsilon_1, \ldots, |(y_k, x)| < \varepsilon_k\}$ for some $y_1, \ldots, y_k \in Y$; $\varepsilon_1, \ldots, \varepsilon_k \in \mathbf{R}^+$. Without loss of generality suppose $y_1 \ldots, y_k$ are linearly independent. Let $k_{\{y_1, \ldots, y_k\}}(X) = \{x \in X \mid (y_k, x) = 0$ for $y_1, \ldots, y_k\}$. If $x \in k_{\{y_1, \ldots, y_k\}}(X)$, then $\mu x \in N$ for all μ so $l(\mu x) = 0$ for all μ, i.e. $l(x) = 0$. Thus l lifts to $X/k_{\{y_1, \ldots, y_k\}}(X)$ which is finite dimensional of dimension k. Its dual space is spanned by $y_1, \ldots, y_k$ so l is a linear combination of $y_1, \ldots, y_k$ and so in Y.

(e) (Remark: If $X = Z$ is a Hilbert space, with (,) the inner product, this result says $Y^{\perp\perp} = \bar{Y}$). That Y^{00} is closed is simple as is $Y \subset Y^{00}$, so $\bar{Y} \subset Y^{00}$. Suppose $x \notin \bar{Y}$. Then by the Hahn–Banach theorem (admittedly for locally convex spaces!), we can find a $\sigma(X, Z)$-continuous function l with $l(x) = 1$ and $l(y) = 0$ for all $y \in \bar{Y}$. Then $l \in Z$ by (c) and so $l \in Y^0$. Since $l(x) \neq 0$, $x \notin Y^{00} \subset \bar{Y}$. This proves $\bar{Y} = Y^{00}$. ∎

And we recall without proof some facts about duals of normed linear spaces:

THEOREM 6.2. (a) *Let X be a normed linear space and X_c its completion. Then X^*, the Banach space of continuous linear functionals on X is isometrically isomorphic to $(X_c)^*$ under the map $r: X_c{}^* \to X^*$ given by restricting $l \in X_c{}^*$ to X.*

(b) *Let X be a normed linear space and Y a closed subspace. Let X/Y be the quotient normed linear space with $\|[x]\| = \inf_{y \in Y} \|x + y\|$. Let $k_Y(X^*) = \{l \in X^* \mid l(y) = 0$ for all $y \in Y\}$. Then $(X/Y)^*$ is isometrically isomorphic to $k_Y(X^*)$ under the map $r: (X/Y)^* \to k_Y(X^*)$ given by $r(l) = l \circ \pi$ where π is the canonical projection $\pi: X \to X/Y; \pi(x) = [x]$.*

(c) *Let X and Y be as in (b). Let X_c be the completion of X, Y_c the closure of Y in X_c and $(X/Y)_c$ the completion of X/Y. Then $(X/Y)_c$ is isometrically isomorphic to X_c/Y_c under the natural map. In particular $(X_c/Y_c)^*$ is isometrically isomorphic to $(X/Y)^*$.*

So much for the abstract nonsense.

We now need one special case of the general theorem we will prove for von Neumann algebras:

THEOREM 6.3. *Let $\mathscr{H}$ be a Hilbert space and let $X = L(\mathscr{H})$, the C^*-algebra of all bounded operators on $\mathscr{H}$. Let Z be the Banach dual space of X. Let Y be the set of elements of Z of the form $A \to \langle\psi, A\phi\rangle$ for $\psi, \phi \in \mathscr{H}$ and let $[Y]$ be the linear span of Y in Z. Then*

(a) *The $\sigma(X, Y) = \sigma(X, [Y])$ topology on X is the weak vector topology.*

(b) *$[Y]$ is the set of $\sigma(X, Y)$–continuous functionals on X.*

(c) *X is isometrically isomorphic (as a Banach space with the operator norm) to the Banach dual space of $[Y]$ (with the norm $\|y\| = \sup_{\substack{A\in X\\ A\neq 0}}|y(A)|/\|A\|$) under the association $r: X \to [Y]^*$ by $r(A)(y) = y(A)$.*

Remarks. 1. $[Y]$ is not a Banach space but only a normed linear space.

2. The reader may have seen this theorem in a slightly different light: $[Y]$ has a natural realization as all finite rank operators, with the norm on $[Y]$ the trace class norm. The completion of $[Y]$ is then I_1, the trace ideal so (c) says $I_1{}^* = L(\mathscr{H})$. (See Reed and Simon (In press, Vol. I)).

Proof. (a) is trivial and (b) is Theorem 6.1 (c). Let us prove (c). X maps naturally into $[Y]^*$ and $\|r(A)\|_{Y^*} \leqslant \|A\|_X$ (trace through the definitions). On the other hand

$$\|A\|_X = \sup_{\|\phi\|,\ \|\psi\| = 1} |(\phi, A\psi)| \leqslant \|r(A)\| \sup_{\|\phi\|\ \|\psi\| = 1} \|(\phi, \,.\, \psi)\| \leqslant \|r(A)\|$$

so r is isometric. It is thus sufficient to show every $l \in [Y]^*$ is of the form $r(A)$ for some A. Since $l \in [Y]^*$ and $\|(\phi, .\psi)\|_{(Y)} \leqslant \|\phi\|\ \|\psi\|$ (actually equal!), $B(\phi, \psi) \equiv l(\langle\phi, .\, \psi\rangle)$ obeys $|B(\phi, \psi)| \leqslant \|l\|\ \|\phi\|\ \|\psi\|$. $B(., .)$ is clearly conjugate linear in the first variable, linear in the second and thus is of the form $B(\phi, \psi) = (\phi, A\psi)$, for some $A \in L(\mathscr{H})$ by the Reisz lemma. Thus l and $r(A)$ agree when applied to elements of Y. Since Y generates $[Y]$, $l = r(A)$. ∎

We are now ready to prove:

THEOREM 6.4. (Dixmier's theorem). *Let $\mathfrak{A}$ be a von Neumann algebra on a Hilbert space $\mathscr{H}$. Let $W_{\mathfrak{A}}$ be the family of weakly continuous linear functions on $\mathfrak{A}$. $W_{\mathfrak{A}}$ is a subspace of $\mathfrak{A}^*$ (since the weak topology is weaker than the norm topology) and so $W_{\mathfrak{A}}$ is a normed linear space $\mathfrak{A}$ is isometrically isomorphic to $W_{\mathfrak{A}}{}^*$ under the duality map $r: \mathfrak{A} \to W_{\mathfrak{A}}{}^*$; $[r(A)](y) = y(A)$.*

Remarks 1. Dixmier's theorem was first proven in Dixmier (1953).

2. This theorem is usually stated in terms of the ultraweak topology, of which more later.

3. An example, the reader might like to work out. Let $\mathscr{H} = L^2([0, 1])$; $\mathfrak{A} = L^\infty([0, 1])$ acting as multiplication operators. Then $W_{\mathfrak{A}}$ (with its norm) is $L^1([0, 1])$ and $\mathfrak{A} = W_{\mathfrak{A}}{}^*$. In this case, $W_{\mathfrak{A}}$ is a Banach space which is not generally the case.

4. This theorem is really a consequence of the Hahn–Banach theorem, Reisz lemma and abstract nonsense.

Proof. Let Z be the set of norm continuous functionals on $L(\mathscr{H})$. Let Y and $[Y]$ be as in the last theorem. Let $k_{\mathfrak{A}}(Y) = \{y \in [Y] \mid y(A) = 0$ for all $A\,[\mathfrak{A}]\}$. [The reader should use the Hahn–Banach theorem to convince himself that $k_{\mathfrak{A}}(Y) \neq \varnothing$ if $\mathfrak{A} \neq L(\mathscr{H})$]. Then, by Theorem 6.1 (d), $W_{\mathfrak{A}} \equiv [Y]/k_{\mathfrak{A}}(Y)$. Let $\hat{k}([Y]^*) = \{l \in [Y]^* \mid l(\alpha) = 0$ for all $\alpha \in k_{\mathfrak{A}}(Y)\}$. By Theorem 6.2 (b), $W_{\mathfrak{A}}{}^* \cong \hat{k}([Y]^*)$ which is a subset of $[Y]^*$. By Theorem 6.3, $[Y]^* = L(\mathscr{H})$, so $\hat{k}([Y]^*)$ is a subset of $L(\mathscr{H})$ (check that the norms come out right!). Explicitly

$$\hat{k}[Y]^* = \{A \in L(\mathscr{H}) \mid y(A) = 0 \quad \text{for all} \quad y \in k_{\mathfrak{A}}(Y)\}.$$

It is thus the double annihilator according to the $\langle L(\mathscr{H}), [Y]\rangle$ pairing of $\mathfrak{A}$. It is thus the $\sigma(L(\mathscr{H}), [Y])$ ($\equiv$ weak!) closure of $\mathfrak{A}$ by Theorem 6.1 (e) and thus is $\mathfrak{A}$. ∎

Thus any von Neumann algebra is the dual of some Banach space (namely $(W_{\mathfrak{A}})_c$ by Theorem 6.2(a)). Sakai (1956) has found a beautiful converse of this theorem yielding a theorem of Gel'fand Naimark type:

THEOREM 6.5 (Sakai). *An abstract C^*-algebra, Y, which is the dual of a Banach space, B, is isometrically isomorphic to a von Neumann algebra.*

Remarks. 1. We will not prove this result. The basic idea is to show that B contains enough positive linear functionals on $\mathfrak{A}$ to separate points.

2. The theorem actually says more; namely there is a representation π of $\mathfrak{A}$ with $\pi(\mathfrak{A})$ von Neumann so that $(W_{\pi(\mathfrak{A})})_c$ is isomorphic to B under π^*.

3. Not every C^*-algebra can be represented as a von Neumann algebra. For the spectral theorem implies any von Neumann algebra contains lots of projections. The reader should use this idea to show $C([0, 1])$ is not isometrically isomorphic to any von Neumann algebra. This provides a proof that $C([0, 1])$ is not the dual of any Banach space (the Krein–Milman theorem proof is easier!).

In the "usual" textbook presentation of the theory of von Neumann's algebras, the ultraweak and ultrastrong topologies play a featured rôle in the proof of von Neumann's double commutant theorem:

THEOREM 6.6 (von Neumann). *Let* $\mathfrak{A}$ *be a *-algebra of operators on a Hilbert space with* $1 \in \mathfrak{A}$ *(*$\mathfrak{A}$ *is not assumed closed in any topology). Let* $\mathfrak{A}' = \{A \in L(\mathscr{H}) \mid AB = BA \text{ for all } B \in \mathfrak{A}\}$.

Let $\mathfrak{A}'' = (\mathfrak{A}')'$. *Then*

$$\mathfrak{A}'' = \mathfrak{A}_W = \mathfrak{A}_S$$

where $\mathfrak{A}_{W(S)}$ *is the weak (strong) closure of* $\mathfrak{A}$.

von Neumann actually proved more; namely $\mathfrak{A}'' = \mathfrak{A}_W = \mathfrak{A}_S = \mathfrak{A}_{US} = \mathfrak{A}_{UW}$ where UW and US are the ultra-topologies we will shortly turn to. If one only wants to prove Theorem 6.6, the ultraweak topology does not simplify things. (See Reed and Simon (In press, Vol. III)). So far I have tried to explain where the ultraweak topology is often used but is not particularly crucial. What is it and why is it so important?

As we have remarked before, $W_{\mathfrak{A}}$ is not always a Banach space, i.e. it is not always closed in $\mathfrak{A}^*$. Thus:

Definition. The norm closure in $\mathfrak{A}^*$ of $W_{\mathfrak{A}}$ will be denoted by $U_{\mathfrak{A}}$. The *ultraweak topology* on $\mathfrak{A}$ is the $\sigma(\mathfrak{A}, U_{\mathfrak{A}})$ topology. An ultraweak continuous state on $\mathfrak{A}$ is called a *normal state*.

Remarks 1. By Theorem 6.1 (c), $U_{\mathfrak{A}}$ is the set of ultraweakly continuous functionals on $\mathfrak{A}$.

2. By Theorem 6.1 (a), and Dixmier's theorem, $\mathfrak{A}$ is the Banach space dual of $U_{\mathfrak{A}}$. This is the more usual formulation of Dixmier's theorem.

3. In the notation of Theorems 6.3 and 6.4 $U_{\mathfrak{A}} = (\overline{[Y]/k_{\mathfrak{A}}(Y)})_c$ and $[Y]$ generates the ultraweak topology on $L(\mathscr{H})$. Thus by Theorem 6.1 (c), the restriction of the $L(\mathscr{H})$-ultraweak topology to $\mathfrak{A}$ is the $\mathfrak{A}$-ultraweak topology. We thus speak of the ultraweak topology without specifying the algebra.

4. **WARNING.** The ultraweak topology on $\mathfrak{A}$ is not weaker than the weak topology. It is stronger.

5. $U_{\mathfrak{A}}$ is easy to describe. To each $l \in U_{L(\mathscr{H})}$ is associated an operator $\rho \in L(\mathscr{H})$ with $\operatorname{Tr}(\rho) < \infty$ and $l(A) = \operatorname{tr}(\rho A)$. Thus the ultraweakly continuous states on $L(\mathscr{H})$ are density matrix states in a physicist's language. By Remark 3, and a Hahn–Banach type of argument, ultraweakly continuous states on any von Neumann algebra, $\mathfrak{A}$, are restrictions of density matrix states.

6. By a simple density argument, the weak and ultraweak topologies coincide on bounded sets of $\mathfrak{A}$.

We have already seen the ultraweak topology is not needed to prove Dixmier's theorem or the double commutator theorem. Why is it useful? Two simple answers we can give already:

(a) In quantum statistical mechanics, the natural local states are density matrix states which are not weakly continuous but they are ultraweakly continuous.

(b) In infinite volume limits, we often take limits of states. On the whole algebra, these limits are only weak *-limits but on nice subalgebras, they are limits in norm (Glimm and Jaffe (1970a)). Thus the natural sets of states are norm closed. $U_{\mathfrak{A}}$ is norm closed by (our) definition. $W_{\mathfrak{A}}$ is not always norm closed.

The last reason involves the "invariance" of the ultraweak topology under weakly (or ultraweakly) continuous isomorphism. To put this final result in perspective, recall that any isomorphism of C^*-algegras was norm isometric (look at the proof of Theorem 5.3); in particular, any norm continuous bijection of C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ has a norm continuous inverse. In this sense, the norm topology is natural. The following example shows that the weak topology is *not* natural in this sense:

Example. Let $\mathscr{H}_1$ be a Hilbert space with $\mathfrak{A}_1 = L(\mathscr{H}_1)$. Let $\mathscr{H}_2 = \oplus_{n=1}^{\infty} \mathscr{H}_1^{(n)} = \mathscr{H} \otimes l^2$ where each $\mathscr{H}_1^{(n)}$ is a copy of $\mathscr{H}_1$. Let $\mathfrak{A}_2 = \mathfrak{A}_1 \otimes 1_2$, i.e. each operator in $\mathscr{H}_2$ is of the form $\phi(A)[\psi_1, \ldots, \psi_n, \ldots] = [A\psi_1, \ldots, A\psi_n, \ldots]$ for some $A \in \mathfrak{A}_1$. Then $\mathfrak{A}_2$ is a von Neumann algebra and ϕ^{-1} is an isomorphism of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. ϕ^{-1} is weakly continuous but its inverse is not weakly continuous. It is however ultraweakly continuous with ultraweakly continuous inverse. This is no coincidence:

THEOREM 6.7. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be von Neumann algebras. Let $\phi: \mathfrak{A}_1 \to \mathfrak{A}_2$ be an isomorphism with the property that $\phi(a_\alpha) \to \phi(a)$ in the $\mathfrak{A}_2$-weak topology if $a_\alpha \to a$ in the $\mathfrak{A}_1$-ultraweak topology. Then*

(a) *ϕ is ultraweakly continuous,*

(b) *ϕ^{-1} is ultraweakly continuous.*

In particular, an ultraweakly continuous isomorphism is bicontinuous.

Remarks. 1. If ϕ is weakly or ultraweakly continuous, the hypothesis and thus the conclusions of the theorem hold.

2. The last statement follows from Remark 1.

3. This theorem shows that the ultraweak topology is a natural (i.e. intrinsic) and not "representation dependent" topology.

Proof. Let $W_{\mathfrak{A}_i}$ and $U_{\mathfrak{A}_i}$ be the weakly and ultraweakly continuous functionals on $\mathfrak{A}_i$. Consider ϕ^*: $\mathfrak{A}_2{}^* \to \mathfrak{A}_1{}^*$. The continuity hypothesis assures us that $\phi^*[W_{\mathfrak{A}_2}] \subset U_{\mathfrak{A}_1}$. Since ϕ is automatically norm continuous and $U_{\mathfrak{A}_1}$ is norm closed, $\phi^*[\overline{W}_{\mathfrak{A}_2}] \equiv \phi^*[U_{\mathfrak{A}_2}]$. Moreover, $(\phi^* \upharpoonright U_2)^*: \mathfrak{A}_1 \to \mathfrak{A}_2$ is just ϕ by the duality definitions. Since ϕ is surjective, $\phi^* \upharpoonright U_{\mathfrak{A}_2}$ is injective. Since ϕ is injective, ran $(\phi^* \upharpoonright U_{\mathfrak{A}_2})$ is dense in $U_{\mathfrak{A}_1}$. But ϕ^* is norm preserving so ran $(\phi^* \upharpoonright U_{\mathfrak{A}_2})$ is closed. Thus ϕ^* is a bijection of $\mathfrak{A}_1 \to \mathfrak{A}_2$. Since it is isometric, it has an inverse, $(\phi^* \upharpoonright \mathfrak{A}_2)^{-1} = \psi$. $\psi: \mathfrak{A}_2 \to \mathfrak{A}_2$ is automatically continuous from the weak $*$-topology on $\mathfrak{A}_{U_1}$to the weak $*$-topology on $\mathfrak{A}_1$. That is ψ^* is ultraweakly continuous. Similarly ϕ is ultraweakly continuous since $\psi^* = \phi^{-1}$, the theorem is proven. ∎

Remark. In a sense we missed the boat in the above discussion. This is the penalty for having a non-expert on operator algebra tell you about operator algebras. There is another definition of $U_{\mathfrak{A}}$ which is purely algebraic and shows that if $\mathfrak{A}$ and $\mathfrak{B}$ are von Neumann algebra and $\phi: \mathfrak{A} \to \mathfrak{B}$ is an algebraic isomorphism, then ϕ^* is a bijection of $U_{\mathfrak{B}}$ and $U_{\mathfrak{A}}$. So $U_{\mathfrak{A}}$ is a purely algebraic concept. If P_1, P_2 are projections in $\mathfrak{A}$, we say $P_1 \geqslant P_2$ if $P_1 - P_2$ is a projection. If P_α is a net of increasing projections and $\mathfrak{A}$ is a von Neumann algebra, $\sup_\alpha P_\alpha$ exists as a projection in $\mathfrak{A}$. $l \geqslant 0$ is in $U_{\mathfrak{A}}$ if and only if

$$l\left(\sup_\alpha P_\alpha\right) = \sup_\alpha l(P_\alpha).$$

This is a purely algebraic definition.

7. THE CCR AND CAR

In this final section, we present an approach of J. Slawny (In press) allowing us neat proofs of some basic facts about the canonical commutation relations and the canonical anti commutation relations.

A. Quasi-equivalence of Representations

As a technical preamble, we describe the notion of quasi-equivalent representations. Let $\mathfrak{A}$ be a $*$-algebra with unit not *a priori* normed or "complete". We are interested in $*$-representations of $\mathfrak{A}$ on Hilbert Spaces taking the unit into 1. In the usual way, one defines unitary equivalence.

Definition. Let π_1 andπ_2 be representations of $\mathfrak{A}$. We say that π_1 is *quasi-equivalent* to π_2 if and only if there are identity operators $I_{\mathscr{H}_1}$ and $I_{\mathscr{H}_2}$ (i.e. $I_{\mathscr{H}_1}$ is the realization with $\pi(A) = 1$ operating on $\mathscr{H}_1$) so that $\pi_1 \otimes I_{\mathscr{H}_1}$ is equivalent to $\pi_2 \otimes I_{\mathscr{H}_2}$.

The basic proposition is:

THEOREM 7.1. *If π_1 is irreducible and π_2 is quasi-equivalent to π_1, then π_2 is unitarily equivalent to some direct sum of copies of π_1 (equivalently to $\pi_1 \otimes I_{\mathscr{H}_3}$ for some $\mathscr{H}_3$).*

Proof. π_2 is a subrepresentation of $\pi_2 \otimes I_2$ so it is enough to prove that every subrepresentation of $\pi_1 \otimes I_1$ is unitarily equivalent to some direct sum of copies of π_1. Since $\mathfrak{A}$ is a *-algebra, every representation is a direct sum of cyclic representations so it is enough to show that every cyclic subrepresentation of $\pi_1 \otimes I_1$ is a direct sum of copies of π_1. Let ψ be a vector in $V_1 \otimes \mathscr{H}_1$ where V_1 is the representation space for π_1 and let $\mathscr{H}(\psi)$ be the closed subspace generated by $\{\pi(A) \otimes I\psi \mid A \in \mathfrak{A}\}$. We will show that $\mathscr{H}(\psi) = V_1 \otimes \tilde{\mathscr{H}}$ for some subspace $\tilde{\mathscr{H}}$ and so $\pi \restriction \mathscr{H}(\psi)$ is a sum of copies π_1. In a standard way we can write $\psi = \sum_{n=1}^{N} \alpha_n \phi_n \otimes \eta_n$ where the $\alpha_n > 0$ and the ϕ_n are orthonormal in V_1, the η_n orthonormal in $\mathscr{H}_1$. Let $\tilde{\mathscr{H}}$ = the linear spin of η_n. Obviously, $\mathscr{H}(\psi) \subset V_1 \otimes \tilde{\mathscr{H}}$. We will prove the converse. Since η is irreducible, the strong closure of $\{\pi_1(\mathfrak{A})\}$ is all of $L(V_1)$. Thus we can obtain P_1 the projection onto ϕ_1 as a limit of operators $\pi_1(A_\alpha)$ with $\sup_{\infty} \|\pi_1(A_\alpha)\| \leqslant 1$ (this follows from a density theorem for algebra called the Kaplansky density theorem; alternately, one can use the ultrastrong topology). Then $\lim \pi(A_\alpha)\psi = P_1\psi = \alpha_1 \phi_1 \otimes \eta_1 \in \mathscr{H}(\psi)$. Since $L(V)$ is the weak closure of $\pi_1(\mathfrak{A})$, $V_1 \otimes \eta_1 \in \mathscr{H}(\psi)$. Similarly, $V_1 \otimes \eta_2, V \otimes \eta_3, \ldots \subset \mathscr{H}(\psi)$ so $V_1 \otimes \tilde{\mathscr{H}} \subset \mathscr{H}(\psi)$. ∎

B. Multipliers: The CCR

Formally, the CCR built up from some *real* inner product space, S, are operators $\phi(h)$, $\pi(h)$ for all $h \in S$ obeying ϕ, π linear on S and $[\pi(h), \phi(g)] = -i(h,g)$; $[\pi(h), \pi(g)] = [\phi(h), \pi(g)] = 0$. Since the π, ϕ are unbounded, one avoids pathologies by dealing with exponentiated formulae:

Definition. Let S be a real inner product space. A **representation of the CCR over S** is a set of unitary operators $U(h)$, $V(h)$, for all $h \in S$ obeying:

(a) $U(0) = V(0) = 1$,

(b) $U(h_1 + h_2) = U(h_1)\, U(h_2)$,

(c) $V(h_1 + h_2) = V(h_1)V(h_2)$,

(d) $U(h_1)V(h_2) = e^{i(h_1,h_2)}\, V(h_2)U(h_1)$

(e) $t \to U(th)$ is strongly continuous for $t \in \mathbf{R}$.

(Think of $U(h) = e^{i\phi(h)}$; $V(h) = e^{i\pi(h)}$). One often wants to take stronger continuity notions than (e). In most of our discussion, (e) will play no role so we ignore it temporarily.

We immediately generalize this setting.

Definition. A **multiplier** b on a locally compact abelian group, G, is a continuous function $b: G \times G$ to $\{z \in \mathbf{C} \mid |z| = 1\}$ so that

(a) $b(g, 1) = b(1, g') = 1$,

(b) $b(g,g')\, b(g + g',g'') = b(g,g' + g'')\, b(g',g'')$.

A **projective representation of G with multiplier b** is a family of unitary operators U_g on a Hilbert space with

(i) $U_g U_{g'} = b(g,g')\, U_{g+g'}$,

(ii) $U_e = 1$,

(iii) $g \to U_g$ is strongly continuous.

Remarks. 1. We are not going to be complete in our history of the general ideas we discuss, but we will give references for our specific approach. Other general references are Gel'fand *et al.* (1961), Reed (1969) and Segal (1959a).

2. The conditions on a multiplier are precisely consistency conditions for projective representation, (*b*) being a statement of the associative law.

Let U be a projective representation for G with multiplier b; let V be a strict representation of G. Then $U \otimes V$ defined by $(U \otimes V)_g = U_g \otimes V_g$ is a projective representation of G with multiplier b.

Example. Let S be a real inner product space. Let $G = S \oplus S$ *with the discrete* topology. Then $b(s_1, s_2; s_1', s_2') = e^{-i(s_1', s_2)}$ is a multiplier and projective representations of G with this multiplier are precisely representations of the CCR without the continuity condition $[U(s_1)V(s_2) \equiv U(s_1, s_2)]$.

C. Fell's Remark

In all our considerations, the following simple remark of Fell (1962) will play a crucial role.

THEOREM 7.2. *Let G be a locally compact Abelian group with multiplier b. Define the regular b-representation B on $L^2(G)$ by*

$$(B(g)f)(g') = b(g',g)f(g+g').$$

Let U be a b-representation and let R be the regular representation of G

$$(R(g)f)(g') = f(g+g').$$

Then $U \otimes R$ is quasi-equivalent to B.

Remark. That B is a b-representation follows from the computation:

$$\begin{aligned}(B(g)B(g')f)(g'') &= b(g'',g)\,b(g''+g,g')f(g+g'+g'')\\ &= b(g,g')\,b(g'',g+g')f(g+g'+g'')\\ &= b(g,g')\,(B(g+g')f)(g'').\end{aligned}$$

Proof. $U \otimes R$ acts on $\mathscr{H} \otimes L^2(G)$ which we think of as $L^2(G;\mathscr{H})$, the functions on G with values in $\mathscr{H}$. So $[(U \otimes R)_g f](g') = U_g(f(g+g'))$. Define $A = L^2(G;\mathscr{H}) \to L^2(G;\mathscr{H})$ by $(Af)(g) = U_g f(g)$. Then

$$\begin{aligned}A[(U \otimes R)_g f](g') &= U_{g'}((U \otimes R)_g f)(g')\\ &= U_{g'}U_g f(g+g')\\ &= b(g',g)\,U_{g'+g}f(g+g')\\ &= b(g',g)\,(Af)(g+g')\\ &= [(B(g) \otimes I_{\mathscr{H}})(Af)](g').\end{aligned}$$

Since A is unitary, $U \otimes R$ is equivalent to $B(g) \otimes I_{\mathscr{H}}$ and quasi-equivalent to B. ∎

D. The Associated Bicharacter

The measure of the non-commutativity of the U_g in a projective representation is the function

$$\beta(g,g') = b(g,g')\,b(g',g)^{-1}$$

since

$$U(g)\,U(g') = \beta(g,g')\,U(g')\,U(g).$$

Definition. Let b be a multiplier on a group G. The function $\beta(g,g') = b(g,g')\,b(g',g)^{-1} = b(g,g')\overline{b(g',g)}$ is called the **associated bicharacter**

THEOREM 7.3. *β is a multiplier; it is antisymmetric (i.e. $\beta(g,g')^{-1} = \beta(g',g)$) and for each fixed g, $\beta(g,.)$ and $\beta(.,g)$ are characters.*

Proof. Let us prove $\beta(g,.)$ is a character. The rest is easy:

$$\begin{aligned}
\beta(g,g'+g'') &= b(g,g'+g'')\,b(g'+g'',g)^{-1} \\
&= [b(g,g'+g'')\,b(g',g'')]\,[b(g',g'')\,b(g'+g'',g)]^{-1} \\
&= b(g,g')\,b(g+g',g'')\,b(g',g''+g)^{-1}\,b(g'',g)^{-1} \\
&= b(g,g')\,b(g'+g,g'')\,b(g',g+g'')^{-1}\,b(g'',g)^{-1} \\
&= b(g,g')\,b(g',g)^{-1}\,[b(g',g)\,b(g'+g,g'') \\
&\qquad\qquad \times b(g',g+g'')^{-1}]\,b(g'',g)^{-1} \\
&= b(g,g')\,b(g',g)^{-1}\,b(g,g'')\,b(g'',g)^{-1} \\
&= \beta(g,g')\,\beta(g,g''). \blacksquare
\end{aligned}$$

β (and hence b) induces a natural homomorphism χ: $G \to \hat{G}$, the dual group by $\chi_g(g') = \beta(g,g')$. We call this the **natural map**.

Example. Let $S = \mathbf{R}^n$ with the usual inner product $(x, y) = \sum_{m=1}^{n} x_m y_m$ and with the usual topology. The multiplier associated with the CCR on S is $b(p,q;p',q') = e^{ip'\cdot q}$ where $\langle p,q\rangle$ is a general point of $S \oplus S = \mathbf{R}^{2n}$ with $p, q, \in \mathbf{R}^n$. Then $B(p,q;p',q') = e^{i(p'\cdot q - q'\cdot p)}$. Under the natural association of $\mathbf{R}^{2n}$ with $\mathbf{R}^{2n}$ by $x \to l_x(y) = {}^{ix\cdot y}$, the natural map associated to b is $\chi_{\langle p,q\rangle} = \langle q, -p\rangle$ (which is the standard canonical transformation!). Thus χ is a topological homomorphism.

Example. Let S be a real vector space which is infinite dimensional. We cannot put a locally convex topology on S, and keep it locally compact, so we put the discrete topology on S. The map χ: $S \oplus S \to S \oplus S$ is not bijective but since $\chi_{\langle p,q\rangle} = \langle q, -p\rangle$ [still (!)], χ is injective. Moreover it has a dense range by the following fact:

It is an amusing exercise to prove that when G is a discrete abelian group and H is a subgroup of $\hat{G}$ separating points then H is dense in $\hat{G}$. One slick way of proving this is as follows. Let P be the family of normalized positive definite functions on G. $\hat{G}$ are the extreme points of P. Essentially by the Stone–Weierstrass theorem on G the closed convex hull of H is all of P. By the "converse" of the Krein–Milman theorem, H is dense in $\hat{G}$. In any event, in the two cases where we need this result, i.e., where G is a vector group and where G is a sum of copies of Z_2 it can be proven directly.

E. von Neumann's Theorem

von Neumann's uniqueness theorem now has a simple and elegant proof and formulation.

THEOREM 7.4. *Suppose b is a multiplier on G for which the associated natural map is a bicontinuous bijection. Then all b-representations of G are quasi-equivalent.*

Remarks. 1. In general, using the Krein–Milman theorem, one can show that G has irreducible b-representations. From Theorem 7.1, we conclude there exists a unique irreducible representation and all others are direct sums of that irreducible. We don't give details for this argument since we can write down an irreducible b-representation explicitly in the case of interest.

2. If $G = H \oplus \hat{H}$ where H is a locally compact abelian group and $b(h_1, \chi_1; h_2, \chi_2) = \chi_1(h_2)\overline{\chi_2(h_1)}$ this result was proven by Mackey (1949).

3. We see explicitly how the hypotheses fail to extend to infinite dimensional CCR's.

Proof. Since quasi-equivalence is an equivalence relation and we have Theorem 7.2, we need only prove U and $R \otimes U$ are quasi-equivalent. R is unitarily equivalent to its Fourier transform $\hat{R}$ on $L^2(\hat{G})$ by $(\hat{R}_g \psi)(\chi) = \chi(g)\psi(\chi)$ for $\chi \in \hat{G}$. We show $\hat{R} \otimes U$ and U are quasi-equivalent. Again realize $L^2(\hat{G}) \otimes \mathscr{H}$ as $L^2(\hat{G}, \mathscr{H})$ so

$$((\hat{R} \otimes U)_g h)(\chi) = \chi(g)\, U_g h(\chi).$$

The intuition now is the following. Each χ is of the form χ_g. Since $U_{g'} U_g U_{g'}^{-1} = \beta(g', g) U_g = \chi_{g'}(g) U_g$, U and $\chi_{g'} U$ are unitarily equivalent for all $\chi_{g'}$. Thus we define $C: L^2(\hat{G}, \mathscr{H}) \to L^2(\hat{G}, \mathscr{H})$ by $(CF)(\chi_g) = U_g^{-1} f(\chi_g)$; since $\chi_g \to g$ is continuous, C is well defined. Then,

$$\begin{aligned} C[(\hat{R} \otimes U)_g h](\chi_{g'}) &= (U_{g'}^{-1} \chi_{g'}(g) U_g)(h(\chi_{g'})) \\ &= U_g U_{g'}^{-1} h(\chi_{g'}) \\ &= [(I_{L^2(G)} \otimes U)_g Ch](\chi_{g'}). \end{aligned}$$

Thus $C(\hat{R} \otimes U)_g C^{-1} = (I \otimes U)_g$ so U and $\hat{R} \otimes U$ are quasi-equivalent. ∎

This proof, then, proceeded by looking at $R \otimes U$ and first trivializing in the U direction so we obtained $B \otimes I$ and trivializing in the R direction so we obtained $I \otimes U$ concluding that $B \otimes I$ and $I \otimes U$ are unitarily equivalent.

As a corollary of this theorem and Theorem 7.1:

THEOREM 7.5 (von Neumann Uniqueness theorem). *Let* $U(a), V(b)$ *be continuous representations of* $\mathbf{R}^n$ *on* $\mathscr{H}$ *obeying,* $U(a)\,V(b) = e^{ia\cdot b}\,V(b)\,U(a)$. *Then there is a unitary operator* $D\colon \mathscr{H} \to L^2(R^n, X; dx)$, *the square-integrable functions on* $\mathbf{R}^n$ *with values in some Hilbert space,* X, *so that*

$$\big(D\,U(a)\,D^{-1}f\big)(x) = e^{ia\cdot x}f(x)$$

$$\big(D\,V(b)\,D^{-1}\big)(x) = f(x+b).$$

Proof. The "usual" Schrödinger representation on $L^2(\mathbf{R}^n, \mathbf{C}, dx)$ is an irreducible representation. The hypotheses of Theorem 7.4 are valid for the multiplier $\beta(a_1, b_1; a_2, b_2) = e^{-ia_2\cdot b_1}$ on $\mathbf{R}^{2n}$ so every representation is a direct sum of Schrödinger representations. ∎

F. The Weyl Algebra: Simplicity of the Weyl Algebra

THEOREM 7.6. *Let* G *be a discrete abelian group with a multiplier* b *whose associated bicharacter* β *is non-degenerate, i.e. the natural map of* G *into* $\hat{G}$ *is injective. Let* $U^{(1)}$, $U^{(2)}$ *be two* b*-representations of* G *and let* $\mathfrak{A}^{(1)}, \mathfrak{A}^{(2)}$ *be the* C^**-algebra generated by the* $\{U_g^{(1)}\}$ *and* $\{U_g^{(2)}\}$. *Then there is a (unique) isomorphism* $\alpha\colon \mathfrak{A}_1 \to \mathfrak{A}_2$ *so that* $\alpha(U_g^{(1)}) = U_g^{(2)}$.

Proof. Quasi-equivalent representations generate isometrically isomorphic algebras, for if $U^{(2)} = U^{(1)} \otimes I_{\mathscr{H}}$, $\mathfrak{A}^{(2)} = \mathfrak{A}^{(1)} \otimes I$ and α can be defined by $\alpha(A) = A \otimes I$. By this remark and Theorem 7.2 we need only consider the case where $U^{(2)} = R \otimes U^{(1)}$ or more conveniently $\hat{R} \otimes U^{(1)}$. It is enough to show that for any function f on G with finite ($\equiv$ compact) support,

$$\Big\|\sum_g f(g)\,U^{(1)}\Big\|_{\mathscr{H}^{(1)}} \quad \text{and} \quad \Big\|\sum_g f(g)\,(\hat{R} \otimes U)_g\Big\|_{L^2(\hat{G},\mathscr{H})}$$

$$= \operatorname*{ess\,sup}_{\chi\in\hat{G}} \Big\|\sum_g f(g)\,\chi(g)\,U_g^{(1)}\Big\|,$$

are equal.

We first note that the function $\sum_g f(g)\,\chi(g)\,U_g$ is norm continuous on $\hat{G}$ so we need only show $\|\sum_g f(g)\,U_g^{(1)}\|$ and $\|\sum_g \chi(g) f(g)\,U_g^{(1)}\|$ are equal for a dense set of χ's in $\hat{G}$. Let $\chi = \chi_{g'}$ for some $g' \in G$ under the natural map of $G \to \hat{G}$. Such $\chi_{g'}$ are dense since β is non-degenerate (see remark in (iv)). As we remarked in (v), $U_{g'}\, U_g\, U_{g'}^{-1} = \chi_{g'}(g)\, U_g$ so

$$\left\|\sum_g \chi_{g'}(g) f(g)\, U_g^{(1)}\right\| = \left\|U_{g'}\left(\sum_g f(g)\, U_g^{(1)}\right) U_{g'}^{-1}\right\| = \left\|\sum_g f(g)\, U_g^{(1)}\right\|.$$

This proves the theorem. ■

Remark. Thus the C^*-algebra generated by $\{U_g\}$ for the CCR over S is independent of the representation† (even independent of continuity conditions). This abstract C^*-algebra is called the *Weyl algebra over S.*

We have a very useful property of the Weyl algebra which is a corollary of the above:

THEOREM 7.7. *The Weyl-algebra is simple, i.e. has no non-trivial two-sided star ideals. Every representation is faithful.*

Proof. If I is a two-sided non-trivial ideal, its closure is also, so we need only show $\mathfrak{A}$ has no two-sided closed ideals. If I is such an ideal $\mathfrak{A}/I$ is an algebra and a Banach space and it is easy to see it is a C^*-algebra. Let π be a representation of $\mathfrak{A}/I$. It yields a representation π of $\mathfrak{A}$ in a natural way, with $I \subset \pi(\mathfrak{A})$. Since unitaries go into unitaries under $*$-representations, π yields a representation, U, of the CCR. The C^*-algebra generated by U is manifestly not isomorphic to $\mathfrak{A}$ under a map α with $\alpha(\tilde{U}_g) = U_g$. The contradiction shows that $\mathfrak{A}$ is simple. ■

G. The CAR: Reformulation

The CAR are defined by:

Definition. Let S be a real inner product space. A set of bounded operators $b(f)$, one for each $f \in S$ obeying

$$\begin{aligned}
&b(f+g) = b(f) + b(g)\\
&b^*(f)\, b^*(g) + b^*(g)\, b^*(f) = 0\\
&b^*(f)\, b(g) + b(g)\, b^*(f) = (f,g)\\
&b(f)\, b(g) + b(g)\, b(f) = 0
\end{aligned}$$

is called **a representation of the CAR over S.**

One consequence of the basic relation is that $b^*(f)\, b(f) + b(f)\, b^*(f) = \|f\|^2$ and since $b(f)\, b^*(f) \geqslant 0, b^*(f)\, b(f) \leqslant \|f\|^2$ or $\|b(f)\| \leqslant \|f\|$. This has several nice immediate consequences.

(a) b is continuous automatically

† It is independent of separability or inseparability of $\mathscr{H}$.

(b) By continuity, b is automatically linear

(c) b can be extended automatically to the completion of S so we may as well suppose S is a Hilbert space

(d) If S is a Hilbert space and ϕ_n is an orthonormal basis, then $b(f) = \Sigma \langle \phi_n, f \rangle b(\phi_n)$ which is norm convergent, so we can recover b from the $b_n \equiv b(\phi_n)$. Thus we may as well look at the discrete CAR

$$\begin{aligned} b_n^* b_m^* + b_m^* b_n^* &= 0 \\ b_n^* b_m^* + b_m b_n^* &= \delta_{mn} \\ b_n b_m + b_m b_n &= 0. \end{aligned} \tag{7.1}$$

Remarks. 1. We will only deal with the case where S is finite dimensional or a countably infinite dimensional Hilbert space. Using index sets, the general case is easy.

2. (c) and (d) above are false in the case of the CCR and are one reason the CAR are easier to deal with in many ways.

We would like to prove a uniqueness theorem in case $\dim S < \infty$ and that the generated C^*-algebra is simple and independent of representation in case $\dim S = \infty$. These results have the flavour of our general setting above so we must reformulate the CAR to look like a projective representation of a group. Trivial computation shows:

PROPOSITION. *Let $\{b_n\}_{n=1}^N$ obey the relation 7.1. (N may be infinite). Let $U_{2n} = b_n + b_n^*$; $U_{2n-1} = 1/i\,(b_n - b_n^*)$. The U's are unitary and obey*

$$\begin{aligned} U_n U_m &= - U_m U_{nz}\, m \neq n \\ U_m^z &= 1. \end{aligned} \tag{7.2}$$

Conversely if $\{U_m\}_{m=1}^{2N}$ is a family of unitaries obeying (7.2), then $b_n = \frac{1}{2}[U_{2n} + i\,U_{2n-1}]$ obey (7.1).

Now comes the trick. Let G be the group $G = \oplus_{m=1}^{2n} Z_2$, i.e. $g \in G$ is a $2N$-triple (or sequence) of 0 and l's with only finitely many l's; $g = (g_1, g_2, \ldots)$. Let $g \in G$ and $\{u_m\}_{m=1}^{2N}$ obeying (7.2). Let $U_g = \prod_{m=1}^{2n} (U_m)^{g_m}$. (7.2) can be expressed by

$$U_g U_{g'} = \pm\, U_{g+g'} \tag{7.3}$$

where $\pm$ is a simple function $b\ (g,g')$ of g and g' we would write down explicitly but don't need to for the time being. To see that b is a multiplier we need only show some representation of the CAR's exist. This is left as an exercise in tensor products of Pauli σ matrices.

H. Jordan–Wigner Theorem

We want to show that in case $N < \infty$, (7.2) has a unique irreducible representation. If we can show β induces an isomorphism of G and $\hat{G}$ we can apply Theorems 7.1 and 7.4. G is a finite group so it is naturally isomorphic to $\hat{G}$ under the map $g \to \eta_g$ where $\eta_g(g') = (-1)^{\sum_{i=1}^{2n} g_i g_i'}$. What is β? Letting δ_n be the obvious element of $g, \beta(\delta_n, \delta_m) = -1$ if $n \neq m$; by (7.2). Since β is a bicharacter $\beta(g,g') = (-1)^{\sum_{i \neq j} g_i g_j'}$. (Notice, if $g = \delta_n$, $g' = \delta_m$, this gives the right answer and is a bicharacter). Let G be the element $G = (1, 1, \ldots)$. Then letting $g \to \chi$ be the mapping induced by β,

$$\chi_g = \eta_{(\#g)G - g}$$

where $\#g = \sum_{i=1}^{2N} g_i$. (Again, check it for δ_n). But since $2N$ is even (!), $(\#g)\,G - g = 0$ implies $g = 0$. Thus β does induce an injective map and since $G = \hat{G}$ is finite, an isomorphism. By Theorems 7.1 and 7.4, we conclude:

THEOREM 7.8 (Jordan–Wigner theorem). *If $N < \infty$, every representation of the* CAR *over* $\mathbf{R}^N$ *is a direct sum of copies of a basic irreducible representation (of dimension 2^N it turns out).*

Remark. Representations of (7.2) when $\{U_n\}_{n=1}^M$ has an odd number of elements are not unique. Notice where the above proof breaks down.

I. The CAR Algebra: Its Simplicity

In case N is infinite, $G = \oplus_{m=1}^{2N} Z_2$ is no longer self-dual, but β is still given by $\beta(g,g') = (-1)^{\sum_{i \neq j} g_i g_j'}$. β is non-degenerate trivially $[\beta(g, \delta_n) = -1$ if $\#g$ is odd and $g_n = 0$ and if $\#g$ is even and $g_n = 1]$. Thus using Theorem 7.6 and the method of proof of Theorem 7.7, we conclude:

THEOREM 7.9. *All representations of the* CAR *over a fixed inner product space S generates isomorphic C^*-algebras. The resulting algebra (which) is only dependent on the Hilbert space dimension of the completion of S is simple.*