

## Orthogonal Polynomials with Exponentially Decaying Recursion Coefficients

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**ABSTRACT.** We review recent results on necessary and sufficient conditions for measures on  $\mathbb{R}$  and  $\partial\mathbb{D}$  to yield exponential decay of the recursion coefficients of the corresponding orthogonal polynomials. We include results on the relation of detailed asymptotics of the recursion coefficients to detailed analyticity of the measures. We present an analog of Carmona's formula for OPRL. A major role is played by the Szegő and Jost functions.

### 1. Introduction: Szegő and Jost functions

In broad strokes, spectral theory concerns the connection between the coefficients in differential or difference equations and the spectral measures associated to those equations. The process of going from coefficients to the measures is the direct problem, and the other direction is the inverse spectral problem. The gems of spectral theory are ones that set up one-one correspondences between classes of measures and coefficients with some properties. Examples are Verblunsky's form of Szegő's theorem [25] and the Killip-Simon theorem [12]. In this paper, our goal is to describe (mainly) recent results involving such gems for orthogonal polynomials whose recursion coefficients decay exponentially. These are technically simpler systems than the  $L^2$  results just quoted but have more involved details.

The two classes we discuss are orthogonal polynomials on the real line (OPRL) and on the unit circle (OPUC). For the OPRL case, we have a probability measure,  $d\rho$ , on  $\mathbb{R}$  of bounded but infinite support whose orthonormal polynomials,  $p_n(x)$ , obey

$$(1.1) \quad xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_n p_{n-1}(x)$$

with  $b_n \in \mathbb{R}$  and  $a_n \in (0, \infty)$  and called Jacobi parameters.  $\{a_n, b_n\}_{n=1}^{\infty}$  is a description of  $d\rho$  in that there is a one-one correspondence between bounded sets of such Jacobi parameters and such  $d\rho$ s. For background discussion of OPRL, see [4, 9, 18, 24].

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This is the final form of the paper.

For the OPUC case,  $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ , and  $d\mu$  is a probability measure on  $\partial\mathbb{D}$  whose support is not a finite set. The orthonormal polynomials,  $\varphi_n(z)$ , obey the Szegő recursion relation:

$$(1.2) \quad z\varphi_n(z) = \rho_n\varphi_{n+1}(z) + \bar{\alpha}_n\varphi_n^*(z)$$

$$(1.3) \quad \varphi_n^*(z) = z^n\varphi_n(1/\bar{z})$$

$$(1.4) \quad \rho_n = (1 - |\alpha_n|^2)^{1/2}$$

with  $\alpha_n \in \mathbb{D}$  and called Verblunsky coefficients.  $\{\alpha_n\}_{n=0}^\infty$  is a description of  $d\mu$  in that there is a one-one correspondence between sequences of  $\alpha_n$  obeying  $|\alpha_n| < 1$  and such  $d\mu$ s. For background discussion of OPUC, see [10, 17-19, 24].

The measure theoretic side of the equivalences will be in terms of a derived object, rather than the measures themselves. For OPUC, the object is  $D(z)$ , the Szegő function [18, Section 2.4]. One says the Szegő condition holds if and only if

$$(1.5) \quad d\mu(\theta) = w(\theta) \frac{d\theta}{2\pi} + d\mu_s$$

where  $d\mu_s$  is singular and

$$(1.6) \quad \int \log(w(\theta)) \frac{d\theta}{2\pi} > -\infty$$

(which is known to be equivalent to  $\sum_{n=0}^\infty |\alpha_n|^2 < \infty$ ). In that case,  $D(z)$  is defined by

$$(1.7) \quad D(z) = \exp\left(\int \frac{e^{i\theta} + z}{e^{i\theta} - z} \log(w(\theta)) \frac{d\theta}{4\pi}\right)$$

which obeys

$$(1.8) \quad \varphi_n^*(z) \rightarrow D(z)^{-1}$$

if  $|z| < 1$ .

$D(z)$  does not uniquely determine  $d\mu$ , but it does if  $d\mu_s = 0$ , as it will be in our cases of interest, since

$$(1.9) \quad w(\theta) = \lim_{r \uparrow 1} |D(re^{i\theta})|^2$$

for a.e.  $\theta$ .

For OPRL, the object is the Jost function. The situation is not as clean as the OPUC case in that there are not simple necessary and sufficient conditions for existence in terms of the measure. There are necessary and sufficient conditions in terms of the Jacobi parameters (see [5; 19, Section 13.9]) but not for the measure. However, there are sufficient conditions for existence that suffice for us here. Suppose

$$(1.10) \quad d\rho(x) = f(x) dx + d\rho_s(x)$$

where  $f$  is supported on  $[-2, 2]$ , and outside this set, the singular part,  $d\rho_s$ , has only pure points  $\{E_j^\pm\}_{j=0}^{N_\pm}$  with

$$(1.11) \quad E_1^- < E_2^- < \dots < -2 < 2 < \dots < E_2^+ < E_1^+$$

and suppose

$$(1.12) \quad \sum_{j,\pm} (|E_j^\pm| - 2)^{1/2} < \infty$$

and that

$$(1.13) \quad \int_{-2}^2 \log(f(x))(4-x^2)^{-1/2} dx > -\infty$$

Then (originally in Peherstorfer-Yuditskii [15]; see also Simon-Zlatoš [23] and [19, Theorem 13.8.9]) there is an analytic function  $u$  on  $\mathbb{D}$  so that its zeros are precisely those points  $z_j^\pm$  in  $\mathbb{D}$  given by

$$(1.14) \quad z_j^\pm + (z_j^\pm)^{-1} = E_j^\pm$$

and if  $B$  is the Blaschke product (convergent by (1.12))

$$(1.15) \quad B(z) = \prod_{j=\pm 1} \frac{z - z_j^\pm}{1 - \bar{z}_j^\pm z}$$

then  $Bu^{-1} \in H^2$  and the boundary values of  $u$  obey

$$(1.16) \quad |u(e^{i\theta})|^2 \operatorname{Im} M(e^{i\theta}) = \sin \theta$$

where

$$(1.17) \quad M(z) = \int \frac{d\rho(x)}{z + z^{-1} - x}$$

(so  $\operatorname{Im} M(e^{i\theta})$  is related to  $f(2\cos \theta)$ ).

These properties determine  $u$  uniquely. Unlike the OPUC case,  $u$  does not determine  $d\rho$  even if  $d\rho_s \upharpoonright [-2, 2] = 0$  for  $u$  only determines  $f$  and the localization of the pure points of  $d\rho$  on  $\mathbb{R} \setminus [-2, 2]$ . To recover  $d\rho$ , we also need to know the weights of the pure points; equivalently, the residues of the poles of  $M$  at the  $z_j^\pm$ .

The theme of this review is that detailed results on exponential decay of recursion coefficients are equivalent to analyticity results on  $D^{-1}$  in the OPUC case and  $u$  in the OPRL case. That exponential decay implies analyticity has been in the physics literature for Schrödinger operators for over fifty years. The subtle aspect is the strict equivalence—an idea that appeared first in Nevai-Totik [14].

In Section 2, we discuss some aspects of finite range potentials, and in Section 3, following Nevai-Totik [14] and Damanik-Simon [6], the initial equivalence. In Section 4, following Simon [20, 21], we discuss detailed exponential asymptotics and meromorphic  $S$  and  $u$ .

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Stas Molchanov is a leading figure in spectral theory. It is a pleasure to present this birthday bouquet to him.

## 2. Finite range

In this section, we present new results on approximation by finite range “potentials.” We begin with an OPRL analog of Carmona’s result [3] on boundary condition averaging for Schrödinger operators. We will also see that Bernstein-Szegő measures for OPUC can be viewed through the Carmona lens. Carmona’s proof relies on computing derivatives of Prüfer variables—our proof here is spectral averaging making the relation to [22] transparent.

Let  $J$  be the semi-infinite Jacobi matrix

$$(2.1) \quad J = \begin{pmatrix} b_1 & a_1 & 0 & \dots \\ a_1 & b_2 & a_2 & \dots \\ 0 & a_2 & b_3 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

associated to an OPRL with measure  $d\rho$ ;  $J_{n,F}$ , the matrix obtained from the top  $n$  rows and  $n$  left columns; and  $J_n^b$ , the matrix  $J_{n,F}$  with  $b_n$  replaced by  $b_n + b$ . Here  $b \in \mathbb{C}$ . Notice that if  $\text{Im } b \leq 0$ , then  $\text{spec}(J_n^b) \subset \mathbb{C} \setminus \mathbb{C}_+$ , so

$$m_n^{(b)}(z) = \langle \delta_0, (J_n^{(b)} - z)^{-1} \delta_0 \rangle$$

is analytic for  $b \in \overline{\mathbb{C}}_-$  and  $z$  fixed in  $\mathbb{C}_+$ .

If  $d\rho$  is a determinate moment problem, then  $J$  is essentially selfadjoint on finite sequences [16], so

$$(2.2) \quad m_n^{(b)}(z) \rightarrow m(z) \equiv \int \frac{d\rho(x)}{x - z}$$

for any  $b$ . Thus, if  $d\nu_n$  is defined by

$$(2.3) \quad \begin{aligned} \tilde{m}_n(z) &\equiv \frac{1}{\pi} \int_{-\infty}^{\infty} m_n^{(b)}(z) \frac{db}{1 + b^2} \\ &= \int \frac{d\nu^{(n)}(x)}{x - z} \end{aligned}$$

then

$$(2.4) \quad d\nu^{(n)} \rightarrow d\rho$$

weakly.  $d\nu^{(n)}$  is thus the average over  $b$  of the pure point spectral measures of  $J_n^{(b)}$ .

**Theorem 2.1.** *If  $p_n(x)$  are the orthonormal OPRL, then*

$$(2.5) \quad d\nu^{(n)}(x) = \frac{dx}{\pi(a_n^2 p_n^2(x) + p_{n-1}^2(x))}$$

In particular, the right-hand side of (2.5) converges weakly to  $d\rho$ . More is true, for Gaussian quadrature implies that if  $m_n^{(b)}(z) = \int d\rho_n^{(b)}(x)(x - z)^{-1}$ , then  $\int x^\ell d\rho_n^{(b)}(x) = \int x^\ell d\rho(x)$  for  $\ell \leq 2n - 2$ , and thus,

$$(2.6) \quad \int x^\ell d\nu^{(n)}(x) = \int x^\ell d\rho, \quad \ell = 0, \dots, 2n - 2$$

Of course,  $d\nu^{(n)}$  does not have all moments finite; indeed,  $\int |x|^\ell d\nu^{(n)} = \infty$  for  $\ell \geq 2n - 1$ .

**PROOF OF THEOREM 2.1.** It is well known (see [18, Section 1.2]) that

$$(2.7) \quad \det(z - J_{n,F}) = P_n(z)$$

the monic OPRL, and if  $J_{n,F}^{(1)}$  is the matrix obtained by removing the top row and leftmost column (i.e., 11 minor), then

$$(2.8) \quad \det(z - J_{n,F}^{(1)}) = Q_n(z)$$

the monic second kind polynomial of degree  $n - 1$ .

By expanding  $\det(z - J_n^{(b)})$  in minors, we see

$$(2.9) \quad \begin{aligned} \det(z - J_n^{(b)}) &= P_n(z) - bP_{n-1}(z) \\ &= (a_1 \cdots a_{n-1})(a_n p_n(z) - b p_{n-1}(z)) \end{aligned}$$

and thus,

$$(2.10) \quad m_n^{(b)}(z) = -\frac{(a_n q_n(z) - b q_{n-1}(z))}{(a_n p_n(z) - b p_{n-1}(z))}$$

As noted above, if  $\text{Im } b \leq 0$ ,  $m_n^{(b)}(z)$  has its poles in  $\text{Im } z \leq 0$  and thus, if  $\text{Im } z > 0$ ,  $m_n^{(b)}(z)$  is analytic in  $\text{Im } b \leq 0$ . Thus, we can close the contour in the lower half-plane and find for  $\text{Im } z > 0$ ,

$$(2.11) \quad \tilde{m}_n(z) = m_n^{(b=-i)}(z) = -\frac{(a_n q_n(z) + i q_{n-1}(b))}{(a_n p_n(z) + i p_{n-1}(z))}$$

Thus,  $\tilde{m}_n$  is analytic on  $\overline{\mathbb{C}}_+$ , so

$$d\nu_n(x) = \pi^{-1} \text{Im } \tilde{m}_n(x) dx$$

Since  $p_n, p_{n-1}, q_n, q_{n-1}$  are real on  $\mathbb{R}$ ,

$$(2.12) \quad \text{Im } \tilde{m}_n(x) = \frac{a_n(p_{n-1}(x)q_n(x) - p_n(x)q_{n-1}(x))}{(a_n^2 p_n(x)^2 + p_{n-1}(x)^2)}$$

which is (2.7) by a standard Wronskian calculation (see [18, (1.2.51)]). □

By this same calculation, one can recover Carmona's formula for the Schrödinger operator case.

One can ask about the analog of this for OPUC. Given a nontrivial measure,  $d\mu$ , on  $\partial\mathbb{D}$  and  $\omega = e^{i\theta} \in \partial\mathbb{D}$ , we define  $d\mu_n^{(\omega)}$  to be the trivial measure with Verblunsky coefficients

$$\begin{aligned} \alpha_j &= \alpha_j(d\mu), \quad j = 0, \dots, n - 1 \\ \alpha_n &= \omega \end{aligned}$$

Then  $d\mu_n^{(\omega)}$  is the measure with  $n + 1$  pure points at the zeros of the paraorthogonal polynomial (POPUC),

$$(2.13) \quad \Phi_{n+1}^{(\omega)}(z) = z\Phi_n(z) - \bar{\omega}\Phi_n^*(z)$$

**Theorem 2.2.**  $d\mu_n \equiv \int d\theta / (2\pi) d\mu_n(e^{i\theta})$  is the Bernstein-Szegő measure

$$(2.14) \quad d\mu_n = \frac{d\theta}{2\pi|\varphi_n(e^{i\theta})|^2}$$

**PROOF.** If  $\psi_n$  are the second kind polynomials, Geronimus' formula for  $F(z)$  (see [18, Theorem 3.2.4]) implies  $(F(z); d\mu) = \int \frac{\psi_n^*(z) - \omega z \psi_n(z)}{e^{i\theta} - z} d\mu(\theta)$

$$(2.15) \quad F(z; d\mu_n^{(\omega)}) = \frac{\psi_n^*(z) - \omega z \psi_n(z)}{\varphi_n^*(z) - \omega z \varphi_n(z)}$$

Averaging  $\omega$  over  $\frac{d\theta}{2\pi}$  gives the value at  $\omega = 0$  since this function is analytic in  $\omega$  for  $z$  fixed in  $\mathbb{D}$ . It follows that

$$(2.16) \quad F(z; d\mu_n) = \frac{\psi_n^*(z)}{\varphi_n^*(z)}$$

and yields (2.14) by [18, (3.2.35)].  $\square$

The Bernstein–Szegő approximation also has the property of being the measure associated to extending the  $a$ s up to  $n$  to be free beyond  $n$  (i.e.,  $\alpha_j = 0$  for  $j \geq n$ ). One can ask about the analogous approximation for OPRL. We will get the function  $S_n$  used by Dombrowski–Nevai [8]:

Let  $J_\ell$  be the Jacobi matrix with parameters

$$(2.17) \quad a_n(J_\ell) = \begin{cases} a_n(J) & n = 1, \dots, \ell - 1 \\ 1 & n \geq \ell \end{cases}$$

$$(2.18) \quad b_n(J_\ell) = \begin{cases} b_n(J) & n = 1, \dots, \ell \\ 0 & n \geq \ell \end{cases}$$

According to Theorem 13.6.1 (with  $a_\ell$  replaced by 1), its Jost function is ( $x = z + 1/z$ )

$$(2.19) \quad g_\ell(z) = z^\ell \left( p_\ell \left( z + \frac{1}{z} \right) - z p_{\ell-1} \left( z + \frac{1}{z} \right) \right)$$

Define  $S_\ell(x)$  by

$$(2.20) \quad S_\ell \left( z + \frac{1}{z} \right) \equiv g_\ell(z) g_\ell \left( \frac{1}{z} \right)$$

Then, by (2.19),

$$(2.21) \quad S_\ell(x) = p_\ell(x)^2 + p_{\ell-1}(x)^2 - x p_\ell(x) p_{\ell-1}(x)$$

Taking into account the different normalization (for us, “free” is  $a_k = 1$ ; for them,  $a_k = \frac{1}{2}$ ), this is the function  $S_\ell(x)$  of Dombrowski–Nevai [8]. The approximating measure has a.c. part related to  $dx/|g_\ell(z)|^2$  on  $[-2, 2]$  which is  $dx/S_\ell(x)$ . The eigenvalues of  $J_\ell$  are zeros of  $S_\ell(x)$  but not all zeros since  $S_\ell$  also vanishes if  $g_\ell(1/z) = 0$ , that is, at antibound state and resonance energies.

For most purposes, (2.7) is a more useful representation than the one associated to  $S_\ell$ .

### 3. Necessary and sufficient conditions on exponential decay

The starting point of the recent results on exponential decay is the following result of Nevai–Totik for OPUC:

**Theorem 3.1** ([14]). *Let  $d\mu$  be a nontrivial probability measure on  $\partial\mathbb{D}$  and  $R > 1$ . Then the following are equivalent:*

(a)

$$(3.1) \quad \limsup_{n \rightarrow \infty} |\alpha_n(d\mu)|^{1/n} \leq R^{-1}$$

(b)  $d\mu_s = 0$ , the Szegő condition (1.6) holds, and  $D(z)^{-1}$  has an analytic continuation to  $\{z \mid |z| < R\}$ .

**Remark.** Since  $R^{-1} < 1$ , (3.1) is an expression of exponential decay.

The proof is easy. If (3.1) holds, Szegő recursion first implies inductively that for  $|z| = 1$ ,

$$(3.2) \quad |\Phi_{n+1}(e^{i\theta})| \leq (1 + |\alpha_n|) |\Phi_n(e^{i\theta})|$$

so

$$(3.3) \quad \sup_{n, |z| \leq 1} |\Phi_n^*(z)| = \sup_{n, \theta} |\Phi_n^*(e^{i\theta})| \quad (\text{by the maximum principle}) \\ \leq \prod_{j=0}^{\infty} (1 + |\alpha_j|) \equiv C < \infty$$

and thus, for  $|z| > 1$ ,

$$(3.4) \quad |\Phi_n(z)| \leq C|z|^n$$

Iterating

$$(3.5) \quad \Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z \Phi_n(z)$$

we get

$$(3.6) \quad \Phi_n^*(z) = 1 - \sum_{j=0}^{n-1} \alpha_j z \Phi_j(z)$$

(3.1), (3.4), and (3.6) imply that for any  $\varepsilon > 0$ ,

$$\sup_{n, |z| < R - \varepsilon} |\Phi_n^*(z)| < \infty$$

which implies that  $\varphi_n^*(z)$  has a limit for  $|z| < R$ . This limit defines the analytic continuation of  $D(z)^{-1}$ .

For the other direction, one can use either of two similar-looking but distinct formulas relating  $D$  to  $\alpha_n$ . One can use a formula of Geronimus [10] and Freud [9] as Nevai–Totik [14] do (it requires  $d\mu_s = 0$ )

$$(3.7) \quad \alpha_n = -\kappa_\infty \int \overline{\Phi_{n+1}(e^{i\theta})} D(e^{i\theta})^{-1} d\mu(\theta)$$

or the following formula of Simon [20] derived from iterated Szegő recursion:

$$(3.8) \quad \alpha_n = -\kappa_\infty^{-1} \kappa_n^2 \int \overline{\Phi_n(e^{i\theta})} [D(e^{i\theta})^{-1} - D(0)^{-1}] e^{-i\theta} d\mu(\theta)$$

In these formulas,

$$\kappa_n = \prod_{j=0}^{n-1} (1 - |\alpha_j|^2)^{-1/2} \quad \kappa_\infty = \lim_{n \rightarrow \infty} \kappa_n$$

To get exponential decay of  $\alpha_n$  from (3.7) or (3.8), one uses  $\int \overline{\Phi_n(e^{i\theta})} e^{-ij\theta} d\mu(\theta) = 0$  for  $j < n$  and the Taylor series for  $D^{-1}$  to see that  $\alpha_n$  is bounded by the tail of the Taylor series of  $D(z)^{-1}$  which, of course, decays exponentially if  $D(z)^{-1}$  is analytic in  $|z| < R$ .

For OPRL, the analogs of Theorem 3.1 are due to Damanik–Simon [6]. The result is simpler if there are no bound states or resonances where

**Definition.** We say a measure  $d\rho$  on  $\mathbb{R}$  has no bound states or resonances if

$$(3.9) \quad d\rho(x) = f(x) dx + d\rho_s$$

where

$$(3.10) \quad \text{supp}(d\rho) \subset [-2, 2]$$

and

$$(3.11) \quad \int (4 - x^2)^{-1} f(x) dx < \infty$$

**Theorem 3.2** ([6]). Let  $R > 1$ . Suppose  $d\rho$  has no bound states or resonances. Then  $u(z)$  has an analytic continuation to  $\{z \mid |z| < R\}$  if and only if

$$(3.12) \quad \limsup [|a_n(d\rho) - 1| + |b_n(d\rho)|]^{1/2n} \leq R^{-1}$$

[6] has several proofs, but the simplest one is in [21]. When (3.11) holds, there is a measure  $d\mu$  on  $\partial\mathbb{D}$  given by

$$(3.13) \quad d\mu(\theta) = w(\theta) \frac{d\theta}{2\pi} + d\mu_s$$

where

$$(3.14) \quad w\left(\arccos\left(\frac{x}{2}\right)\right) = c(4 - x^2)^{-1/2} f(x)$$

for suitable  $c$  and  $d\mu_s$ . The Verblunsky coefficients  $\alpha_n$  for  $d\mu$  and Jacobi parameters for  $d\rho$  are related by ([2; 11; 19, Section 13.2])

$$(3.15) \quad b_{n+1} = \alpha_{2n} - \alpha_{2n+2} - \alpha_{2n+1}(\alpha_{2n} + \alpha_{2n+2})$$

$$(3.16) \quad a_{n+1}^2 - 1 = \alpha_{2n+1} - \alpha_{2n+3} - \alpha_{2n+2}^2(1 - \alpha_{2n+3})(1 + \alpha_{2n+1}) - \alpha_{2n+3}\alpha_{2n+1}$$

and the Jost function for  $d\rho$  and Szegő function for  $d\mu$  by

$$(3.17) \quad u(z) = (1 - |\alpha_0|^2)(1 - \alpha_1)D(z)^{-1}$$

From this, it is easy to derive Theorem 3.2 from Theorem 3.1.

To understand the situation when  $J$  has bound states, we note the analytic continuation of (1.16) says

$$(3.18) \quad u(z)u\left(\frac{1}{z}\right) \left[ M(z) - M\left(\frac{1}{z}\right) \right] = z - z^{-1}$$

(this uses also  $u, M$  real on  $\mathbb{R}$ ). Recall that if  $z_0 \in \mathbb{D}$  is such that  $z_0 + z_0^{-1}$  is an eigenvalue of  $J$ , then  $u(z_0) = 0$ . An argument shows that if  $|z_0| > R^{-1}$  and  $|a_n - 1| + |b_n| \leq CR^{-2n}$ , then  $u(z_0^{-1}) \neq 0$  and  $M(1/z)$  is regular at  $z_0$ . Thus, (3.18) implies a relation between  $u'(z_0)$ ,  $u(1/z_0)$ , and the residue of the pole of  $M(z)$  at  $z_0$ . This leads to

**Definition.** Suppose  $u$  is analytic in  $\{z \mid |z| < R\}$  for some  $R > 1$  and  $z_0 \in \mathbb{D}$  with  $u(z_0) = 0$  and  $|z_0| > R^{-1}$ . We say the weight of the point mass at  $z_0 + z_0^{-1}$  is canonical if

$$(3.19) \quad \lim_{z \rightarrow z_0} (z - z_0)M(z) = (z_0 - z_0^{-1}) \left[ u'(z_0)u\left(\frac{1}{z_0}\right) \right].$$

**Theorem 3.3** ([6]). Fix  $R > 1$ . Then (3.13) holds if and only if

(i)  $u(z)$  has an analytic continuation to  $\{z \mid |z| < R\}$ .

(ii) The point mass at each  $z_0 \in \mathbb{D}$  with  $|z_0| > R^{-1}$  and  $u(z_0) = 0$  is a canonical weight.

If  $u$  is entire and has  $m$  zeros in  $\mathbb{D}$ , the set of measures with that  $u$  has dimension  $m - 1$ . A single point on this space has decay at rate faster than any exponential. Similarly, if  $u$  is a polynomial,  $\{a_n - 1, b_n\}$  has finite support if and only if all weights are canonical.

#### 4. Detailed asymptotics

Let  $S$  be defined by

$$(4.1) \quad S(z) = - \sum_{j=0}^{\infty} \alpha_{j-1} z^j$$

where  $\alpha_{-1} = -1$ . Of course, when  $D$  exists, both  $D(z)^{-1}$  and  $S(z)$  are analytic near  $z = 0$ . Theorem 3.1 can be rephrased.

**Theorem 4.1.** The Taylor series of  $D(z)^{-1}$  and  $S(z)$  have the same radius of convergence.

Barrios Rolanía et al. [1] extend this to show  $S(z)$  is meromorphic in  $\{z \mid |z| < R + \varepsilon\}$  with a single simple pole at  $z = R$  if and only if  $D(z)^{-1}$  is meromorphic in a similar region. This condition on  $S$  is, of course, equivalent to

$$(4.2) \quad \alpha_n = CR^{-n} + O(R^{-n(1+\delta)})$$

which is how they phrased their result. To go further, it is useful to define

$$(4.3) \quad r(z) = \overline{D(1/\bar{z})} D(z)^{-1}$$

which is analytic in  $\{z \mid 1 - \varepsilon < |z| < R\}$  if (3.1) holds. Simon [18] proved that  $r(z) - S(z)$  is analytic in  $\{z \mid 1 - \varepsilon < |z| < R^2\}$  when (3.1) holds, thereby generalizing [1]. The ultimate result of this genre was found independently by Deift - Östensson [7] and Martínez-Finkelshtein et al. [13]; an alternate proof was then found by Simon [20].

**Theorem 4.2.** If (3.1) holds for some  $R > 1$ , then  $r(z) - S(z)$  is analytic in  $\{z \mid 1 - \varepsilon < |z| < R^3\}$ .

This is optimal in that there are examples [13, 20] where  $S$  (and  $r$ ) have a simple pole at  $z = R$  but  $S - r$  has a pole at  $z = R^3$ .

Motivated by this, Simon [20] proved:

**Theorem 4.3.**  $S(z)$  is an entire meromorphic function if and only if  $D(z)^{-1}$  is.

One can even relate the poles. Given a set  $S$  in  $\{z \mid |z| > 1\}$  which is discrete, one defines  $G(S)$  to be the set of all products  $z_1 \dots z_{n+1} \bar{z}_{n+2} \dots \bar{z}_{2n+1}$  where  $z_j \in S$ . Then

**Theorem 4.4** ([20]). Let  $S(z)$  be entire meromorphic and let  $P$  be the poles of  $D(z)^{-1}$  and  $T$  the set of poles of  $S(z)$ . Then  $P \subset G(T)$  and  $T \subset G(P)$ .

Simon [21] studies analogs of the results for OPRL. In the Jacobi case, define

$$(4.4) \quad B(z) = 1 - \sum_{n=0}^{\infty} [b_{n+1} z^{2n+1} + (a_{n+2} - 1) z^{2n+2}]$$

The analog of Theorem 4.2 is

**Theorem 4.5** ([21]). *Suppose  $R > 1$  and*

$$\limsup_{n \rightarrow \infty} (|a_n^2 - 1| + |b_n|)^{1/2n} = R^{-1}$$

*Then  $(1 - z^2)u(z) + z^2u(1/z)B(z)$  is analytic in  $\{z \mid R^{-1} < |z| < R^2\}$ .*

As explained there,  $R^2$  is optimal. The analog of Theorem 4.3 is

**Theorem 4.6.**  *$B(z)$  is an entire meromorphic function if and only if  $u(z)$  is.*

The connection between poles, that is, the analog of Theorem 4.4 is complicated but appears in [21].

#### Notes added in proof

(1) Theorem 2.1 is not new. It appeared already (at least if  $a_n = 1$ ) as Proposition 2.1 in: D. Krutikov and C. Remling, *Schrödinger operators with sparse potentials: Asymptotics of the Fourier transform of the spectral measure*, Comm. Math. Phys. **223** (2001), no. 3, 509–532. Remling informs me that their proof using non-selfadjoint boundary conditions (essentially equivalent to our calculation) was motivated by earlier work of Atkinson and Clark.

(2) A. Zlatoš has pointed out that in the discussion of the formulas of Dombrowski–Nevai, I left out a factor of  $\sqrt{(4 - x^2)}$ .

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