THE GLIMM-JAFFE ϕ-BOUND : A MARKOV PROOF

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§1. Introduction

One of the most useful estimates in the control of the thermodynamic limit for $P(\phi)_2$ is the ϕ-bound of Glimm-Jaffe (1972) [henceforth GJ]:

$$\pm \phi(h) < |||h||| \, (\hat{H}_\ell + 1) \tag{1}$$

for suitable h and a suitable norm, $|||\ \ |||$. Here $\hat{H}_\ell$ is defined by:

$$H_\ell = H_0 + \int_{-\ell/2}^{\ell/2} :P(\phi(x)):dx \tag{2}$$

$$E(A) = \text{inf spec } (A) \tag{3a}$$

$$\hat{A} = A - E(A) \tag{3b}$$

Shortly after the appearance of GJ, Guerra, Rosen, and Simon (1972) [henceforth GRS] provided an abbreviated proof of bounds of the form (1). The GRS bounds were weaker than the GJ bounds in the types of functions, h , allowed and in the norm, $|||\ \ |||$, used. In particular, GJ allow $|||\ \ |||$ to be the L^1 norm and GRS do not. For the original applications, this distinction did not matter but recently Fröhlich (1973) exploited the L^1-bound to prove the existence of equal time VEV's in the infinite volume limit. One of our goals is the extension of the GRS proof to cover these L^1-bounds.

It is possible to merely modify one step in the GRS proof. However, we wish to rephrase the GRS proof in a way that we think makes the mechanism of proof more transparent. To explain our point, we recall the GRS proof: one rewrites the bound (1) as a set of bounds on matrix elements of the semigroup $\exp[-t(H_\ell \pm \phi(h)]$ and then uses Nelson's symmetry. In this new form, one bounds the matrix element as the norm of an operator times the product of the norms of two vectors. Nelson's symmetry is then applied to each of the vector norms. Our improved proof can be phrased as applying Nelson's symmetry also to the operator norm. But then we have exploited Nelson's symmetry twice which suggests that the two uses of the symmetry "cancel" and that somehow the symmetry is not needed.

In a narrow sense, this is the case: what we wish to demonstrate is that what is really critical is the Markov property for constant space planes which

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provides a sort of decoupling of spatial regions. GRS (or at least a subset of them!) did not really understand the Markov property and so used Nelson's symmetry to reduce to the semigroup property in time-like directions. While we will emphasize the ϕ bound, our remark applies equally well to the other material in GRS. Of course in a deeper sense, "Nelson's symmetry" is involved as the critical element in a Euclidean invariant path integral.

What we will prove (in the third section) is the following result which is a large part of Spencer's (1973) $N_{\tau,loc}$-bounds [which generalize (1)]:

Theorem 1 Let F be any function of the (time zero) fields smeared in $[a,a+1]$. Then for any ℓ. with $[a,a+1] \subset [-\ell/2,\ell/2]$:

$$- F \leq \hat{H}_\ell + c_1 - c_2 \; E(H_0 + c_2^{-1} F) \tag{4}$$

for suitable constants c_1, c_2 independent of ℓ and F.

Corollary 2 Let $||f||_{-1} = (\int |\hat{f}(k)|^2 (k^2+m^2)^{-1} dk)^{\frac{1}{2}}$. Then for a suitable constant c and any f with $\text{supp}\, f \subset [a,a+1] \subset [-\ell/2,\ell/2]$

$$\pm \phi(f) \leq \hat{H}_\ell + c(||f||_{-1}^2 + 1) \tag{5}$$

In particular

$$\pm \phi(f) \leq c' ||f||_{L^1} (\hat{H}_\ell + 1) \tag{6}$$

We prove (5) for Theorem 1 in the next section. (6) follows from (5) as in GJ. We also have:

Corollary 3 Let $||f||_{-\frac{1}{2}} = (\int |\hat{f}(k)|^2 (k^2+m^2)^{-\frac{1}{2}} dk)^{\frac{1}{2}}$. Then for a suitable constant c

$$\phi(f)^2 \leq c ||f||_{-\frac{1}{2}}^2 (\hat{H}_\ell + 1) \tag{5'}$$

In particular

$$\phi(f)^2 \leq c' ||f||_{L^1}^2 (\hat{H}_\ell + 1) \tag{6'}$$

§2. Nelson's Bound

Our notation for the free Euclidean field follows Simon (1974); see also Guerra et al. (1973). If $\Lambda \subset \mathbb{R}^2$, we say F, a function of the Euclidean fields, is Λ-measurable if F is measurable with respect to the σ-field generated by $\{\phi(f) | f \in N;\ \text{supp}\, f \in \Lambda\}$. If $\Lambda \subset \mathbb{R}$, we use $\Lambda \times \mathbb{R}$ (resp. $\mathbb{R} \times \Lambda$) to denote $\{(x,s) | x \in \Lambda\}$ (resp. $\{(x,s) | s \in \Lambda\}$). Later when we deal with the time zero fields and $\Lambda \subset \mathbb{R}$, Λ-measurable will denote measurable with respect to the σ-field generated by $\{\phi_F(f) | f \in F,\ \text{supp}\, f \subset \Lambda\}$. Finally J_a (resp. $\tilde{J}_a$) will denote the isometry of $\mathcal{F}$ into $\mathcal{N}$ induced by the map $j_a: F \to N$ (resp. $\tilde{j}_a$) given by $j_a f(x,s) = f(x)\delta(s-a)$ (resp. $(\tilde{j}_a f)(x,s) = \delta(x-a)f(s)$).

A basic role is played by:

<u>Theorem 4</u> (Nelson's Bound) Let m be the mass of the free Euclidean field. Let $p = 2/1 - \exp[-m(b-a)]$. Then as a map from $\mathcal{F}$ to $\mathcal{F}$

$$||J_a^* \; V \; J_b|| \leq ||V||_p \qquad (\text{resp. } ||\tilde{J}_a^* \; V \; \tilde{J}_b|| \leq ||V||_p) \tag{7}$$

if V is $\mathbb{R} \times [a,b]$-measurable (resp. $[a,b] \times \mathbb{R}$-measurable).

<u>Proof</u> This is just an expression of hypercontractivity and Hölder's inequality. The basic idea is Nelson's (1973a) although we have used a result from the later Nelson (1973b). For details see Guerra et al. (1973) or Simon (1974). ▨

<u>Proof of Corollary 2</u> By Nelson's bound and the FKN formula:

$$\begin{aligned} e^{-E(H_0+\phi(f))} &\leq ||\exp(-\textstyle\int_0^1 ds \; \phi(f,s))||_p \\ &= [\textstyle\int d\mu_0 \exp(-p\phi(f \otimes \chi_{(0,1)}))]^{1/p} \\ &\leq \exp(c||f \otimes \chi_{(0,1)}||_N^2) \leq \exp(c'||f||_{-1}^2) \end{aligned}$$

From this and (4) we immediately conclude (5). Thus for all f with $||f||_{-1} = 1$ and supp $f \subset [a,a+1]$,

$$\pm \; \phi(f) \leq \hat{H}_\ell + d \leq d(\hat{H}_\ell + 1)$$

for some fixed $d \geq 1$. By homogenity,

$$\pm \; \phi(f) \leq d||f||_{-1}(\hat{H}_\ell + 1)$$

(6) follows from this. ▨

<u>Remark</u> By simply modifying the above one shows that for any $\varepsilon > 0$, there is a $d(\varepsilon)$ with

$$\pm \; \phi(f) \leq ||f||_{L^1}(\varepsilon \; \hat{H}_\ell + d(\varepsilon))$$

<u>Proof of Corollary 3</u> This is similar to that of Corollary 2. We use that fact that

$$\begin{aligned} e^{-E(H_0-\phi(f)^2)} &\leq ||\exp(+ \textstyle\int_0^1 ds(J_\sigma\phi(f))^2 ds)||_p \\ &\leq ||\exp(+ \; \phi(f)^2)||_p \\ &\leq \text{const} \end{aligned}$$

so long as $||f||_F \leq d$ for d sufficiently small. ▨

§3. The Proof of Theorem 1

Theorem 1 depends on the following result of some independent interest:

Theorem 5 Let V_1, V_2, V_3 be functions of the time zero fields which are respectively $(-\infty,a],[a,a+1]$ and $[a+1,\infty)$ measurable. Let V_1 (resp. V_3) be the reflection of V_1 (resp. V_3) in the point $x = a$ (resp. $x = a+1$). Let $\lambda_0 = 2/1 - \exp(-m)$. Then

$$-E(H_0+V_1+V_2+V_3) \leqslant -1/2\ E(H_0+V_1+\widetilde{V}_1) - 1/2\ E(H_0+V_3+\widetilde{V}_3) - 1/\lambda_0\ E(H_0+\lambda_0 V_2) \tag{8}$$

Proof We need only show that

$$<\Omega_0, e^{-t(H_0+V_1+V_2+V_3)}\Omega_0> \leqslant <\Omega_0 e^{-t(H_0+V_1+\widetilde{V}_1)}\Omega_0>^{\frac{1}{2}} \ldots \tag{9}$$

By the FKN formula and the Markov property:

$$<\Omega_0, e^{-t(H_0+V_1+V_2+V_3)}\Omega_0> = \int F_1F_2F_3\ d\mu_0 = \int (\widetilde{J}^*_a F_1)(J^*_a F_2 J_{a+1})(J^*_{a+1}F_3)d\mu_0$$

with $F_i = \exp(-\int_0^t (J_s V_i)ds)$. On account of Nelson's bound, this last quantity is bounded by:

$$(\int (\widetilde{J}^*_a\ \overline{F}_1)^2 d\mu_0)^{\frac{1}{2}}\ (\int F_2^{\lambda_0})^{1/\lambda_0}(\int (\widetilde{J}^*_{a+1}F_3)^2 d\mu_0)^{\frac{1}{2}}$$

and by the Markov property again

$$\int (\widetilde{J}^*_a F_1)^2 d\mu_0 = \int \widetilde{F}_1 F_1\ d\mu_0$$

so (8) follows. ▨

Remark Among other things, (8) implies the linear lower bound of Glimm-Jaffe (1970) that $-E_\ell \leqslant d\ell$ for ℓ large. For (8) implies that (with $V_2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} P\ dx$ and $V_1+V_2+V_3 = \int_{-\ell-\frac{1}{2}}^{\ell+\frac{1}{2}} P\ dx$) $- E(2\ell+1) \leqslant -1/2\ E(2\ell) - 1/2\ E(2\ell) + c = -E(2\ell) + \text{const.}$

Proof of Theorem 1 Let $V_1 = \int_{-\ell/2}^{a} :P(\phi(x)):dx$; $V_3 = \int_{a+1}^{\frac{1}{2}} :P(\phi(x)):dx$ and let $V_2 = F + \int_a^{a+1} :P(\phi(x)):dx$. Then, by (8):

$$-E(H_\ell+F) \leqslant -1/2\ E(H_{\ell+2a}) - 1/2\ E(H_{\ell-2a-1}) - 1/\mu_0\ E(H_0+\mu_0 V_2) \tag{10}$$

Now, by bounds of GRS:

$$\alpha_\infty \ell + \beta_\infty \leqslant -E(H_\ell) \leqslant \alpha_\infty \ell$$

so that:

$$-\frac{1}{2}E(H_{\ell+2a}) - 1/2\ E(H_{\ell-2a-1}) \leqslant -E(H_\ell) - \beta_\infty - \alpha_\infty$$

Moreover, by the convexity of the $-E(A)$ in A :

$$-E(H_0+\mu_0 F+\mu_0 \int_a^{a+1} :P:) \leq -1/2\ E(H_0+2\mu_0 F) - 1/2\ \acute{E}(H_0+2\mu_0 \int_0^1 :P)$$

so that (10) becomes:

$$-E(H_\ell+F) \leq -E(H_\ell) - 1/2\mu_0\ E(H_0+2\mu_0 F) + C$$

which implies (4). ▨

§4. Fröhlich's Bounds

Fröhlich (1973) remarked a very convenient form of the ϕ-bound which provides the neatest form of the bounds needed to complete the proof of Nelson's convergence theorem for the half-Dirichlet Schwinger functions (see Nelson's and Rosen's lectures). Since Rosen uses these bounds in his lecture, we sketch their proof:

Theorem 6 (Fröhlich(1973)) Let $d\nu_\Lambda$ denote the spatially cutoff Markov measure for a fixed $P(\phi)_2$ model. Then for any bounded region D , there exists ℓ-independent constants c_1 and c_2 so that for any f in $L^2(D)$, real-valued:

$$\lim_{t\to\infty} \int e^{\phi(f)} d\nu_{[-\ell/2,\ell/2]\times[-t/2,t/2]} \leq c_1 \exp(c_2||f||_2^2) \qquad (11)$$

so long as $D \subset [-\ell/2,\ell/2] \times \mathbb{R}$.

Proof If Ω_ℓ is the vacuum for $\hat{H}_\ell$, the right side of (11) is

$$\langle\Omega_\ell,\exp(-\int_{-\infty}^{\infty} \hat{H}_\ell-\phi(f_t))\Omega_\ell\rangle \qquad (12)$$

Without loss of generality, take $D = [-1/2,1/2] \times [-1/2,1/2]$. Then (12) is bounded by

$$\exp\left(-\int_{-\frac{1}{2}}^{\frac{1}{2}} E(\hat{H}_\ell-\phi(f_t))\right) \leq \exp(c\int_{-\frac{1}{2}}^{\frac{1}{2}}(||f_t||_{-1}^2+1))$$

$$\leq \exp(c\int_{-\frac{1}{2}}^{\frac{1}{2}}(||f_t||_{L^2}^2+1))$$

which is (11). ▨

Theorem 7 (Fröhlich (1973)) Let $d\nu_\Lambda^{HD}$ deonote the spatially cutoff half-Dirichlet theory for a fixed $P(\phi)_2$ model with $P(X) = Q(X) - \mu X$ with Q even. Then for each bounded region D , there are constants c_1 and c_2 so that for all $\Lambda \supset D$ with Λ bounded and $f \in L^2(D)$, complex valued:

$$\left|\int e^{\phi(f)} d\nu_\Lambda^{HD}\right| \leq c_1 \exp(c_2||f||_2^2) \qquad (13)$$

Proof Clearly $|\int e^{\phi(f)} d\nu_\Lambda^{HD}| \leqslant \int e^{\phi(\mathrm{Re} f)} d\nu_\Lambda^{HD}$. By the first GKS inequality. $\int e^{\phi(\mathrm{Re} f)} d\nu_\Lambda^{HD} \leqslant \int e^{\phi(|\mathrm{Re} f|)} d\nu_\Lambda^{HD} \leqslant \int e^{\phi(|f|)} d\nu_\Lambda^{HD}$. Let $\Lambda \subset [-\ell/2,\ell/2] \times [-t/2,t/2]$. Then by Nelson's Monotonicity theorem and the bound $S_\Lambda^{HD} \leqslant S_\Lambda^{Free}$:

$$\int e^{\phi(|f|)} d\nu_\Lambda^{HD} \leq \lim_{t\to\infty} \int e^{\phi(|f|)} d\nu^{HD}_{[-\ell/2,\ell/2]\times[-t/2,t/2]}$$

$\leqslant \lim_{t\to\infty} \int e^{\phi(|f|)} d\nu_{[-\ell/2,\ell/2]\times[-t/2,t/2]}$. (11) now implies (13) . ▨

§5. The Lower Bound on the Wave Function Renormalization

We have reported on one simplification of the paper GRS. There is a new bound which helps illuminate another of their results, namely that $\ln(\Omega_\ell,\Omega_0) \geqslant - c\ell$.

Theorem 8 For any vector Ω in $L^2(Q_F,d\mu_0)$ with $||\Omega||_2 = 1$

$$||\Omega||_1 \geqslant \exp[-(\Omega,N\Omega)] \tag{14}$$

Remarks

1. This bound in a disguised form was first shown me by C. Newman who proved it by the method GRS use to prove $\ln(\Omega_\ell,\Omega_0) \geqslant -c\ell$.

2. Since
$$\begin{aligned}(\Omega,N\Omega) &\leqslant \frac{1}{m_0}(\Omega,H_0\Omega) \\ &= \frac{1}{m_0}(\Omega,2(H_0+V_\ell)\Omega) - \frac{1}{m_0}(\Omega,(H_0+2V_\ell)\Omega) \\ &\leqslant \frac{1}{m_0}[2\alpha_\ell(1) - \alpha_\ell(2)]\ell\end{aligned}$$

(14) implies the $\ln(\Omega_\ell,\Omega_0) \geqslant -c\ell$ bound.

Proof By the infinitesimal form of hypercontractivity of Gross (1973):

$$-(\Omega,N\Omega) \leqslant -\int |\Omega|^2 \ln|\Omega| \, d\mu_0. \tag{15}$$

Since $|\Omega|^2 d\mu_0$ is a probability measure,

$$\begin{aligned}\exp(-\int |\Omega|^2 \ln|\Omega| d\mu_0) &\leqslant \int \exp(-\ln|\Omega|)|\Omega|^2 d\mu_0 \\ &= \int |\Omega| \, d\mu_0\end{aligned} \tag{16}$$

(14) follows from (15) and (16) . ▨

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