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Finite gap Jacobi matrices: A review

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1. Introduction

Perhaps the most common theme in Fritz Gesztesy's broad opus is the study of problems with periodic or almost periodic finite gap differential and difference equations, especially those connected to integrable systems. The present paper reviews recent progress in the understanding of finite gap Jacobi matrices and their perturbations. We'd like to acknowledge our debt to Fritz as a collaborator and friend. We hope Fritz enjoys this birthday bouquet!

We consider Jacobi matrices, J, on $\ell^2(\{1,2,\dots,\})$ indexed by $\{a_n, b_n\}_{n=1}^{\infty}$, $a_n > 0$, $b_n \in \mathbb{R}$, where $(u_0 \equiv 0)$

$$(Ju)_n = a_n u_{n+1} + b_n u_n + a_{n-1} u_{n-1} \tag{1.1}$$

or its two-sided analog on $\ell^2(\mathbb{Z})$ where a_n, b_n, u_n are indexed by $n \in \mathbb{Z}$ and J is still given by (1.1) (we refer to "Jacobi matrix" for the one-sided objects and "two-sided Jacobi matrix" for the $\mathbb{Z}$ analog). Here the a's and b's parametrize the operator J and $\{u_n\} \in \ell^2$.

We recall that associated to each bounded Jacobi matrix, J, there is a unique probability measure, μ, of compact support in $\mathbb{R}$ characterized by either of the equivalent

(a) J is unitarily equivalent to multiplication by x on $L^2(\mathbb{R}, d\mu)$ by a unitary with $(U\delta_1)(x) \equiv 1$.

(b) $\{a_n, b_n\}_{n=1}^{\infty}$ are the recursion parameters for the orthogonal polynomials for μ.

We'll call μ the spectral measure for J.

By a finite gap Jacobi matrix, we mean one whose essential spectrum is a finite union

$$\sigma_{\mathrm{ess}}(J) = \mathfrak{e} \equiv [\alpha_1, \beta_1] \cup \cdots \cup [\alpha_{\ell+1}, \beta_{\ell+1}] \tag{1.2}$$

where

$$\alpha_1 < \beta_1 < \cdots < \alpha_{\ell+1} < \beta_{\ell+1} \tag{1.3}$$

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ℓ counts the number of gaps.

We will see that for each such $\mathfrak{e}$, there is an ℓ-dimensional torus of two-sided J's with $\sigma(J) = \mathfrak{e}$ and J almost periodic and regular in the sense of Stahl–Totik [56]. We'll present the theory of perturbations of such J that decay but not too slowly. Our interest will be in spectral types, Lieb–Thirring bounds on the discrete eigenvalues and on orthogonal polynomial asymptotics. We begin in Section 2 with a discussion of the case $\ell = 0$ where we may as well take $\mathfrak{e} = [-2, 2]$, in which the (0-dimensional) torus is the single point with $a_n \equiv 1$, $b_n \equiv 0$. We'll discuss the theory in that case as background.

Section 3 describes the isospectral torus. Section 4 discusses the results for general finite gap sets with a mention of the special results that occur if each $[\alpha_j, \beta_j]$ has rational harmonic measure, in which case the isospectral torus contains only periodic J's. Section 5 discusses a method for the general finite gap case which relies on the realization of $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$ as the quotient of the unit disk in $\mathbb{C}$ by a Fuchsian group—a method pioneered by Peherstorfer-Sodin-Yuditskii [42, 55], who were motivated by earlier work of Widom [64] and Aptekarev [4].

While we focus on the finite gap case, we note there are some results on general compact $\mathfrak{e}$'s in $\mathbb{R}$ with various restrictive conditions on $\mathfrak{e}$ (e.g., Parreau-Widom). Peherstorfer–Yuditskii [42] discuss homogeneous sets and Christiansen [8, 9] proves versions of Theorems 4.3 and 4.5 below for suitable infinite gap $\mathfrak{e}$'s. See [16, 65] for discussion of properties of some $\mathfrak{e}$'s and examples relevant to this area.

These works suggest forms of two conditions in the finite gap case suitable for generalization. Let $\rho_{\mathfrak{e}}$ be the equilibrium measure for $\mathfrak{e}$ and $G_{\mathfrak{e}}(z)$ its Green's function ($-\mathcal{E}(\rho_{\mathfrak{e}}) - \Phi_{\rho_{\mathfrak{e}}}(z)$ in terms of (3.1)/(3.2)). Then (4.5) should read

$$\sum_{n=1}^{N} G_{\mathfrak{e}}(x_n) < \infty \tag{1.4}$$

(which for finite gap $\mathfrak{e}$ is equivalent to (4.5)). Similarly, (4.6) should read

$$\int \log[f(x)]\, d\rho_{\mathfrak{e}}(x) > -\infty \tag{1.5}$$

(again, for finite gap $\mathfrak{e}$ equivalent to (4.6)).

2. The Zero Gap Case

The Jacobi matrix, J_0, with $a_n \equiv 1$, $b_n \equiv 0$ is called the free Jacobi matrix. It is easy to see that the solutions of $J_0 u = \lambda u$ are given by solving

$$\alpha + \alpha^{-1} = \lambda \tag{2.1}$$

for $\lambda \in \mathbb{C}$ and setting

$$u_n = \frac{1}{2i}\,(\alpha^n - \alpha^{-n}) \tag{2.2}$$

This is polynomially bounded in n if and only if $|\alpha| = 1$. If $\alpha = e^{ik}$, then

$$\lambda = 2\cos k, \qquad u_n = \sin(kn) \tag{2.3}$$

Thus,

$$\sigma(J_0) = [-2, 2], \quad \lambda \in (-2, 2) \Rightarrow \text{all eigenfunctions bounded} \tag{2.4}$$

(by all eigenfunctions here, we mean without the boundary condition $u_0 = 0$).

In identifying the spectral type, the following is useful:

THEOREM 2.1. *Let J be a Jacobi matrix with $a_n + a_n^{-1} + |b_n|$ bounded. Suppose all solutions of $(Ju)_n = \lambda u_n$ (where u_0, u_1 are arbitrary) are bounded for $\lambda \in S \subset \mathbb{R}$. Then the spectrum of J on S is purely a.c. in the sense that if μ is the spectral measure of J and $|\cdot|$ is Lebesgue measure, then*

$$\mu_s(S) = 0, \quad T \subset S \text{ and } |T| > 0 \Rightarrow \mu_{ac}(T) > 0 \tag{2.5}$$

REMARK. The modern approach to this theorem would use the inequalities of Jitomirskaya–Last [**28, 29**] or Gilbert–Pearson subordinacy theory [**23, 24, 30, 40**] to handle μ_s and the results of Last–Simon [**36**] for the a.c. spectrum. The simplest proof for this special case (where the above ideas are overkill) is perhaps Simon [**49**].

A simple variation of parameters in the difference equation implies that under ℓ^1 perturbations, eigenfunctions remain bounded when $\lambda \in (-2, 2)$, that is,

THEOREM 2.2. *Let J be a Jacobi matrix with*

$$\sum_{n=1}^{\infty} |a_n - 1| + |b_n| < \infty \tag{2.6}$$

Then $\sigma_{ess}(J) = [-2, 2]$ and the spectrum on $(-2, 2)$ is purely a.c.

REMARK. The continuum analog of Theorem 2.2 goes back to Titchmarsh [**60**].

Thus, the spectrum outside $[-2, 2]$ is a set of eigenvalues $\{x_n\}_{n=1}^N$ where $N \in \mathbb{N} \cup \{\infty\}$. (2.6) has implications for these eigenvalues.

THEOREM 2.3. *Let $\{x_n\}_{n=1}^N$ be the eigenvalues of a Jacobi matrix. Then*

$$\sum_{n=1}^{N} (x_n^2 - 4)^{1/2} \leq \sum_{n=1}^{\infty} |b_n| + 4 \sum_{n=1}^{\infty} |a_n - 1| \tag{2.7}$$

REMARKS. 1. This implies

$$\sum_{n=1}^{N} \operatorname{dist}(x_n, [-2, 2])^{1/2} \leq \frac{1}{2} \left(\sum_{n=1}^{\infty} |b_n| + 4 \sum_{n=1}^{\infty} |a_n - 1| \right) \tag{2.8}$$

2. The analog of (2.8) in the continuum case is due to Lieb–Thirring [**37**] who proved it when the power $1/2$ is replaced by $p > 1/2$ and the right side is replaced by $|b_n|^{p+1/2}$, $|a_n - 1|^{p+1/2}$ and $1/2$ by a suitable constant. They proved the analog is false if $p < 1/2$ and conjectured the result if $p = 1/2$. This conjecture was proven by Weidl [**63**] with an alternate proof and optimal constant by Hundermark–Lieb–Thomas [**25**]. (2.8) and its $p > 1/2$ analogs are called Lieb–Thirring inequalities after [**37**].

3. This theorem is a result of Hundertmark–Simon [**26**] who used a method inspired by [**25**].

4. (2.7) is optimal in the sense that its $p < 1/2$ analog is false and one cannot put a constant $\gamma < 1$ in front of neither the b sum nor the $a - 1$ sum. The same also applies to (2.8).

5. (2.7) implies $p > 1/2$ analogs by an argument of Aizenman–Lieb [**3**].

6. The one-half power in (2.7)/(2.8) is especially significant for the following reason:

$$x(z) = z + z^{-1} \tag{2.9}$$

maps $\mathbb{D}$ to $\mathbb{C}\cup\{\infty\}\setminus[-2,2]$. Its inverse

$$z(x) = \tfrac{1}{2}\left(x - \sqrt{x^2-4}\right) \tag{2.10}$$

has a square root singularity at $x = \pm 2$. Thus, the finiteness of the left side of (2.7)/(2.8) is equivalent to a Blaschke condition

$$\sum_{n=1}^{N} (1 - |z(x_n)|) < \infty \tag{2.11}$$

THEOREM 2.4. *Let J be a Jacobi matrix with $\sigma_{ess}(J) = [-2,2]$ and Jacobi parameters $\{a_n, b_n\}_{n=1}^{\infty}$. Suppose its spectral measure has the form*

$$d\mu = f(x)\,dx + d\mu_s \tag{2.12}$$

where $d\mu_s$ is singular with respect to dx. Suppose that $\{x_n\}_{n=1}^{N}$ are its pure points outside $[-2,2]$. Consider the three conditions:

$$\text{(a)} \quad \sum_{n=1}^{N} \operatorname{dist}(x_n,[-2,2])^{1/2} < \infty \tag{2.13}$$

$$\text{(b)} \quad \int_{-2}^{2} (4-x^2)^{-1/2} \log[f(x)]\,dx > -\infty \tag{2.14}$$

$$\text{(c)} \quad \lim_{n\to\infty} a_1 \dots a_n \text{ exists in } (0,\infty) \tag{2.15}$$

Then any two conditions imply the third. Moreover, in that case,

$$\text{(d)} \quad \sum_{n=1}^{\infty} (a_n - 1)^2 + b_n^2 < \infty \tag{2.16}$$

$$\text{(e)} \quad \lim_{K\to\infty} \sum_{n=1}^{K} (a_n - 1) \text{ and } \lim_{K\to\infty} \sum_{n=1}^{K} b_n \text{ exist} \tag{2.17}$$

REMARKS. 1. (2.13) is called a critical Lieb–Thirring inequality. (2.14) is the Szegő condition.

2. Since $f \in L^1$, the integral in (2.14) can only diverge to $-\infty$. That is, the integral over $\log_+$ is always finite and (2.14) is equivalent to the integral converging absolutely.

3. By a result of Ullman [**62**], $\sigma_{ess}(J) = [-2,2]$ and $f(x) > 0$ for a.e. x in $[-2,2]$ implies $\lim_{n\to\infty}(a_1 \dots a_n)^{1/n} = 1$, so (2.15) can be thought of as a second term in the asymptotics of $\frac{1}{n}\log(a_1 \dots a_n)$.

4. Condition (c) can be thought of as three statements: $\limsup < \infty$, $\liminf > 0$, and $\limsup = \liminf$. The full strength of (c) is not always needed. For example, (a) plus $\limsup > 0$ implies (b) and the rest of (c).

5. This result can be thought of as an analog of a theorem of Szegő for OPUC [**57**] (see also [**50**, Ch. 2]). That (b) $\Rightarrow$ (c), if there are no eigenvalues, is due to Shohat [**47**] and that (b) $\Leftrightarrow$ (c), if there are finitely many x's, is due to Nevai [**38**]. The general (a) + (b) $\Rightarrow$ (c) is due to Peherstorfer–Yuditskii [**41**] and the essence of this theorem is from Killip–Simon [**32**], although the precise theorem is from Simon–Zlatoš [**54**].

COROLLARY 2.5. *If* (2.6) *holds, then so does* (2.14).

PROOF. (2.6) implies $\prod_{n=1}^{\infty} a_n$ converges absolutely and, by Theorem 2.3, it implies (2.13). Thus, (2.14) holds by Theorem 2.4. □

REMARKS. 1. This result was a conjecture of Nevai [39].

2. It was proven by Killip–Simon [32]. It was the need to complete the proof of this that motivated Hundertmark–Simon [26].

There is a close connection between these conditions and asymptotics of the OPRL:

THEOREM 2.6. *Let $\{p_n(x)\}_{n=0}^{\infty}$ be the orthonormal polynomials for a Jacobi matrix, J, obeying the conditions* (a)–(c) *of Theorem* 2.4. *Then uniformly for x in compact subsets of $\mathbb{C} \cup \{\infty\} \setminus [-2,2]$,*

$$\lim_{n\to\infty} \frac{p_n(x)}{\left[\frac{1}{2}(x+\sqrt{x^2-4})\right]^n} \tag{2.18}$$

exists and is analytic with zeros only at the x_n's.

REMARKS. 1. When there are no x_n's, this is essentially a result of Szegő [57, 58]. For the general case, see Peherstorfer–Yuditskii [41].

2. This is called Szegő asymptotics.

3. The reason for the different sign in (2.10) and (2.18) is that, as $n \to \infty$, $p_n(x) \to \infty$, $|z(x)| < 1$ so $z(x)^n p_n(x)$ is bounded. The other solution of (2.9) is $z(x)^{-1}$ and it is that solution that appears in the denominator of (2.18).

While conditions (a)–(c) of Theorem 2.4 are sufficient for Szegő asymptotics, they are not necessary:

THEOREM 2.7. *Let J be a Jacobi matrix whose parameters obey* (2.16) *and* (2.17). *Then* (2.18) *holds on compact subsets of $\mathbb{C} \cup \{\infty\} \setminus [-2,2]$. Conversely, if* (2.18) *holds uniformly on the circle $|x| = R$ for some $R > 2$, then* (2.16) *and* (2.17) *hold.*

REMARKS. 1. This is a result of Damanik–Simon [14].

2. There exist examples where (2.16) and (2.17) hold but both (2.13) and (2.14) fail.

THEOREM 2.8. *For a Jacobi matrix, J, with parameters $\{a_n, b_n\}_{n=1}^{\infty}$, spectral measure obeying* (2.12), *and discrete eigenvalues $\{x_n\}_{n=1}^{N}$, one has*

$$\sum_{n=1}^{\infty} (a_n - 1)^2 + b_n^2 < \infty \tag{2.19}$$

if and only if

$$\sigma_{ess}(J) = [-2,2] \tag{2.20}$$ (a)

$$\sum_{n=1}^{N} \operatorname{dist}(x_n, [-2,2])^{3/2} < \infty \tag{2.21}$$ (b)

$$\int_{-2}^{2} (4-x^2)^{+1/2} \log[f(x)]\, dx > -\infty \tag{2.22}$$ (c)

REMARKS. 1. This theorem is due to Killip–Simon [**32**]. They call (a) Blumenthal–Weyl, (b) Lieb–Thirring, and (c) quasi-Szegő.

2. The continuous analog of (2.19) $\Rightarrow$ (2.21) is due to Lieb–Thirring [**37**].

THEOREM 2.9. *Let J be a Jacobi matrix with $\sigma_{ess}(J) = [-2,2]$ and spectral measure, $d\mu$, given by* (2.12). *Suppose $f(x) > 0$ for a.e. x in $[-2,2]$. Then*

$$\lim_{n\to\infty} |a_n - 1| + |b_n| = 0 \tag{2.23}$$

REMARK. This is often called the Denisov–Rakhmanov theorem after [**15, 44, 45**]. The result is due to Denisov. Rakhmanov had the analog for OPUC which implies the weak version of Theorem 2.9, where $\sigma_{ess}(J) = [-2,2]$ is replaced by $\sigma(J) = [-2,2]$. That the result as stated was true was a long-standing conjecture settled by Denisov.

Conditions on the spectrum combined with weak conditions on the Jacobi parameters have strong consequences. For example, the existence of $\lim_{n\to\infty} a_1 \dots a_n$ clearly has no implication for the b's, but if combined with $\sigma(J) = [-2,2]$ implies, by Theorems 2.4 and 2.8, that $\sum_{n=1}^{\infty} b_n^2 < \infty$. Similarly, one has

THEOREM 2.10. *Suppose $\sigma_{ess}(J) = [-2,2]$ and*

$$\lim_{n\to\infty} (a_1 \dots a_n)^{1/n} = 1 \tag{2.24}$$

Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (a_n - 1)^2 + b_n^2 = 0 \tag{2.25}$$

REMARKS. 1. (2.24) says that the underlying measure is regular in the sense of Ullman–Stahl–Totik; see the discussion in Section 3.

2. This theorem is a result of Simon [**52**].

3. The Isospectral Torus

Let $\mathfrak{e}$ be a finite gap set with ℓ gaps and $\ell + 1$ components, $\mathfrak{e}_j = [\alpha_j, \beta_j]$, $j = 1, \dots, \ell + 1$. There is associated to $\mathfrak{e}$ a natural ℓ-dimensional torus, $\mathcal{T}_{\mathfrak{e}}$, of almost periodic Jacobi matrices. If $\{a_n, b_n\}_{n=-\infty}^{\infty}$ are almost periodic sequences, they are determined by their values for $n \geq 1$ so we can view the elements of $\mathcal{T}_{\mathfrak{e}}$ as either one- or two-sided Jacobi matrices. There are at least three different ways to think of $\mathcal{T}_{\mathfrak{e}}$:

(a) As reflectionless two-sided Jacobi matrices, J, with $\sigma(J) = \mathfrak{e}$. This is the approach of [**5, 7, 21, 22, 42, 53, 55, 59**].

(b) As one-sided Jacobi matrices whose m-functions are minimal Herglotz functions on the Riemann surface of $\left[\prod_{j=1}^{\ell+1}(z-\alpha_j)(z-\beta_j)\right]^{1/2}$. This is the approach of [**10**].

(c) As two-sided almost periodic J which are regular in the sense of Stahl–Totik [**56**] with $\sigma(J) = \mathfrak{e}$. This is the approach of [**35**].

In understanding these notions, some elementary aspects of potential theory are relevant, so we begin by discussing them. For discussion of potential theory ideas in spectral theory, see Stahl–Totik [**56**] or Simon [**51**].

On our finite gap set, $\mathfrak{e}$, there is a unique probability measure, $\rho_\mathfrak{e}$, called the equilibrium measure which minimizes

$$\mathcal{E}(\rho) = \int \log|x-y|^{-1}\, d\rho(x) d\rho(y) \tag{3.1}$$

among all probability measures supported on $\mathfrak{e}$. The corresponding equilibrium potential is

$$\Phi_{\rho_\mathfrak{e}}(x) = \int \log|x-y|^{-1}\, d\rho_\mathfrak{e}(x) \tag{3.2}$$

The capacity, $C(\mathfrak{e})$, is defined by

$$C(\mathfrak{e}) = \exp(-\mathcal{E}(\rho_\mathfrak{e})) \tag{3.3}$$

A Jacobi matrix with $\sigma_{\text{ess}}(J) = \mathfrak{e}$ has

$$\limsup(a_1 \dots a_n)^{1/n} \leq C(\mathfrak{e}) \tag{3.4}$$

J is called regular if one has equality in (3.4). We call a two-sided Jacobi matrix regular if each of the (one-sided) Jacobi matrices

$$J_+ \text{ (resp. } J_-) \text{ with parameters } \{a_n, b_n\}_{n=1}^\infty \text{ (resp. } \{a_{-n}, b_{-n+1}\}_{n=1}^\infty) \tag{3.5}$$

is regular. $\rho_\mathfrak{e}$ is the density of zeros for any regular J with $\sigma_{\text{ess}}(J) = \mathfrak{e}$.

The $\ell+1$ numbers $\rho_\mathfrak{e}([\alpha_j, \beta_j])$, $j = 1, \dots, \ell+1$, which sum to 1 are called the harmonic measures of the bands. We also recall that a bounded function, ψ, on $\mathbb{Z}$ is called almost periodic if $\{S^k\psi\}_{k\in\mathbb{Z}}$, where $(S^k\psi)_n = \psi_{n-k}$, has compact closure in ℓ^∞ (see the appendix to Section 5.13 in [**53**] for more on this class). Such ψ's are associated to a continuous function, Ψ, on a torus of finite or countably infinite dimension so that

$$\psi_n = \Psi(e^{2\pi i n\omega_1}, e^{2\pi i n\omega_2}, \dots) \tag{3.6}$$

The set of $\{n_0 + \sum_{k=1}^K n_k\omega_k : n_0, n_k \in \mathbb{Z},\ \sum_{k=1}^K |n_k| < \infty\}$ is called the frequency module of ψ when there is no proper submodule (over $\mathbb{Z}$) that includes all the nonvanishing Bohr–Fourier coefficients. This set for arbitrary $\{\omega_k\}_{k=1}^K$ is called the frequency module generated by $\{\omega_k\}_{k=1}^K$.

With $J_\pm$ given by (3.5), we define $m_\pm(z)$ for $z \in \mathbb{C} \setminus \mathbb{R}$ by

$$m_\pm(z) = \langle \delta_1, (J_\pm - z)^{-1}\delta_1\rangle \tag{3.7}$$

One has for a two-sided Jacobi matrix that

$$\langle \delta_0, (J-z)^{-1}\delta_0\rangle = -(a_0^2 m_+(z) - m_-(z)^{-1})^{-1} \tag{3.8}$$

An important fact is that $J_\pm$ are determined by $m_\pm$, essentially because $m_\pm$ determine the spectral measures $\mu_\pm$ via their Herglotz representations,

$$m_\pm(z) = \int \frac{d\mu_\pm(x)}{x-z} \tag{3.9}$$

and $\mu_\pm$ determine the a's and b's via recursion coefficients for OPRL. Alternatively, the Jacobi parameters can be read off a continued fraction expansion of $m_\pm(z)$ at $z = \infty$.

It is sometimes useful to let $\tilde{J}_-$ have parameters $\{a_{-n-1}, b_{-n}\}_{n=1}^\infty$, in which case

$$\langle \delta_0, (J-z)^{-1}\delta_0\rangle = -(z - b_0 + a_0^2 m_+(z) + a_{-1}^2 \tilde{m}_-(z))^{-1} \tag{3.10}$$

We can now turn to the descriptions of the isospectral torus. A two-sided Jacobi matrix, J, is called reflectionless on $\mathfrak{e}$ if for a.e. $\lambda \in \mathfrak{e}$ and all n,

$$\operatorname{Re}\langle \delta_n, (J - (\lambda + i0))^{-1} \delta_n \rangle = 0 \tag{3.11}$$

($g(\lambda + i0)$ means $\lim_{\varepsilon \downarrow 0} g(\lambda + i\varepsilon)$). It is known that this is equivalent to

$$a_0^2 m_+(\lambda + i0)\, \overline{m_-(\lambda + i0)} = 1 \text{ for a.e. } \lambda \in \mathfrak{e} \tag{3.12}$$

First Definition of the Isospectral Torus. A two-sided Jacobi matrix, J, is said to lie in the isospectral torus, $\mathcal{T}_\mathfrak{e}$, for $\mathfrak{e}$ if $\sigma(J) = \mathfrak{e}$ and J is reflectionless on $\mathfrak{e}$.

$G_{00}(z) = \langle \delta_0, (J - z)^{-1} \delta_0 \rangle$ is determined by $\operatorname{Im} \log(G_{00}(x + i0))$ via an exponential Herglotz representation. This argument is $\pi/2$ on $\mathfrak{e}$, 0 on $(-\infty, \alpha_1)$, and π on $(\beta_{\ell+1}, \infty)$. G_{00} is real in each gap and monotone, so G_{00} has at most one zero and that zero determines $\operatorname{Im} \log(G_{00}(x + i0))$ on that gap. If $G_{00} > 0$ on (β_j, α_{j+1}) we'll say the zero is at β_j and if $G_{00} < 0$ on (β_j, α_{j+1}) the zero is at α_{j+1}. Thus, the zeros of G_{00} determine G_{00} and so $\operatorname{Im} G_{00}(\lambda + i0)$ on $\mathfrak{e}$.

By (3.10), G_{00} has a zero at λ_0 if and only if m_+ or $\tilde{m}_-$ has a pole at λ_0, and one can show that m_+ and $\tilde{m}_-$ have no common poles. The residue of the pole is determined by the derivative of G_{00} at $\lambda = \lambda_0$. The reflectionless condition determines $\operatorname{Im} m_+$ and $\operatorname{Im} \tilde{m}_-$ on $\mathfrak{e}$, so $a_0, a_{-1}, b_0, m_+, \tilde{m}_-$, and thus J, are uniquely determined by knowing the position of the zero and if they are in the gaps (as opposed to the edges) whether the poles are in m_+ or $\tilde{m}_-$. Hence, for each gap, we have the two copies of (β_j, α_{j+1}) glued at the ends, that is, a circle. Thus, given that one can show each possibility occurs, $\mathcal{T}_\mathfrak{e}$ is a product of ℓ circles, that is, a torus. It is not hard to show that the Jacobi parameters depend continuously on the positions of the zeros of G_{00} and $m_+/\tilde{m}_-$ data.

We turn to the second approach. Any G_{00} as above is purely imaginary on the bands which implies, by the reflection principle, that it can be meromorphically continued to a matching copy of $\mathcal{S}_+ \equiv \mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$. This suggests meromorphic functions on $\mathcal{S}$, two copies of $\mathcal{S}_+$ glued together along $\mathfrak{e}$, will be important. $\mathcal{S}$ is precisely the compactified Riemann surface of $\sqrt{R(z)}$, where

$$R(z) = \prod_{j=1}^{\ell+1} (z - \alpha_j)(z - \beta_j) \tag{3.13}$$

$\mathcal{S}$ is a Riemann surface of genus ℓ. Meromorphic functions on the surface that are not functions symmetric under interchange of the sheets (i.e., meromorphic on $\mathbb{C}$) have degree at least $\ell + 1$.

By a minimal meromorphic Herglotz function, we mean a meromorphic function of degree $\ell + 1$ on $\mathcal{S}$ that obeys

(i) $\operatorname{Im} f > 0$ on $\mathcal{S}_+ \cap \mathbb{C}_+$ ($\mathbb{C}_+ = \{z\colon \operatorname{Im} z > 0\}$)

(ii) f has a zero at ∞ on $\mathcal{S}_+$ and a pole at ∞ on $\mathcal{S}_-$.

Such functions must have their ℓ other poles on $\mathbb{R}$ in the gaps on one sheet or the other and are uniquely, up to a constant, determined by these ℓ poles, one per gap. Each "gap," when you include the two sheets and branch points at the gap edges, is a circle. So if we normalize by $m(z) = -z^{-1} + O(z^{-2})$ near ∞ on $\mathcal{S}_+$, the set of such minimal normalized Herglotz functions is an ℓ-dimensional torus. Each

such Herglotz function can be written on $\mathcal{S}_+ \cap \mathbb{C}_+$ as

$$m(z) = \int \frac{d\mu(x)}{x - z} \tag{3.14}$$

where μ is supported on $\mathfrak{e}$ plus the poles of m in the gaps on $\mathcal{S}_+$. μ then determines a Jacobi matrix.

Second Definition of the Isospectral Torus. The isospectral torus, $\mathcal{T}_\mathfrak{e}$, is the set of one-sided J's whose m-functions are minimal Herglotz functions on the compact Riemann surface $\mathcal{S}$ of $\sqrt{R}$ given by (3.13).

The relation between the two definitions is that the restrictions of the two-sided J's to the one-sided are these J given by minimal Herglotz functions. In the other direction, each J is almost periodic and so has a unique almost periodic two-sided extension.

Third Definition of the Isospectral Torus. The isospectral torus is the almost periodic two-sided J's with $\sigma(J) = \mathfrak{e}$ and which are regular.

This is equivalent to the reflectionless definition since regularity implies the Lyapunov exponent is zero and then Kotani theory [**33**, **48**] implies J is reflectionless.

As noted, the J's in the isospectral torus are all almost periodic. Their frequency module is generated by the harmonic measures of the bands. In particular, the elements of the isospectral torus are periodic if and only if all harmonic measures are rational. Their spectra are purely a.c. and all solutions of $Ju = \lambda u$ are bounded for any $\lambda \in \mathfrak{e}^{\text{int}}$.

Szegő asymptotics is more complicated than in the $\ell = 0$ case. One has for the OPRL associated to a point in the isospectral torus (thought of as a one-sided Jacobi matrix) that for all $z \in \mathbb{C} \setminus \sigma(J)$,

$$p_n(z) \exp(-n\Phi_{\rho_\mathfrak{e}}(z)) \tag{3.15}$$

is asymptotically almost periodic as a function of n with magnitude bounded away from 0 for all n. The frequency module is z-dependent (as written, this is even true if $\ell = 0$ as can bee seen from the free case): the frequencies come from the harmonic measures of the bands plus one that comes from the conjugate harmonic function of $\Phi_{\rho_\mathfrak{e}}(z)$ in $\mathbb{C}_+$ (which gives the z-dependence of the frequency module). The limit of (3.15) on $\mathfrak{e}$, where $\Phi_{\rho_\mathfrak{e}}(x) = 0$, yields the boundedness of solutions of $(J - \lambda)u = 0$. There is also a limit at $z = \infty$: $a_1 \dots a_n / C(\mathfrak{e})^n$ which is almost periodic.

4. Results in the Finite Gap Case

As we've seen, if $\tilde{J}$ is in the isospectral torus for $\mathfrak{e}$ and $\lambda \in \mathfrak{e}^{\text{int}}$, then all solutions of $\tilde{J}u = \lambda u$ are bounded. This remains true under ℓ^1 perturbations by a variation of parameters, so Theorem 2.1 is applicable and we have

THEOREM 4.1. *Let $\mathfrak{e}$ be a finite gap set and $\tilde{J}$, with parameters $\{\tilde{a}_n, \tilde{b}_n\}_{n=1}^\infty$, an element of $\mathcal{T}_\mathfrak{e}$, the isospectral torus for $\mathfrak{e}$. Let J be a Jacobi matrix with*

$$\sum_{n=1}^{\infty} |a_n - \tilde{a}_n| + |b_n - \tilde{b}_n| < \infty \tag{4.1}$$

Then $\sigma_{ess}(J) = \mathfrak{e}$ and the spectrum on $\mathfrak{e}^{int}$ is purely a.c.

REMARK. We are not aware of this appearing explicitly in the literature, although it follows easily from results in [10, 42].

As for eigenvalues in $\mathbb{R} \setminus \mathfrak{e}$:

THEOREM 4.2. *There is a constant C depending only on $\mathfrak{e}$ so that for any Jacobi matrix, J, obeying (4.1) for some $\tilde{J} \in \mathcal{T}_{\mathfrak{e}}$, we have, with $\{x_n\}_{n=1}^N$ the eigenvalues of J,*

$$\sum_{n=1}^{N} \operatorname{dist}(x_n, \mathfrak{e})^{1/2} \leq C_0 + C\Big(\sum_{n=1}^{\infty} |a_n - \tilde{a}_n| + |b_n - \tilde{b}_n| \Big) \tag{4.2}$$

where

$$C_0 = \sum_{j=1}^{\ell} \left| \frac{\alpha_{j+1} - \beta_j}{2} \right|^{1/2} \tag{4.3}$$

REMARKS. 1. This result is essentially in Frank–Simon [18]. They are only explicit about perturbations of two-sided Jacobi matrices where $\tilde{J}$ has no eigenvalues. They mention that one can use interlacing to then get results for the one-sided case—this makes that idea explicit.

2. Prior to [18], Frank–Simon–Weidl [19] proved such a bound on the x_n in $\mathbb{R} \setminus [\alpha_1, \beta_{\ell+1}]$ and Hundertmark–Simon [27] if $1/2$ in the power of $\operatorname{dist}(\dots)^{1/2}$ is replaced by $p > 1/2$ and 1 in the power of $|a_n - \tilde{a}_n|$ and $|b_n - \tilde{b}_n|$ by $p + 1/2$, that is, noncritical Lieb–Thirring bounds.

THEOREM 4.3. *Let J be a Jacobi matrix with $\sigma_{ess}(J) = \mathfrak{e}$ and Jacobi parameters $\{a_n, b_n\}_{n=1}^{\infty}$. Suppose its spectral measure has the form*

$$d\mu = f(x)\, dx + d\mu_s \tag{4.4}$$

where $d\mu_s$ is singular with respect to dx. Suppose $\{x_n\}_{n=1}^N$ are the pure points of $d\mu$ outside $\mathfrak{e}$. Consider the three conditions:

$$\text{(a)} \qquad \sum_{n=1}^{N} \operatorname{dist}(x_n, \mathfrak{e})^{1/2} < \infty \tag{4.5}$$

$$\text{(b)} \qquad \int_{\mathfrak{e}} \operatorname{dist}(x, \mathbb{R} \setminus \mathfrak{e})^{-1/2} \log[f(x)]\, dx > -\infty \tag{4.6}$$

$$\text{(c)} \qquad \text{For some constant } R > 1,\ R^{-1} \leq \frac{a_1 \dots a_n}{C(\mathfrak{e})^n} \leq R \tag{4.7}$$

Then any two imply the third, and if they hold, there exists $\tilde{J} \in \mathcal{T}_{\mathfrak{e}}$ so that

$$\lim_{n \to \infty} |a_n - \tilde{a}_n| + |b_n - \tilde{b}_n| = 0 \tag{4.8}$$

Moreover,

$$\text{(d)} \qquad \lim_{n \to \infty} \frac{a_1 \dots a_n}{\tilde{a}_1 \dots \tilde{a}_n} \text{ exists in } (0, \infty) \tag{4.9}$$

$$\text{(e)} \qquad \lim_{K \to \infty} \sum_{n=1}^{K} (b_n - \tilde{b}_n) \text{ exists in } \mathbb{R} \tag{4.10}$$

REMARKS. 1. Depending on which implications one looks at, only part of (c) is needed. For example, if (a) holds,

$$\text{(4.11)} \qquad \text{(b)} \Leftrightarrow \limsup_{n\to\infty} \frac{a_1 \dots a_n}{C(\mathfrak{e})^n} > 0$$

(that is, indeed, lim sup and not lim inf).

2. As stated, this theorem (except for (e); see below) is due to Christiansen–Simon–Zinchenko [**11**], but parts of it were known. While [**11**] focus on Szegő asymptotics (see below), the work of Widom [**64**] and Aptekarev [**4**] implied if there are no or finitely many x_n's, then (b) $\Rightarrow$ (c), and Peherstorfer–Yuditskii [**42**] proved (a) + (b) $\Rightarrow$ (c) (and as noted to us privately by Peherstorfer, combining their results and an idea of Garnett [**20**] yields (4.11)).

3. That (e) holds does not seem to have been noted before, although it follows easily from the results in [**11**]. For $g_n(z) \equiv p_n(z)/\tilde{p}_n(z)$ has a limit as $n \to \infty$ on $\mathbb{C} \setminus [\alpha_1, \beta_{\ell+1}]$ and that limit also exists and is analytic and nonzero at infinity (see Theorem 4.5 below). Since

$$\text{(4.12)} \qquad z^{-n} p_n(z) = (a_1 \dots a_n)^{-1} \left(1 - \left(\sum_{j=1}^{n} b_j\right) z^{-1} + O(z^{-2})\right)$$

near $z = \infty$,

$$\text{(4.13)} \qquad \log(g_n(z)) = -\log\left(\frac{a_1 \dots a_n}{\tilde{a}_1 \dots \tilde{a}_n}\right) - \left[\sum_{j=1}^{n} (b_j - \tilde{b}_j)\right] z^{-1} + O(z^{-2})$$

so convergence of the analytic functions uniformly near ∞ implies convergence of the $O(z^{-1})$ term.

Theorems 4.2 and 4.3 immediately imply:

COROLLARY 4.4. *If* (4.1) *holds, so does* (4.6).

PROOF. Since $\tilde{a}_1 \dots \tilde{a}_n / C(\mathfrak{e})^n$ is almost periodic bounded away from 0 and ∞, and $\sum_{n=1}^{\infty} |a_n - \tilde{a}_n| < \infty$ and $\tilde{a}_n$, $\tilde{a}_n^{-1}$ bounded imply $\sum_{n=1}^{\infty} |1 - a_n/\tilde{a}_n| < \infty$, we have (4.9), which implies (4.7). By Theorem 4.2, (4.1) $\Rightarrow$ (4.5), so Theorem 4.3 implies (4.6). □

REMARK. This is a result of [**18**], although [**11**] conjectured Theorem 4.2 and noted it would imply this corollary.

THEOREM 4.5. *If the conditions* (a)–(c) *of Theorem* 4.3 *hold, then for all* $z \in \mathbb{C} \cup \{\infty\} \setminus [\alpha_1, \beta_{\ell+1}]$, $\lim_{n\to\infty} p_n(z)/\tilde{p}_n(z)$ *exists and the limit is analytic with zeros only at the* x_n *in* $\mathbb{R} \setminus [\alpha_1, \beta_{\ell+1}]$.

REMARKS. 1. In this form, this result is from [**11**], although earlier it appeared implicitly in Peherstorfer–Yuditskii [**42,43**], and special cases (with stronger assumptions on the x_n's) are in [**4,64**]. See also [**53**].

2. There is also an asymptotic result on $\mathfrak{e}$ not pointwise but in $L^2(d\mu)$ sense; see [**11**].

3. Asymptotics results for orthogonal polynomials on finite gap sets have been pioneered by Akhiezer and Tomčuk [**1,2**].

We do not know an analog of the "if and only if" statement of Theorem 2.7, but there is one direction:

THEOREM 4.6. *Let $\{\tilde{a}_n, \tilde{b}_n\}_{n=1}^{\infty}$ be an element of the isospectral torus, $\mathcal{T}_\mathfrak{e}$, of a finite gap set, $\mathfrak{e}$. Let $\{a_n, b_n\}_{n=1}^{\infty}$ be another set of Jacobi parameters and $\delta a_n, \delta b_n$ given by*

$$\delta a_n = a_n - \tilde{a}_n, \qquad \delta b_n = b_n - \tilde{b}_n$$

Suppose that

(a)

$$\sum_{n=1}^{\infty} |\delta a_n|^2 + |\delta b_n|^2 < \infty \tag{4.14}$$

(b) *For any $\mathbf{k} \in \mathbb{Z}^\ell$,*

$$\sum_{n=1}^{N} e^{2\pi i(\mathbf{k}\cdot\omega)n} \delta a_n \quad \text{and} \quad \sum_{n=1}^{N} e^{2\pi i(\mathbf{k}\cdot\omega)n} \delta b_n \tag{4.15}$$

have (finite) limits as $N \to \infty$.

(c) *For every $\varepsilon > 0$,*

$$\sup_N \left\{ \left| \sum_{n=1}^{N} e^{2\pi i(\mathbf{k}\cdot\omega)n} \delta a_n \right| + \left| \sum_{n=1}^{N} e^{2\pi i(\mathbf{k}\cdot\omega)n} \delta b_n \right| \right\} \le C_\varepsilon \exp(\varepsilon|\mathbf{k}|) \tag{4.16}$$

Let $p_n(z)$ (resp. $\tilde{p}_n(z)$) be the orthonormal polynomials for $\{a_n, b_n\}_{n=1}^{\infty}$ (resp. $\{\tilde{a}_n, \tilde{b}_n\}_{n=1}^{\infty}$). Then for any $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\lim_{n\to\infty} \frac{p_n(z)}{\tilde{p}_n(z)} \tag{4.17}$$

exists and is finite and nonzero.

REMARKS. 1. Here $\omega = (\omega_1, \dots, \omega_\ell)$ is the ℓ-tuple of harmonic measures (i.e., $\omega_j = \rho_\mathfrak{e}([\alpha_j, \beta_j])$) and $\mathbf{k} \cdot \omega = \sum_{j=1}^{\ell} k_j \omega_j$. We thus require infinitely many conditions.

2. This result is from [**12**].

3. If the torus consists of period p elements (i.e., each $\rho_\mathfrak{e}([\alpha_j, \beta_j])$ is k_j/p, where there is no common factor for $p, k_1, \dots, k_\ell$), then the infinity of conditions (4.15) reduces to the finitely many conditions that for $j = 1, 2, \dots, p$, $\sum_{n=0}^{N} \delta a_{np+j}$ and $\sum_{n=0}^{N} \delta b_{np+j}$ have finite limits and (4.16) becomes automatic.

4. [**12**] uses this theorem to construct examples where Szegő asymptotics holds, but both (4.5) and (4.6) fail to hold.

An analog of Theorem 2.8 is not known for general $\mathfrak{e}$ but is known in one special case. We say $\mathfrak{e}$ is p-periodic with all gaps open if $\ell = p - 1$, and for $j = 1, \dots, p$, $\rho_\mathfrak{e}([\alpha_j, \beta_j]) = 1/p$.

We also need a notion of approach to the isospectral torus rather than a single element. Given two Jacobi matrices, we define

$$d_m(J, J') = \sum_{k=0}^{\infty} e^{-|k|} (|a_{m+k} - a'_{m+k}| + |b_{m+k} - b'_{m+k}|) \tag{4.18}$$

and

$$d_m(J, \mathcal{T}_\mathfrak{e}) = \inf_{J' \in \mathcal{T}_\mathfrak{e}} d_m(J, J') \tag{4.19}$$

THEOREM 4.7. *Let $\mathfrak{e}$ be p-periodic with all gaps open. Let J be a Jacobi matrix with spectral measure obeying* (4.4) *and eigenvalues $\{x_n\}_{n=1}^N$ outside $\mathfrak{e}$. Then*

$$\sum_{m=1}^{\infty} d_m(J, \mathcal{T}_{\mathfrak{e}})^2 < \infty \tag{4.20}$$

if and only if

$$\text{(a)} \quad \sigma_{ess}(J) = \mathfrak{e} \tag{4.21}$$

$$\text{(b)} \quad \sum_{n=1}^{N} \operatorname{dist}(x_n, \mathfrak{e})^{3/2} < \infty \tag{4.22}$$

$$\text{(c)} \quad \int_{\mathfrak{e}} \operatorname{dist}(x, \mathbb{R} \setminus \mathfrak{e})^{+1/2} \log[f(x)]\, dx > -\infty \tag{4.23}$$

REMARK. This theorem is due to Damanik–Killip–Simon [**13**]. Their method is specialized to the periodic case, and in that case, proves some of the earlier results of this section, such as Theorem 4.2.

THEOREM 4.8. *Suppose J is a Jacobi matrix with $\sigma_{ess}(J) = \mathfrak{e}$ and so that the f of* (4.4) *is a.e. strictly positive on $\mathfrak{e}$. Then*

$$\lim_{m\to\infty} d_m(J, \mathcal{T}_{\mathfrak{e}}) = 0 \tag{4.24}$$

REMARKS. 1. This is a result of Remling [**46**]. For the periodic case, it was proven earlier by [**13**], who conjecture the result for general $\mathfrak{e}$.

2. Remling replaces (4.24) by the assertion that every right limit of J (i.e., limit point of $\{a_{n+r}, b_{n+r}\}_{n=1}^{\infty}$ as $r \to \infty$) is in $\mathcal{T}_{\mathfrak{e}}$. By a compactness argument, it is easy to see that this is equivalent to (4.24).

THEOREM 4.9. *Let $\mathfrak{e}$ be a finite gap set and J a Jacobi matrix so that*

$$\text{(a)} \quad \sigma_{ess}(J) = \mathfrak{e} \tag{4.25}$$

$$\text{(b)} \quad J \text{ is regular, i.e., } \lim_{n\to\infty} (a_1 \dots a_n)^{1/n} = C(\mathfrak{e}) \tag{4.26}$$

Then

$$\lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} d_m(J, \mathcal{T}_{\mathfrak{e}})^2 = 0 \tag{4.27}$$

REMARKS. 1. This result was proven in case all harmonic measures are rational by Simon [**52**], who conjectured the result in general. It was proven by Krüger [**34**].

2. By the Schwarz inequality, (4.27) is equivalent to

$$\lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} d_m(J, \mathcal{T}_{\mathfrak{e}}) = 0 \tag{4.28}$$

We close this section on results with a list of some open questions:

(1) Do (a)–(c) of Theorem 4.3 imply that

$$\sum_{n=1}^{\infty} (a_n - \tilde{a}_n)^2 + (b_n - \tilde{b}_n)^2 < \infty \tag{4.29}$$

as is true in the case $\mathfrak{e} = [-2, 2]$?

(2) Is there an extension of Theorem 4.7 to the general $\mathfrak{e}$ case?
(3) Is there a converse to Theorem 4.6? This would be interesting even in the periodic case.

5. Methods

The theory of regular Jacobi matrices says one expects the leading growth of $P_n(z)$ as $n \to \infty$ to be $\exp(n\Phi_{\rho_\mathfrak{e}}(z))$. $\Phi_{\rho_\mathfrak{e}}$ is harmonic on $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$ so we can locally define a harmonic conjugate and so $\widetilde{\Phi}_{\rho_\mathfrak{e}}(z)$ analytic with $\operatorname{Re} \widetilde{\Phi}_{\rho_\mathfrak{e}} = \Phi_{\rho_\mathfrak{e}}$. If you circle around x, $\log(z-x)$ changes by $2\pi i$, so circling around the band $[\alpha_j, \beta_j]$, we expect $\int \log(z-x)\, d\rho_\mathfrak{e}(x)$ to change by $2\pi i \rho_\mathfrak{e}([\alpha_j, \beta_j])$ and $\exp(-\widetilde{\Phi}_{\rho_\mathfrak{e}}(z))$ to have a change of phase by $\exp(-2\pi i \rho_\mathfrak{e}([\alpha_j, \beta_j]))$. Thus, we are led to consider analytic functions on $\mathbb{C}_+$ which we can continue along any curve in $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$.

To get a single-valued function, we need to lift to the universal covering space of $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$ and $\exp(-\widetilde{\Phi}_{\rho_\mathfrak{e}}(z))$ will transform under the homotopy group via a character of this group.

So long as $\ell \neq 0$, this cover, as a Riemann surface, is the disk, $\mathbb{D}$, and the deck transformations act as a family of fractional linear transformations on the disk, that is, a Fuchsian group. The use of these Fuchsian groups is thus critical to the theory and used to prove several of the theorems of Section 4 (Theorems 4.7, 4.8, and 4.9 are exceptions).

For more on Fuchsian groups, see Beardon [6], Ford [17], Katok [31], Simon [53], and Tsuji [61]. The pioneers in this approach were Sodin–Yuditskii [55]. See [10–12, 42, 53] for applications of these techniques.

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