Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Computational and Applied Mathematics 171 (2004) 411–424
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Analogs of the *m*-function in the theory of orthogonal polynomials on the unit circle

Barry Simon[1]

Department of Mathematics 253-37, California Institute of Technology, Pasadena, CA 91125, USA

Received 1 October 2003; accepted 19 January 2004

Dedicated to Norrie Everitt, on his 80th birthday, a bouquet to the master of the *m*-function

Abstract

We show that the multitude of applications of the Weyl–Titchmarsh *m*-function leads to a multitude of different functions in the theory of orthogonal polynomials on the unit circle that serve as analogs of the *m*-function.

Keywords: *m*-function; Orthogonal polynomials; Verblunsky coefficients

1. Introduction

Use of the Weyl–Titchmarsh *m*-function has been a constant theme in Norrie Everitt's opus, so I decided a discussion of the analogs of these ideas in the theory of orthogonal polynomials on the unit circle (OPUC) was appropriate. Interestingly enough, the uses of the *m*-functions are so numerous that OPUC has multiple analogs of the *m*-function!

m-functions are associated to solutions of

$$-u'' + qu = zu \tag{1.1}$$

with q a real function on $[0, \infty)$ and z a parameter in $\mathbb{C}_+ = \{z \mid \operatorname{Im} z > 0\}$. The most fundamental aspect of the *m*-function is its relation to the spectral measure, ρ, for (1.1) by

$$m(z) = c + \int d\rho(x) \left[\frac{1}{x - z} - \frac{x}{1 + x^2} \right], \tag{1.2}$$

[1] Supported in part by NSF Grant DMS-0140592. For special issue of J. Comp. Appl. Math.

E-mail address: bsimon@caltech.edu (B. Simon).

0377-0427/$ - see front matter
doi:10.1016/j.cam.2004.01.022

where c is determined by (see [3,13]):

$$m(z) = \sqrt{-z} + o(1) \quad \text{as}\, z \to i\infty. \tag{1.3}$$

Eqs. (1.2) plus (1.3) allow you to compute m given $d\rho$, and $d\rho$ is determined by m via

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\pi}\int_a^b m(x+i\varepsilon)\,dx = \frac{1}{2}[\rho((a,b)) + \rho([a,b])]. \tag{1.4}$$

Of course, I have not told you what m or ρ is. This is done by defining m, in which case ρ is defined by (1.4). Under weak conditions on q at ∞, for $z \in \mathbb{C}_+$, (1.1) has a solution $u(x,z)$ which is L^2 at infinity, and it is unique up to a constant multiple. Then, m is defined by

$$m(z) = \frac{u'(0,z)}{u(0,z)}. \tag{1.5}$$

With this definition, $d\rho$ is a spectral measure for $u \mapsto -u'' + qu = Hu$ in the sense that H is unitarily equivalent to multiplication by λ on $L^2(\mathbb{R}, d\rho)$. (1.5) is often written in the equivalent form,

$$\psi(x,z) + m(z)\varphi(x,z) \in L^2,$$

where φ, ψ solve (1.1) with initial conditions $\varphi(0)=0, \varphi'(0)=1, \psi(0)=1, \psi'(0)=0$.

Note that if one defines

$$m(x;z) = \frac{u'(x,z)}{u(x,z)}, \tag{1.6}$$

the m-function for $q_x(\cdot) = q(\cdot + x)$, then m obeys the Riccati equation

$$m' = q - z - m^2. \tag{1.7}$$

It could be said that this is backwards: definition (1.5) should come first, before (1.2). I put it in this order because it is (1.2) that makes m such an important object both in classical results [2,5,7–9,16,23,33] and very recent work [4,10,21,25,27,31].

To describe the third role of the m-function, it will pay to switch to the case of Jacobi matrices. We now have, instead of q, two sequences $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ with $a_n > 0$, $b_n \in \mathbb{R}$ which we will suppose uniformly bounded. Define an infinite matrix

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.8}$$

which is a bounded self-adjoint operator. One defines

$$m(z) = \langle \delta_1, (J-z)^{-1}\delta_1 \rangle. \tag{1.9}$$

In terms of the spectral measure, μ, for δ_1 for J,

$$m(z) = \int \frac{d\mu(x)}{x-z}. \tag{1.10}$$

If u_n is the ℓ^2 solution of $a_{n-1}u_{n-1} + (b_{n-z})u_n + a_n u_{n+1} = 0$ with $\mathrm{Im}\, z > 0$, one has the analog of (1.5)

$$m(z) = \frac{u_1(z)}{u_0(z)}. \tag{1.11}$$

This process of going from a and b to m and then to μ can be reversed. One way is by iterating (1.5) below, which lets one go from μ to m (by (1.10)) and then gets the a's and b's as coefficients in a continued fraction expansion of m. From our point of view, an even more important way of going backwards uses orthogonal polynomials on the real line (OPRL). Given μ (of bounded support), one forms the monic orthogonal polynomials $P_n(x)$ for $d\mu$ and shows they obey a recursion relation

$$P_{n+1}(x) = (x - b_{n+1})P_n(x) - a_n^2 P_{n-1}(x) \tag{1.12}$$

which yields the Jacobi parameters a and b. The orthonormal polynomials, $p_n(x)$, are related to P_n by

$$p_n(x) = (a_1 \dots a_n)^{-1} P_n(x) \tag{1.13}$$

and obey

$$a_{n+1}\, p_{n+1}(x) = (x - b_{n+1})\, p_n(x) - a_n\, p_{n-1}(x). \tag{1.14}$$

Eq. (1.7) has the analog

$$m(z; J) = (b_1 - z - a_1^2 m(z; J^{(1)}))^{-1}, \tag{1.15}$$

where $J^{(1)}$ is the Jacobi matrix with parameters $\tilde{a}_m = a_{m+1} \tilde{b}_m = b_{m+1}$ (i.e., the top row and left column are removed).

If $m(x + i\varepsilon; J)$ has a limit as $\varepsilon \downarrow 0$, (1.15) says that $m(x + i\varepsilon; J^{(1)})$ has a limit, and by (1.15),

$$\frac{\mathrm{Im}\, m(x; J)}{\mathrm{Im}\, m(x; J^{(1)})} = |a_1 m(x; J)|^2. \tag{1.16}$$

$\mathrm{Im}\, m$ is important because if μ is given by (1.10), then

$$d\mu_{\mathrm{ac}} = \frac{1}{\pi}\, \mathrm{Im}\, m(x + i0)\, dx. \tag{1.17}$$

This property of m, that its energy is the ratio of Im's, is a critical element of recent work on sum rules for spectral theory [6,19,28–30].

The interesting point is that, for OPUC, the analogs of the functions obeying (1.2), (1.5), and (1.16) are different! In Section 2, we will give a quick summary of OPUC. In Section 3, we discuss (1.2); in Section 4, we discuss (1.16); and finally, in Section 5, the analog of (1.5).

Happy 80th, Norrie. I hope you enjoy this bouquet.

2. Overview of OPUC

We want to discuss here the basics of OPUC, although we will only scratch the surface of a rich and beautiful subject [29]. The theory reverses the usual passage from differential/difference equations to measures, and instead follows the discussion of OPRL in Section 1. μ is now a probability measure on $\partial\mathbb{D} = \{z \mid |z| = 1\}$. We suppose μ is nontrivial, that is, not supported on a finite set. One can then form, by the Gram–Schmidt procedure, the monic orthogonal polynomials $\Phi_n(z)$ and the orthonormal polynomials, $\varphi_n(z) = \Phi_n(z)/\|\Phi_n\|$ where $\|\cdot\|$ is the L^2 $(\partial\mathbb{D}, \mathrm{d}\mu)$ norm.

Given fixed $n \in \{0, 1, 2, \ldots\}$, we define an anti-unitary operator on $L^2(\partial\mathbb{D}, \mathrm{d}\mu)$ by

$$f^*(z) = z^n\,\overline{f(z)}. \tag{2.1}$$

The use of a symbol without "n" is terrible notation, but it is standard! If Q_n is a polynomial of degree n, Q_n^* is also a polynomial of degree n. Indeed,

$$Q_n^*(z) = z^n\,\overline{Q_n(1/\bar{z})}$$

so if $Q_n(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$, then $Q_n^*(z) = \bar{a}_0 z^n + \bar{a}_1 z^{n-1} + \cdots + \bar{a}_n$.

Since Φ_n is monic, $\Phi_n^*(0) = 1$, and thus, $N(z) \equiv (\Phi_{n+1}^*(z) - \Phi_n^*(z))/z$ is a polynomial of degree n. Since * is anti-unitary,

$$\begin{aligned}\langle z^m, N(z)\rangle &= \langle z^{m+1}, \Phi_{n+1}^* - \Phi_n^*\rangle \\ &= \langle \Phi_{n+1}, z^{n+1-(m+1)}\rangle - \langle \Phi_n, z^{n-m-1}\rangle \\ &= 0,\end{aligned}$$

for $m = 0, 1, \ldots, n-1$. Thus $N(z)$ must be a multiple of $\Phi_n(z)$, that is, for some $\alpha_n \in \mathbb{C}$,

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z \Phi_n(z) \tag{2.2}$$

and its *,

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n \Phi_n^*(z). \tag{2.3}$$

(2.2)/(2.3) are the *Szegő recursion formulae* ([32]); the α_n's are the Verblunsky coefficients (after [34]). The derivation I have just given is that of Atkinson [2].

Since $\Phi_n^* \perp \Phi_{n+1}$, (2.3) implies

$$\|\Phi_{n+1}\|^2 + |\alpha_n|^2\|\Phi_n^*\|^2 = \|z\Phi_n\|^2.$$

Since $\|\Phi_n^*\| = \|z\Phi_n\| = \|\Phi_n\|$, we have

$$\|\Phi_{n+1}\| = (1 - |\alpha_n|^2)^{1/2}\|\Phi_n\|. \tag{2.4}$$

This implies first of all that

$$|\alpha_n| < 1 \tag{2.5}$$

and if

$$\rho_n \equiv (1 - |\alpha_n|^2)^{1/2}, \tag{2.6}$$

then

$$\|\Phi\|_n = \rho_0\rho_1 \ldots \rho_{n-1} \tag{2.7}$$

so

$$\varphi_n = (\rho_0 \ldots \rho_{n-1})^{-1}\Phi_n \tag{2.8}$$

and (2.2), (2.3) becomes

$$z\varphi_n = \rho_n \varphi_{n+1} + \bar{\alpha}_n \varphi_n^*, \tag{2.9}$$

$$\varphi_n^* = \rho_n, \varphi_{n+1}^* + \alpha_n z \varphi_n. \tag{2.10}$$

The α_n's not only lie in $\mathbb{D}$, but it is a theorem of Verblunsky [34] that as μ runs through all nontrivial measures, the set of α's runs through all of $\times_{n=0}^{\infty}\mathbb{D}$. The α's are the analogs of the a's and b's in the Jacobi case or of V in the Schrödinger case.

We will later have reason to consider Szegő's theorem in Verblunsky's form [35].

Theorem 2.1. *Let*

$$\mathrm{d}\mu = w\,\frac{\mathrm{d}\theta}{2\pi} + \mathrm{d}\mu_{\mathrm{s}}. \tag{2.11}$$

Then

$$\prod_{j=0}^{\infty}(1-|\alpha_j|^2) = \exp\left(\int \log(w(\theta))\frac{\mathrm{d}\theta}{2\pi}\right). \tag{2.12}$$

Remark. The log integral can diverge to $-\infty$. The theorem says the integral is $-\infty$ if and only if the product on the left is 0, that is, if and only if $\sum|\alpha_j|^2 = \infty$.

If

$$\sum_{j=0}^{\infty}|\alpha_j|^2 < \infty, \tag{2.13}$$

we say the Szegő condition holds. This happens if and only if

$$\int |\log(w(\theta))|\,\frac{\mathrm{d}\theta}{2\pi} < \infty. \tag{2.14}$$

In that case, we define the Szegő function on $\mathbb{D}$ by

$$D(z) = \exp\left(\int \frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z}\log\left(w(\theta)\right)\frac{\mathrm{d}\theta}{4\pi}\right). \tag{2.15}$$

3. The Carathéodory and Schur functions

Given (1.10) (and (1.2)), the natural "m-function" for OPUC is the Carathéodory function, $F(z)$,

$$F(z) = \int \frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z}\,\mathrm{d}\mu(\theta). \tag{3.1}$$

The Cauchy kernel $(\mathrm{e}^{\mathrm{i}\theta}+z)/(\mathrm{e}^{\mathrm{i}\theta}-z)$ has the Poisson kernel

$$\mathrm{Re}\left(\frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z}\right)\Bigg|_{z=r\mathrm{e}^{\mathrm{i}\varphi}} = \frac{1-r^2}{1+r^2-2\cos(\theta-\varphi)} \tag{3.2}$$

as its real part, and this is positive, so

$$\operatorname{Re} F(z) > 0 \quad \text{for } z \in \mathbb{D}, \quad F(0) = 1. \tag{3.3}$$

This replaces $\operatorname{Im} m > 0$ if $\operatorname{Im} z > 0$.

One might think the "correct" analog of m is

$$R(z) = \int \frac{1}{e^{i\theta} - z} \, d\mu(\theta). \tag{3.4}$$

R and F are related by

$$R(z) = (2z)^{-1}(F(z) - 1). \tag{3.5}$$

If one rotates $d\mu$ and z (i.e., $d\mu(\theta) \to d\mu(\theta - \varphi), z \to e^{i\varphi}z$), F is unchanged but R is multiplied by $e^{-i\varphi}$, so the set of values R can take are essentially arbitrary—which shows F, which obeys $\operatorname{Re} F(z) > 0$, is a nicer object to take. That said, we will see R again in Section 5.

F has some important analogs of m:

(1) $\lim_{r\uparrow 1} F(re^{i\theta})$ exists for a.e. θ, and if (2.11) defines w, then

$$w(\theta) = \operatorname{Re} F(e^{i\theta}). \tag{3.6}$$

(2) θ_0 is a pure point of μ if and only if $\lim_{r\uparrow 1}(1 - r)\operatorname{Re} F(re^{i\theta_0}) \neq 0$ and, in general,

$$\lim_{r\uparrow 1} (1 - r)\operatorname{Re} F(re^{i\theta_0}) = \mu(\{\theta_0\}).$$

(3) $d\mu_s$ is supported on $\{\theta | \lim_{r\uparrow 1} F(re^{i\theta}) = \infty\}$.

In fact, the proof of the analogs of these facts for m proceeds by mapping $\mathbb{C}_+$ to $\mathbb{D}$ and using these facts for F!

These properties provide a strong analogy, but one can note a loss of "symmetry" relative to the ODE case. The m-function maps $\mathbb{C}_+$ to $\mathbb{C}_+$. F though maps $\mathbb{D}$ to $-i\mathbb{C}_+$. One might prefer a map of $\mathbb{D}$ to $\mathbb{D}$. In fact, one defines the Schur function, f, of μ via

$$F(z) = \frac{1 + zf(z)}{1 - zf(z)}, \tag{3.7}$$

then f maps $\mathbb{D}$ to $\mathbb{D}$ and (3.7) sets up a one-one correspondence between F's with $\operatorname{Re} F > 0$ on $\mathbb{D}$ and $F(0) = 1$ and f mapping $\mathbb{D}$ to $\mathbb{D}$ (this fact relies on the Schwarz lemma that f maps $\mathbb{D}$ to $\mathbb{D}$ with $f(0) = 0$ if and only if $f = zg$ where g maps $\mathbb{D}$ to $\mathbb{D}$).

For at least some purposes, f is a "better" analog of m than F, for example, in regard to its analog of the recursion (1.10). If f is the Schur function associated to Verblunsky coefficients $\{\alpha_0, \alpha_1, \ldots\}$ and f_n is the Schur function associated to $\{\alpha_n, \alpha_{n+1}, \ldots\}$, then

$$f = \frac{\alpha_0 + zf_1}{1 + \bar{\alpha}_0 z f_1}, \tag{3.8}$$

a result of Geronimus (see [29] for lots of proofs of this fact).

Interestingly enough, Schur, not knowing of the connection to OPUC, discussed (3.8) for $\alpha_0 = f(0)$ as a map of $f \to (\alpha_0, f_1)$ and, by iteration, to a parametrization of functions of $\mathbb{D}$ to $\mathbb{D}$ by parameters

$\alpha_0, \ldots, \alpha_n, \ldots$. There is, of course, a formula relating F to F_1 that can be obtained from (3.7) and (3.8) or directly [22], but it is more complicated than (3.8).

Finally, in discussing f, we note that there is a natural family $\{d\mu_\lambda\}_{\lambda \in \partial\mathbb{D}}$ of measures related to $d\mu$ (with $d\mu_{\lambda=1} = d\mu$) that corresponds to "varying boundary conditions." We will discuss those more fully in Section 5, but we note

$$f(z; d\mu_\lambda) = \lambda f(z; d\mu), \tag{3.9}$$

while the formula for $F(d\mu_\lambda)$ is more involved.

The Schur function and Schur iterates, f_n, have been used by Khrushchev [14,17,18] as a powerful tool in the analysis of OPUC.

4. The relative Szegő function

As explained in the Introduction, a critical property of m is (1.16), which is the basis of step-by-step sum rules (see [28]). The left side of (1.16) enters as the ratio of a.c. weights of $d\mu_J$ and $d\mu_{J^{(1)}}$. Thus, we are interested in $\operatorname{Im} F(e^{i\theta}; \{\alpha_j\}_{j=0}^\infty)$ divided by $\operatorname{Im} F(e^{i\theta}; \{\alpha_{j+1}\}_{j=0}^\infty)$, that is, $\operatorname{Im} F / \operatorname{Im} F_1$ in the language of the last section. Neither $|F|$ nor $|f|$ is directly related to this ratio, so we need a different object to get an analog of (1.16). The following was introduced by Simon in [29]:

$$(\delta_0 D)(z) = \frac{1 - \bar{\alpha}_0 f}{\rho_0} \frac{1 - z f_1}{1 - z f}. \tag{4.1}$$

It is called the "relative Szegő function" for reasons that will become clear in a moment.

In (4.1), f_1 is the Schur function for Verblunsky coefficients

$$\alpha_j^{(1)} = \alpha_{j+1}. \tag{4.2}$$

Here is the key fact:

Theorem 4.1. *Let* $d\mu$ *and* $d\mu^{(1)}$ *be measures on* $\partial\mathbb{D}$ *with Verblunsky coefficients related by* (4.2). *Suppose* $d\mu = w(\theta) d\theta/2\pi + d\mu_s$ *and* $d\mu^{(1)} = w^{(1)} d\theta/2\pi + d\mu_s$. Then

(1) *For a.e.* $\theta, \lim_{r \uparrow 1} (\delta_0 D)(re^{i\theta}) \equiv \delta_0 D(e^{i\theta})$ *exists.*
(2) *If* $w(\theta) \neq 0$, *then* (*for a.e.* θ *w.r.t.* $d\theta/2\pi$), $w_1(\theta) \neq 0$ *and*

$$\frac{w(\theta)}{w_1(\theta)} = |(\delta_0 D)(e^{i\theta})|^2. \tag{4.3}$$

Sketch of Proof. Each of the functions $1 - \bar{\alpha}_0 f, 1 - z f_1$, and $1 - z f$ takes values in $\{w \mid |w - 1| < 1\}$ on $\mathbb{D}$, so their arguments lie in $[-\pi/2, \pi/2]$, so their logs are in all $H^p, 1 < p < \infty$. That is, they are outer functions, and so $\delta_0 D$ is an outer function, which means that assertion (1) holds (see Rudin [24] for a pedagogic discussion of outer functions).

To get (4.3), we note that (3.7) implies

$$\operatorname{Re} F(z) = \frac{1 - |f|^2 |z|^2}{|1 - z f|^2},$$

so

$$\frac{\mathrm{Re}F(z)}{\mathrm{Re}F_1(z)} = \left|\frac{1-zf_1}{1-zf}\right|^2 \frac{1-|f|^2|z|^2}{1-|f_1|^2|z|^2}. \tag{4.4}$$

On the other hand, (3.8) implies

$$zf_1 = \frac{f-\alpha_0}{1-\bar{\alpha}_0 f}, \tag{4.5}$$

which implies

$$1-|zf_1|^2 = \frac{\rho_0^2(1-|f|^2)}{|1-\bar{\alpha}_0 f|^2} \tag{4.6}$$

so, putting these formulae together,

$$\frac{\mathrm{Re}\,F(z)}{\mathrm{Re}\,F_1(z)} = |(\delta_0 D)(z)|^2 \left(\frac{1-|z|^2|f|^2}{1-|f|^2}\right) \tag{4.7}$$

which, as $|z| \to 1$, yields (4.3). □

In particular, one has the nonlocal step-by-step sum rule that if $w(\theta) \neq 0$ for a.e. θ, then

$$(\delta_0 D)(z) = \exp\left(\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log\left(\frac{w(\theta)}{w_1(\theta)}\right) \frac{d\theta}{4\pi}\right) \tag{4.8}$$

and, in particular, setting $z=0$,

$$\rho_0^2 = \exp\left(\int_0^{2\pi} \log\left(\frac{w(\theta)}{w_1(\theta)}\right) \frac{d\theta}{2\pi}\right) \tag{4.9}$$

which is not only consistent with Szegő's theorem (2.11) but, using semicontinuity of the entropy, can be used to prove it (see [19,29]) as follows:

(1) Iterating (4.9) yields

$$(\rho_0 \dots \rho_{n-1})^2 = \exp\left(\int_0^{2\pi} \log\left(\frac{w(\theta)}{w_n(\theta)}\right) \frac{d\theta}{2\pi}\right). \tag{4.10}$$

(2) Since $\exp(\int_0^{2\pi} \log(w_n(\theta) d\theta/2\pi) \leqslant \int_0^{2\pi} w_n(\theta) d\theta/2\pi \leqslant 1$, (4.10) implies

$$(\rho_0 \dots \rho_{n-1})^2 \geqslant \exp\left(\int_0^{2\pi} \log(w(\theta)) \frac{d\theta}{2\pi}\right). \tag{4.11}$$

(3) If $w^{(n)}$ is the weight associated to the measure with

$$\alpha_j^{(n)} = \begin{cases} \alpha_j, & j \leqslant n-1 \\ 0, & j \geqslant n, \end{cases}$$

(4.10) proves

$$(\rho_0 \dots \rho_{n-1})^2 = \exp \int_0^{2\pi} \log(w^{(n)}(\theta)) \frac{d\theta}{2\pi}. \tag{4.12}$$

(4) $d\mu \to \int_0^{2\pi} \log(w(\theta))d\theta/2\pi$ is an entropy, hence, weakly upper semicontinuous. Since $w^{(n)}d\theta/2\pi \to d\mu$ weakly as $n \to \infty$, this semicontinuity shows

$$\lim_{n\to\infty} (\rho_n \ldots \rho_{n-1})^2 \leqslant \exp\left(\int_0^{2\pi} \log(w(\theta))\frac{d\theta}{2\pi}\right). \tag{4.13}$$

Eqs. (4.11) and (4.13) is Szegő's theorem.

Two other properties of $\delta_0 D$ that we should mention are:

(A) If $\sum_{n=0}^{\infty} |\alpha_n|^2 < \infty$, then

$$(\delta_0 D)(z) = \frac{D(z;\alpha_0,\alpha_1,\alpha_2,\ldots)}{D(z;\alpha_1,\alpha_2,\alpha_3,\ldots)}. \tag{4.14}$$

(B) In general, one has

$$\delta_0 D(z) = \lim_{n\to\infty} \frac{\varphi^*_{n-1}(z;\alpha_1,\alpha_2,\ldots)}{\varphi^*_n(z;\alpha_0,\alpha_1,\ldots)}. \tag{4.15}$$

5. Eigenfunction ratios

Finally, we look at the analogs of m as a function ratio, its initial definition by Weyl and Titchmarsh. The key papers on this point of view are by Geronimo–Teplyaev [11] and Golinskii–Nevai [15]. We will see from one point of view [15] that $F(z)$ plays this role, but from other points of view [11] that other functions are more natural.

The recursion relations (2.9)/(2.10) can be rewritten as

$$\begin{pmatrix} \varphi_{n+1} \\ \varphi^*_{n+1} \end{pmatrix} = A(\alpha_n, z)\begin{pmatrix} \varphi_n \\ \varphi^*_n \end{pmatrix}, \tag{5.1}$$

where

$$A(\alpha, z) = \rho^{-1}\begin{pmatrix} z & -\bar{\alpha}_n \\ -\alpha_n z & 1 \end{pmatrix} \tag{5.2}$$

(with $\rho = (1-|\alpha^2)^{1/2}$). From this point of view, the analog of the fundamental differential/difference equation in the real case is

$$\Xi_n = T_n(z)\Xi_0 \tag{5.3}$$

with

$$T_n(z) = A(\alpha_{n-1}, z)\cdots A(\alpha_0, z). \tag{5.4}$$

The correct boundary conditions for the usual OPUC are $\Xi_0 = \binom{1}{1}$.

One can ask for what other initial conditions the polynomials associated with the top component of $T_n(z)\Xi_0$ are OPUC for some measure. Note that

$$\begin{pmatrix} 1 \\ \lambda \end{pmatrix} = U(\lambda)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.5}$$

with

$$U(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \tag{5.6}$$

and that

$$U(\lambda)^{-1} A(\alpha, z) U(\lambda) = \rho^{-1} \begin{pmatrix} z & -\bar{\alpha}_n \lambda \\ -\alpha_n \lambda^{-1} z & 1 \end{pmatrix}. \tag{5.7}$$

We see from this that $\bar{\lambda} = \lambda^{-1}$, that is, $|\lambda| = 1$ will yield $U(\lambda)^{-1} A(\alpha_1, z) U(\lambda) = A(\bar{\lambda}\alpha, z)$. Changing λ to $\bar{\lambda}$, we see that

Proposition 5.1. *Let* $|\lambda| = 1$. *If* $\varphi_n^{(\lambda)}(z)$ *are the OPUC for Verblunsky coefficients* $\alpha_n^{(\lambda)} = \lambda \alpha_n$, *then*

$$\begin{pmatrix} \varphi_n^{(\lambda)}(z) \\ \bar{\lambda} \varphi_n^{(\lambda)*}(z) \end{pmatrix} = T_n(z; \{\alpha_j\}_{j=1}^{\infty}) \begin{pmatrix} 1 \\ \bar{\lambda} \end{pmatrix}. \tag{5.8}$$

This suggests that one look at the family $d\mu_\lambda$ or measures with

$$\alpha_j(d\mu_\lambda) = \lambda \alpha_j(d\mu) \tag{5.9}$$

called the family of Aleksandrov measures associated to $\{\alpha_j\}_{j=0}^{\infty}$ after [1]. The special case $\lambda = -1$ goes back to Verblunsky [35] and Geronimus [12], and are called the second kind polynomials, denoted $\psi_n(z)$. The following goes back to Verblunsky [35].

Theorem 5.2. *For* $z \in \mathbb{D}$, *uniformly on compact subsets of* $\mathbb{D}$,

$$\lim_{n \to \infty} \frac{\psi_n^*(z)}{\varphi_n^*(z)} = F(z). \tag{5.10}$$

Clearly related to this is the following result of Golinskii–Nevai [15]:

Theorem 5.3. *Let* $z \in \mathbb{D}$. *Then*

$$\sum_{n=0}^{\infty} \left| \begin{pmatrix} \psi_n(z) \\ -\psi_n^*(z) \end{pmatrix} + \beta \begin{pmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{pmatrix} \right|^2 < \infty \tag{5.11}$$

if and only if

$$\beta = F(z). \tag{5.12}$$

From this point of view, F is again the "correct" analog of m! Indeed, the Golinskii–Nevai [15] proof uses Weyl limiting circles to prove the theorem (one is always in limit point case!).

But this is not the end of the story. Define

$$u_k = \psi_k + F(z)\varphi_k, \quad u_k^* = -\psi_k^* + F(z)\varphi_k^* \tag{5.13}$$

so $\binom{u_k}{u_k^*}$ is the unique solution of $\Xi_n = T_n(z)\Xi_0$ which is in ℓ^2. In the OPRL case, the basic vector solution is of the form $\binom{u_n}{u_{n+1}}$, so we have the analog of (1.11),

$$\tilde{m}(z) = \frac{u_0^*}{u_0} = \frac{-1+F}{1+F} = zf. \tag{5.14}$$

So one analog of the m-function is zf.

In particular, (5.14) implies

$$|u_k^*| < |u_k| \tag{5.15}$$

for $z \in \mathbb{D}$, and thus the rate of exponential decay of $\left|\binom{u_k}{u_k^*}\right|$ is that of u_k. If there is such exponential decay in the sense that

$$\gamma_2 = \lim_{n\to\infty} \left[\left\| \begin{pmatrix} u_n \\ u_n^* \end{pmatrix} \right\|^{1/n} \right] \tag{5.16}$$

exists, then, by (5.15),

$$\gamma_2 = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |m_n^+|, \tag{5.17}$$

where

$$m_n^+ = \frac{u_{n+1}}{u_n}. \tag{5.18}$$

For $n=0, u_1 = \psi_1 + F\varphi_1, u_0 = 1+F, \psi_1 = \rho_0^{-1}(z+\bar{\alpha}_0), \varphi_1 = \rho_0^{-1}(z-\bar{\alpha}_0)$, so by a direct calculation,

$$m_0^+(z) = \rho_0^{-1} z(1-\bar{\alpha}_0 f), \tag{5.19}$$

yet another reasonable choice for an m-function.

Indeed, if $\gamma(z) = \lim_{n\to\infty}(1/n)\log\|T_n(z)\|$ exists, the fact that $\det(T_n) = z^n$ implies that $\gamma = \log|\lambda| - \gamma_2$, and one finds in the case of stochastic Verblunsky coefficients that [11,29]

$$\mathbb{E}(\log|m_\omega^+(z)|) = \log|z| - \gamma(z), \tag{5.20}$$

an analog of a fundamental formula of Kotani [20,26] that in his case uses m!

Finally, we turn to the connection of m to whole-line Green's functions. Given V on $(-\infty,\infty)$ and $z \in \mathbb{C}_+$, it is natural to look at the two solutions of (1.1), $u_\pm(x,z)$, which are ℓ^2 on $\pm(0,\infty)$ and the m-functions,

$$m_\pm(z) = \pm \frac{u'_\pm(0,z)}{u_\pm(0,z)}. \tag{5.21}$$

$m_\pm$ are the m-functions for $V(\pm x)[0,\infty)$. Standard Green's function formulae show that the integral kernel, $G(x,y;z)$, of $(-\mathrm{d}^2/\mathrm{d}x^2 + V - z)^{-1}$ is

$$G(x,y;z) = \frac{u_-(x_<)u_+(x_>)}{(u_+(0)u'_-(0) - u'_+(0)u_-(0))},$$

where $x_< = \min(x,y)$ and $x_> = \max(x,y)$. In particular,

$$G(0,0;z) = -(m_+(z) + m_-(z))^{-1}. \tag{5.22}$$

A complete description of the OPUC analog would require too much space, so we sketch the ideas, leaving the details to [29]. Just as the difference equation is associated to a tridiagonal self-adjoint matrix whose spectral measure is the one generating the OPRL, any set of α's is associated to a five-diagonal unitary matrix, called the CMV matrix, whose spectral measure is the $d\mu$ with $\alpha_j\ (d\mu) = \alpha_j$.

The CMV matrix is one-sided, but given $\{\alpha_j\}_{j=-\infty}^{\infty}$, one can define a two-sided CMV matrix, $\mathscr{E}$, in a natural way. If $G(z)$ is the 00 matrix element of $(\mathscr{E} - z)^{-1}$, then (see [11,17,29])

$$G(z) = \frac{f_+(z)f_-(z)}{1 - zf_+(z)f_-(z)}, \tag{5.23}$$

where f_+ is the Schur function for $(\alpha_0, \alpha_1, \alpha_2, \ldots)$ and f_- the Schur function for $(-\bar{\alpha}_{-1}, -\bar{\alpha}_{-2}, \ldots)$. On the basis of the analogy between (5.23) and (5.22), Geronimo–Teplyaev [11] called f_+ and zf_- the m_+ and m_- functions.

6. Summary

We have thus seen that there are many analogs of the m-function in the theory of OPUC:

(1) The Carathéodory function, $F(z)$, given by (3.1), an analog of (1.2) and also related to the classic Weyl definition (5.11)/(5.12).
(2) The Schur function, $f(z)$, given by (3.7) with a recursion, (3.8), closer to the recursion (1.15) for the m-function of OPRL. f also enters via (5.23).
(3) $zf(z)$, the $\tilde{m}$-function of (5.14).
(4) The relative Szegő function, (4.1), which, via (4.3) and (1.16), is an analog of $a_1m(z)$.
(5) The m^+-function, (5.19), which plays the role that m does in Kotani theory.

Note added in proof

After this paper was processed, while finishing up the preparation of [29], I realized there is yet another OPUC analog of the m-function. A key property of the m-function for the Jacobi case is that m has poles at eigenvalues of J and zeros at eigenvalues of the Jacobi matrix obtained by removing one a and one b. An analogous function for OPUC is

$$M(z) = z(1 + \alpha_0)(1 + F(z)) + (\bar{\alpha}_0 + 1)(1 - F(z)).$$

This has poles at poles of F and zeros at point masses for $d\mu_1$, the measure associated to $\{\alpha_{j+1}\}_{j=0}^{\infty}$. There are two exceptions to this statement. It can happen at $z = (1 + \bar{\alpha}_0)/(1 + \alpha_0)$ that both measures have a pure point, in which case M has neither a zero nor a pole (this kind of cancellation does not happen for Jacobi matrices because of interlacing of zeros). M vanishes at $z = 0$. This M-function continued to a hyperelliptic Riemann surface is critical to the analysis of finite gap Verblunsky coefficients; see [29].

References

[1] A.B. Aleksandrov, Multiplicity of boundary values of inner functions, Izv. Akad. Nauk Arm. SSR 22 (1987) 490–503.
[2] F.V. Atkinson, Discrete and Continuous Boundary Problems, Academic Press, New York, 1964.
[3] F.V. Atkinson, On the location of the Weyl circles, Proc. Roy. Soc. Edinburgh Sect. A 88 (1981) 345–356.
[4] G. Borg, Uniqueness theorems in the spectral theory of $y'' + (\lambda - q(x))y = 0$, Proceedings of the 11th Scandinavian Congress of Mathematicians, Trondheim, 1949, Johan Grundt Tanums Forlag, Oslo, 1952, pp. 276–287.
[5] E.A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.
[6] D. Damanik, B. Simon, Jost functions and Jost solutions for Jacobi matrices, I. A necessary and sufficient condition for Szegő asymptotics, in preparation.
[7] M.S.P. Eastham, The Spectral Theory of Periodic Differential Equations, Scottish Academic Press, Edinburgh, 1973.
[8] M.S.P Eastham, Antibound states and exponentially decaying Sturm–Liouville potentials, J. London Math. Soc. 65 (2) (2002) 624–638.
[9] W.N. Everitt, M.K. Kwong, A. Zettl, Oscillation of eigenfunctions of weighted regular Sturm–Liouville problems, J. London Math. Soc. 27 (2) (1983) 106–120.
[10] I.M. Gel'fand, private communication.
[11] J.S. Geronimo, A. Teplyaev, A difference equation arising from the trigonometric moment problem having random reflection coefficients—an operator-theoretic approach, J. Funct. Anal. 123 (1994) 12–45.
[12] J. Geronimus, On the trigonometric moment problem, Ann. Math. 47 (2) (1946) 742–761.
[13] F. Gesztesy, B. Simon, A new approach to inverse spectral theory, II. General real potentials and the connection to the spectral measure, Ann. Math. 152 (2000) 593–643.
[14] L. Golinskii, S. Khrushchev, Cesàro asymptotics for orthogonal polynomials on the unit circle and classes of measures, J. Approx. Theory 115 (2002) 187–237.
[15] L. Golinskii, P. Nevai, Szegő difference equations, transfer matrices and orthogonal polynomials on the unit circle, Comm. Math. Phys. 223 (2001) 223–259.
[16] P. Hartman, Ordinary Differential Equations, Wiley, New York, 1964.
[17] S. Khrushchev, Schur's algorithm, orthogonal polynomials, and convergence of Wall's continued fractions in $L^2(\mathbb{T})$, J. Approx. Theory 108 (2001) 161–248.
[18] S. Khrushchev, Classification theorems for general orthogonal polynomials on the unit circle, J. Approx. Theory 116 (2002) 268–342.
[19] R. Killip, B. Simon, Sum rules for Jacobi matrices and their applications to spectral theory, Ann. Math. 158 (2) (2003) 253–321.
[20] S. Kotani, Lyapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators, Stochastic Analysis, Katata/Kyoto, 1982, North-Holland Math. Library, 32, North-Holland, Amsterdam, 1984, pp. 225–247.
[21] B.M. Levitan, Inverse Sturm–Liouville Problems, VNU Science Press, Utrecht, 1987.
[22] F. Peherstorfer, A special class of polynomials orthogonal on the unit circle including the associated polynomials, Constr. Approx. 12 (1996) 161–185.
[23] W.T. Reid, Ordinary Differential Equations, Wiley, New York, 1971.
[24] W. Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York, 1987.
[25] B. Simon, Functional Integration and Quantum Physics, in: Pure and Applied Mathematics, Vol. 86, Academic Press, New York, 1979.
[26] B. Simon, Kotani theory for one dimensional stochastic Jacobi matrices, Comm. Math. Phys. 89 (1983) 227–234.
[27] B. Simon, A new approach to inverse spectral theory, I. Fundamental formalism, Ann. Math. 150 (2) (1999) 1029–1057.
[28] B. Simon, A canonical factorization for meromorphic Herglotz functions on the unit disk and sum rules for Jacobi matrices, J. Funct. Anal. to appear.
[29] B. Simon, Orthogonal Polynomials on the Unit Circle, AMS Colloquium Publications Series, expected 2004.
[30] B. Simon, A. Zlatoš, Sum rules and the Szegő condition for orthogonal polynomials on the real line, Comm. Math. Phys. 242 (2003) 393–423.

[31] T. Stieltjes, Recherches sur les fractions continues, Anns. Fac. Sci. Univ. Toulouse 8 (1894–1895) J1–J122; T. Stieltjes, Anns. Fac. Sci. Univ. Toulouse 9, A5–A47.
[32] G. Szegő, Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publ., 3rd Edition, Vol. 23, American Mathematical Society, Providence, RI, 1939, 1967.
[33] E.C. Titchmarsh, Eigenfunction Expansions Associated with Second-Order Differential Equations, Part 1, 2nd Edition, Clarendon Press, Oxford, 1962.
[34] S. Verblunsky, On positive harmonic functions: a contribution to the algebra of Fourier series, Proc. London Math. Soc. 38 (1935) 125–157.
[35] S. Verblunsky, On positive harmonic functions (second part), Proc. London Math. Soc. 40 (1936) 290–320.