SPECTRAL THEORY OF ORTHOGONAL POLYNOMIALS

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This is a summary of a talk given at ICMP 2012. It discusses some recent results in spectral theory through the prism of a new-found synergy between the spectral theory and OP communities.

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1. Orthogonal Polynomials

During the past dozen years, a major focus of my research has been the spectral theory of orthogonal polynomials—both orthogonal polynomials on the real line (OPRL) and on the unit circle (OPUC). There has been a flowering of the subject in part because of a cross-fertilization of two communities of researchers. I will discuss some aspects of this subject here; for a lot more, see my recent books on the subject [1–3]. We begin with OPRL.

If μ is a measure on $\mathbb{C}$ with $\int |z|^n \, d\mu < \infty$ and so that μ is not supported on finitely many points, then $\{z^n\}_{n=0}^{\infty}$ are independent in $L^2(\mathbb{C}, d\mu)$ so one can use Gram–Schmidt to obtain monic and also normalized orthogonal polynomials. From the point of view of spectral/operator theory, two cases—OPRL (orthogonal polynomials on the real line) and OPUC (orthogonal polynomials on the unit circle)—are special because they have three-term recurrence relations which make the connection

$$\mu \leftrightarrow \text{recursion coefficients}$$

a problem in spectral theory that can provide guidance for the same problems for Schrödinger operators.

If μ is supported on $\mathbb{R}$, we use $\{P_n\}_{n=0}^{\infty}$ and $\{p_n\}_{n=0}^{\infty}$ for the monic and normalized ($p_n = P_n/\|P_n\|$) OPs. Since multiplication by x is selfadjoint (not true for general μ on $\mathbb{C}$),

$$\langle xP_n, P_j \rangle = \langle P_n, xP_j \rangle = 0 \qquad \text{if } j+1 < n$$

so

$$xP_n = P_{n+1} + b_{n+1}P_n + a_n^2 P_{n-1}$$

$\{a_n, b_n\}_{n=1}^\infty$ are μ's Jacobi parameters. We can write a_n^2 because

$$xP_{n-1} = P_n + \text{lower order}$$

so

$$a_n^2 = \frac{\langle P_{n-1}, xP_n\rangle}{\|P_{n-1}\|^2} = \frac{\langle xP_{n-1}, P_n\rangle}{\|P_{n-1}\|^2} = \frac{\|P_n\|^2}{\|P_{n-1}\|^2}$$

Therefore, if $\mu(\mathbb{R}) = 1$,

$$\|P_n\| = a_1 \dots a_n$$
$$p_n = (a_1 \dots a_n)^{-1} P_n$$

Thus,

$$xp_n = a_{n+1}p_{n+1} + b_{n+1}p_n + a_n p_{n-1}$$

In the ON basis (and it is if $\operatorname{supp}(d\mu)$ is compact), multiplication by x is given by a Jacobi matrix

$$J = \begin{pmatrix} b_1 & a_1 & 0 & \cdots \\ a_1 & b_2 & a_2 & \cdots \\ 0 & a_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

μ is the spectral measure for J and vector δ_1. The inverse problem is easy: given μ, form P_n, and use recursion coefficients or use continued fraction expansion at infinity for $\langle \delta_1, (J-z)^{-1}\delta_1\rangle$.

Indeed, Gel'fand–Levitan said they were motivated by Gram–Schmidt in OPs, and my alternate approach [4] for the inverse problem for $-\frac{d^2}{dx} + V$ was motivated by continued fractions.

Next, we turn to OPUC. Suppose μ is a measure on $\partial\mathbb{D} = \{z\colon |z| = 1,\ z \in \mathbb{C}\}$. Use Φ_n for the monic and φ_n for normalized OPs. For OPRL, we have $P_{n+1} - xP_n \perp \{1, x, \dots, x^{n-2}\}$, so a linear combination of P_n and P_{n-1}. For measures on $\partial\mathbb{D}$, we have

$$\langle zf, g\rangle = \langle f, z^{-1}g\rangle$$

so $\Phi_{n+1} - z\Phi_n \perp \{z^n, \dots, z\}$.

Knowing $\Phi_n \perp \{1, \dots, z^{n-1}\}$, it is not hard to see that

$$\Phi_n^*(z) = z^n\, \overline{\Phi_n(1/\bar{z})}$$

is $\perp \{z^n, \dots, z\}$ and is the only degree n polynomial with this property. Thus, for $\{\alpha_n\}_{n=0}^\infty$ (Verblunsky coefficients), we have the Szegő recursion

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n \Phi_n^*(z)$$

Since $\Phi_{n+1} \perp \Phi_n^*$, taking $-\bar{\alpha}_n\Phi_n^*$ to the other side gets

$$\|\Phi_{n+1}(z)\|^2 + |\alpha_n|^2 \|\Phi_n^*\|^2 = \|\Phi_n\|^2$$

First note that this implies

$$|\alpha_n| < 1$$

Since $\|\Phi_n^*\| = \|\Phi_n\|$, if we define $\rho_n = \sqrt{1 - |\alpha_n|^2}$,

$$\|\Phi_{n+1}\| = \rho_n \|\Phi_n\|$$

Thus (if $\mu(\partial\mathbb{D}) = 1$), $\varphi_n = (\rho_0 \dots \rho_{n-1})^{-1}\Phi_n$ and the φ's obey

$$\begin{aligned} z\varphi_n &= \rho_n \varphi_{n+1} + \bar{\alpha}_n \varphi_n^* \\ \varphi_n^* &= \rho_n \varphi_{n+1}^* + \alpha_n (z\varphi_n) \end{aligned}$$

Going from the α's back to μ is more subtle than in the OPRL case. In fact (Verblunsky's theorem), there is a one-one correspondence between nontrivial probability measures on $\partial\mathbb{D}$ and $\{\alpha_n\}_{n=0}^\infty$. There are various ways to go from α to μ:

(1) The $\{\varphi_n\}_{n=0}^\infty$ are orthonormal but they may not be dense in $L^2(\partial\mathbb{D}, d\mu)$ (e.g., $d\mu = \frac{d\theta}{2\pi}$), but by orthonormalizing $\{1, z, z^{-1}, z^2, z^{-2}, \dots\}$, one gets an ON basis expressible in terms of φ_n and φ_n^* and a five-diagonal matrix for multiplication by z whose elements can be written in terms of α_n (and ρ_n) [CMV matrices]. Its spectral measure for $\varphi_0 = 1$ is μ.

(2) If one defines $f(z)$ by

$$\frac{1 + zf(z)}{1 - zf(z)} = \int \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta)$$

then $f \colon \mathbb{D} \to \mathbb{D}$ (Schur function) and (Geronimus' theorem), there is a continued fraction expansion for f (in terms of Schur parameters) whose parameters are the α_n. So one can construct f from $\{\alpha_n\}_{n=0}^\infty$ and μ as above.

(3) For any nontrivial probability measure, we have that (Bernstein–Szegő approximation)

$$d\mu_n = \frac{d\theta}{2\pi|\varphi_n(z)|^2}$$

is a probability measure whose Verblunsky coefficients are

$$\alpha_j(d\mu_n) = \begin{cases} \alpha_j(d\mu) & j = 0, 1, \dots, n-1 \\ 0 & j \geq n \end{cases}$$

and $\mu_n \to \mu$ weakly. So given $\{\alpha_n\}$, form φ_n, then μ_n, and take limits. These $d\mu_n$ are reminiscent of Carmona's formula (1983) [5]; for OPUC, $\mu_n \to \mu$ goes back to Verblunsky (1936) [6].

2. Why OPs

At a conference on mathematical physics, one can ask: Why study OPs? Of course, one answer has to do with the centrality of OPs in a variety of applications but most of those involve the "algebraic theory," that is, the study of explicit examples, not the "analytic theory" of general connections of μ and recursion coefficients that I focus on here. It is true that the analytic theory of OPUC has application in electronic filter design, information theory, and geophysics. But I guess what I really mean is:

How did I get interested and why should mathematical physicists care? I started out studying $-\Delta + V$ and $-\frac{d^2}{dx^2} + V$, its one-dimensional version. Around 1980, a number of us took to heart Mark Kac's dictum: "Be wise—discretize."

To avoid technical issues, we looked at one-dimensional discrete Schrödinger operators, that is, Jacobi matrices with $a_n \equiv 1$. This was also popular in the condensed matter theory literature. For example, what is often called the almost Mathieu equation (a name I introduced), was heavily studied in the physics literature as "Harper's equation," after a one-band approximation to two-dimensional electrons in a constant magnetic field.

Early on, I realized that if one cares about the "inverse problem," that is, recovering the potential from the spectral measure, one needs to allow general a's because there is no known criteria on measures that tell you their $a_n \equiv 1$. So I began to consider general a_n some of the time.

I have spent a significant part of the first third of my career proving that singular continuous spectrum does not occur (e.g., Perry–Sigal–Simon [7]) and of the second third showing it does! By 2000, it was clear that if $a_n \equiv 1$ and $|b_n| \leq Cn^{-\alpha}$, one has:

- If $\alpha > 1$, J has purely a.c. spectrum in $[-2, 2]$; see, for example, [8, Sect. XIII.8].
- If $\alpha < \frac{1}{2}$, generically, J has purely s.c. spectrum in $[-2, 2]$ [9].
- If $1 > \alpha > \frac{1}{2}$, J always has a.c. spectrum in $[-2, 2]$ but not necessarily purely a.c. [10, 11].
- If $1 > \alpha > \frac{1}{2}$, there are examples where there is also dense point spectrum [12–14].

In my list of "Problems for the 21st Century" [15], I included showing there is mixed singular continuous spectrum for some $\alpha < 1$ (in the continuum Schrödinger case).

Shortly after, Kiselev [16] constructed an example using "standard" methods. And at the same time, Denisov [17] found something weaker in that it was not in terms of power behavior but rather lying in L^2 and it was much stronger in that he allowed much more general kinds of singular continuous components. He relied on a continuum analog of Szegő's theorem for OPUC.

In fact, my "problem for the 21st century," at least the analog for OPUC, had been solved by Verblunsky [6] in 1936! He proved if $d\mu = f(\theta)\frac{d\theta}{2\pi} + d\mu_s$ with $f(\theta) \geq c > 0$ and $d\mu_s$ arbitrary, then $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$. We'll say much more about this later.

The Moral: Analogies between OPUC, OPRL, and Schrödinger operators can be exceedingly fruitful. The OP community and the Schrödinger community can learn from each other, as we'll see.

3. Tales of Two Tribes

I learned there were two distinct mathematical communities studying the same objects: the OP community and part of the Schrödinger operator community; as if there were two tribes hunting the same game, almost unaware of each other's existence.

There was little overlap. Percy Deift, who had worked some on spectral theory problems, revolutionized OPs by introducing Riemann–Hilbert asymptotic methods, but he and they focused on that method so that was little exchange about spectral theory ideas. There was some vague awareness—for example, they knew of Simon–Wolff—but on both sides there was little understanding of the methods of the other side.

This meant that for me working on OPUC was almost a playground. The whole second volume of my OPUC book was translating spectral theory methods into OPUC!

There was a different focus some of the time. For example, OP people rarely considered two-sided or ergodic Jacobi matrices. They found operator techniques strange. On the other side, we rarely used the Christoffel–Darboux formula or Gauss quadrature.

The lack of communication can be seen in some independent discoveries. A first example is Dirichlet decoupling.

In 1989, Tom Spencer and I [18] found a cute argument. In the discrete case, we considered discrete Schrödinger operators and proved that if $b_n \geq 0$ and $\limsup_{n\to\infty} b_n = \infty$, then J has no a.c. spectrum. We used the invariance of a.c. spectrum under trace class perturbations. One picks $n_j \to \infty$ so $\sum |b_{n_j}|^{-1} < \infty$ and compares J to the $\widetilde{J}$ with $b_{n_j} = a_{n_j} = a_{n_j-1} = 0$ and shows that $(J+1)^{-1} - (\widetilde{J}+1)^{-1}$ is trace class. $\widetilde{J}$ is a direct sum of finite matrices. Thus, $\sigma_{\text{ac}}(\widetilde{J}) = \emptyset$, so $\sigma_{\text{ac}}(J) = \emptyset$.

Had we considered general Jacobi matrices, we could have found an even cleaner example of this idea. If $\liminf a_n = 0$, we could have picked n_j so $\sum a_{n_j} < \infty$, let $\widetilde{J}$ be the Jacobi matrix where each a_{n_j} is replaced by 0 so $J - \widetilde{J}$ is trace class and $\sigma_{\text{ac}}(\widetilde{J}) = \emptyset$. The rub is we did this in 1989. Dombrowski [19] in 1978 has found this exact $\liminf a_n = 0$ argument. She had the same use of trace class invariance of σ_{ac}.

A different example involved potentials with bounded variation. In 1986, Dombrowski–Nevai [20] proved that if $a_n \to 1$, $b_n \to 0$, and $\sum_{n=1}^{\infty} |a_{n+1} - a_n| + |b_{n+1} - b_n| < \infty$, then J has purely a.c. spectrum on $[-2, 2]$. Peherstorfer–Steinbauer [21] (2000) and Golinskii–Nevai [22] (2002) found an OPUC analog. But an analog for $-\frac{d^2}{dx^2} + V$ ($V \to 0$, V of bounded variation) was done by Weidmann [23] in 1967!

As a final example of differing discoveries, consider a celebrated formula of Thouless [24] (1974) that says for discrete Schrödinger operators, that if $\gamma(\lambda)$ is the Lyapunov exponent for solutions of $Ju = \lambda u$ (all boundary conditions) and $d\rho$ is the density of zeros, and if

$$\Phi_\rho(z) = \int \log |z - x| \, d\rho(x)$$

then

$$\gamma(z) = \Phi_\rho(z)$$

Some mathematical physicists argue passionately that it should be called the Herbert–Jones [25] formula after their 1971 paper in *J. Phys. C.*

But ideas closely related to this were used by Faber, Fekete, and Szegő in the 1920s, Walsh in the 1930s, and Erdös–Turan in the 1940s. For example, in 1931 Walsh proved the Bernstein–Walsh lemma that if $P(z)$ is a polynomial of degree n and Φ_ρ is the potential (as above) for the equilibrium measure for $\mathfrak{e} \subset \mathbb{R}$ a compact subset, then

$$|P(z)| \leq \big[\sup_{x \in \mathfrak{e}} |P(x)|\big] [\exp(n\Phi_\rho(z))]$$

The new element here is "equilibrium measure" which segues into our next topic.

4. Potential Theory

As I mentioned, potential theory played a role in the work of Szegő, Fekete, and Walsh but came to the forefront with a 1972 paper of Ullman [26] and especially (motivated in part by work of Mhaskar and Saff [27]) a deep book of Stahl–Totik [28] in 1992. I learned the power of the ideas from them.

Let $\mathfrak{e} \subset \mathbb{R}$ be compact. We say $\mathfrak{e}$ has zero capacity if

$$\mathcal{E}(\mu) = \int \log |x - y|^{-1} \, d\mu(x) d\mu(y)$$

is ∞ for every μ supported in $\mathfrak{e}$. Zero capacity sets are really small—in particular, of zero Hausdorff dimension. If $\mathfrak{e}$ doesn't have zero capacity, there is a unique $\mu_{\mathfrak{e}}$ supported on $\mathfrak{e}$ which minimizes $\mathcal{E}(\mu)$. It is called the equilibrium measure for $\mathfrak{e}$ and $\exp(-\mathcal{E}(\mu_{\mathfrak{e}}))$ is $C(\mathfrak{e})$, the capacity of $\mathfrak{e}$.

One consequence of the Thouless formula is that if $\gamma(z) = 0$ on support of ρ, the density of zeros, then $\rho = \mu_{\mathfrak{e}}$, the equilibrium measure, and $\lim_{n\to\infty}(a_1 \dots a_n)^{1/n} = C(\mathfrak{e})$, the capacity of $\mathfrak{e}$. Thus, for example, for purely a.c. ergodic Schrödinger operators (e.g., periodic), the density of states is the equilibrium measure.

Let $J(\omega)$ be an ergodic family of two-sided Jacobi matrices. Then the subadditive ergodic theorem implies that if $T_n(\lambda, \omega)$ is the transfer matrix for solutions of $(J(\omega) - \lambda)u = 0$, then for each λ, there is a Lyapunov exponent $\gamma(\lambda) \geq 0$ so that for a.e. ω,

$$\lim_{n\to\infty} \tfrac{1}{n} \log \|T_n(\lambda, \omega)\| = \gamma(\lambda)$$

For each ω, there can be a set $B(\omega)$ of those λ for which the above is false. We'll call this the Pastur–Ishii bad set.

The name comes from the following argument of Pastur [29] and Ishii [30]. Suppose $\gamma(\lambda) > 0$ for $\lambda \in \mathfrak{e} \subset \mathbb{R}$. Then every solution for $\mathfrak{e} \setminus B(\omega)$ is exponentially growing or decaying, so only eigenvalues. Thus, loosely speaking,

$$\sigma(J(\omega)) \cap \mathfrak{e} \subset \{\text{eigenvalues}\} \cup B(\omega)$$

They noted, by Fubini, for a.e. ω, $\nu\ (B(\omega)) = 0$ for any fixed Baire measure, ν. In particular, if $|\cdot|$ = Lebesgue measure, then, for a.e. ω, $|B(\omega)| = 0$ so $\mathfrak{e}$ has no a.c. spectrum.

Jitomirskaya–Last [31] conjectured that when $\gamma(\lambda) > 0$, the bad set has Hausdorff dimension zero and proved it in some special cases. This, in turn, implies any singular component of the spectral measure on $\mathfrak{e}$ has zero Hausdorff dimension.

Using potential theory techniques ("upper envelope theorem"), I proved the bad set always has capacity zero [32]. In particular, on a set $\mathfrak{e}$ where $\gamma(\lambda) > 0$, the Hausdorff dimension of the spectral measure is 0! Before my work, this was regarded as a very hard problem. But the potential theory ideas make it really easy!

5. Szegő's Theorem

Szegő's theorem provides an illuminating paradigm of the impact of OP/Spectral Theorists "Clash of Civilizations." The term "Szegő's Theorem" is not unique, not only because Gabor Szegő proved many theorems but because, in particular, he proved two about asymptotics of large N Toeplitz determinants, each called "Szegő's Theorem." He found both the leading and second terms in a large N expansion. Remarkably, he found the leading order in 1914 [33] when he was 19 years old and the next term in 1952 [34] when he was 57!

I think it took so long not because it was hard but because it hadn't occurred to Szegő to look at it until Kakutani asked him about it because Onsager had asked Kakutani. Since this second term, which I call the "Strong Szegő Theorem," (see [1, Ch. 6]) is critical in Ising model calculations, it is mentioned more often in the physics literature. But the first term, as a variational statement, is mentioned more often in the mathematics literature. We refer here to the leading term.

In 1920 [35], Szegő realized there was an OPUC translation of his Toeplitz asymptotics—and it's that form we'll focus on. Indeed, he "invented" OPUC because of this connection. Szegő considered probability measures on $\partial\mathbb{D}$ of the form

$$d\mu = w(\theta)\,\frac{d\theta}{2\pi}$$

and proved that

$$\lim_{n\to\infty} \|\Phi_n\| = \exp\biggl(\int \log(w(\theta))\,\frac{d\theta}{2\pi}\biggr)$$

This limit is $\inf\{\|P\|_{L^2(d\mu)} \colon P(0) = 1,\ P \text{ polynomial}\}$ so this has a variational aspect.

In 1935, Verblunsky [6] provided the following variant. Let

$$d\mu = w(\theta)\,\frac{d\theta}{2\pi} + d\mu_{\mathrm{s}}$$

Then

$$\prod_{n=0}^{\infty}(1-|\alpha_n|^2)^{1/2} = \exp\biggl(\int \log(w(\theta))\,\frac{d\theta}{2\pi}\biggr)$$

One new element is the product of ρ's. Of course, this is just $\prod_{j=0}^{n-1}\rho_j = \|\Phi_n\|$ that follows from Szegő recursion, but Szegő recursion was only found in 1939! Verblunsky defined α_n in a different way.

The more significant element is that any $d\mu_{\mathrm{s}}$ is allowed and does not affect the sum rule. Alas, Verblunsky only began to get credit for this about ten years ago!

An immediate consequence (including mixed spectrum) is

$$\int \log(w(\theta))\,\frac{d\theta}{2\pi} > -\infty \Leftrightarrow \sum_{n=1}^{\infty}|\alpha_n|^2 < \infty$$

This includes the OPUC analog of a famous 1999 result of Deift–Killip [11]. And the sum rule can be viewed as a precursor of the KdV sum rules.

The Szegő theorem has generated enormous followup. Function algebra types found a general version in the 1960s! I have a whole book [3] on its descendants in spectral theory.

For OPRL, one class of analogs involves $\lim\|P_n\|$, that is, $\lim(a_1\dots a_n)$. These don't have the sharp if and only if nature nor the arbitrary $d\mu_{\mathrm{s}}$.

In 2003, Killip and I [36] found a different kind of analog (later also for $-\frac{d^2}{dx^2}+V(x)$):

Theorem 5.1. *Let $\{a_n,b_n\}_{n=1}^{\infty}$ be the Jacobi parameters for a measure $d\rho$ on $\mathbb{R}$, $\{x_n\}_{n=1}^{N}$ a listing of the discrete eigenvalues, and $d\mu = f(x)\,dx + d\mu_{\mathrm{s}}$. Then*

$$\sum_{n=1}^{\infty}(a_n-1)^2 + \sum_{n=1}^{\infty}b_n^2 < \infty$$

if and only if

- (Weyl) $\sigma_{\mathrm{ess}}(J) = [-2,2]$
- (Lieb–Thirring) $\sum_{n=1}^{N}(|x_n|-2)^{3/2} < \infty$
- (Quasi-Szegő) $\int_{-2}^{2}(4-|x|^2)^{1/2}\log f(x)\,dx > -\infty$

Motivated by the sum rule for OPUC, we found a sum rule here which is complicated, so I won't write it down. Killip and I were pleased by our realization that the $\log f(x)$ integrals were relative entropies; semicontinuity of the entropy played a role in our proof. We later discovered that while he didn't know it was an entropy, Verblunsky proved and then used a semicontinuity result that is the theorem for the entropy.

6. Clock Spacing and Universality

As a final topic, I want to consider eigenvalue distributions in a box. If a J is truncated to $N \times N$, then $P_N(x) = \det(x\mathbf{1} - J_N)$, so this is the same as spacing of the zeros of OPs. The earliest results are by Erdös–Turan [37] in the 1940s. Recent papers are due to OP people interacting with some Schrödinger-operator types with a touch of random matrix/Riemann–Hilbert.

Let $E_j^{(N)}$ be the zeros of $P_N(x)$ listed in order and $E_j^{(N)}(x)$ the same sequence with j shifted so that

$$E_{-1}^{(N)}(x) < x \leq E_0^{(N)}(x)$$

Let's suppose the density of states is a.c., that is, $\rho(x)\,dx$. Clock spacing says for x, j fixed with $\rho(x) > 0$,

$$E_{j+1}^{(N)}(x) - E_j^{(N)}(x) \sim \frac{1}{N\rho(x)}$$

The name (introduced by me [38]) comes from OPUC where $\rho(\theta) \equiv \frac{1}{2\pi}$ and the zeros look like the numerals on a clock.

After some results by Last and me [39], the subject was revolutionized by two new approaches of Lubinsky [40, 41], both relying on the Christoffel–Darboux kernel (a late nineteenth century invention)

$$K_n(x, y) = \sum_{j=0}^{n} p_n(x)p_n(y)$$

The CD formula says

$$K_n(x, y) = \frac{a_{n+1}(p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x))}{x - y}$$

Thus, if $p_n(x) = 0$, other zeros of $p_n(y)$ are exactly the zeros of $K_n(x, y)$.

Universality for the CD kernel says

$$\frac{K_n(x_0 + \frac{a}{n}, x_0 + \frac{b}{n})}{K_n(x_0, x_0)} \sim \frac{\sin(\pi\rho(x_0)(b - a))}{\pi\rho(x_0)(b - a)}$$

This was proved for analytic weights using Riemann–Hilbert methods (a related result is used in random matrix theory). Lubinsky (2008, 2009) then did his revolutionary work proving this for fairly general a.c. measures on $[-2, 2]$.

Lubinsky included an argument he learned from Eli Levin (it turns out it already appeared in a 1971 book of Freud! [42]). Since the CD formula related zeros of p_n to those of K_n and the universality limit has equally spaced zeros, one has

$$\text{Universality} \Rightarrow \text{Clock Spacing}$$

Totik [43] and I [44] independently used Lubinsky's first method to extend universality (and clock spacing) to fairly general a.c. measures on compact sets $\mathfrak{e} \subset \mathbb{R}$ with $\mathfrak{e}^{\text{int}}$ dense in $\mathfrak{e}$.

Avila, Last, and I [45] used Lubinsky's second method and some ergodic Jacobi matrix machinery to prove universality (and clock spacing) for general ergodic Jacobi matrices in the a.c. spectrum region.

Breuer [46] constructed examples with purely singular continuous spectrum and universality.

Typically (e.g., almost Mathieu below critical coupling), this a.c. spectrum is a nowhere dense Cantor set, making clock spacing striking.

I hope I've shown you the OP/Spectral Theory clash of civilizations has produced intellectual ferment.

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