BOSE FIELD THEORY AS CLASSICAL STATISTICAL MECHANICS.
III. THE CLASSICAL ISING APPROXIMATION

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§1. Introduction and General Strategy

The techniques which have been developed initially or solely for Ising ferromagnets fall generally into two broad categories. One group, which includes correlation inequalities of GKS and FKG type, holds for general kinds of ferromagnets with more or less arbitrary (even) single spin distributions and with many body (ferromagnetic) forces allowed. The other group, which includes the correlation inequalities of GHS and the zero theorem of Lee-Yang, has been proven directly only for spin 1/2 ferromagnets (each spin takes the values ±1 with equal probability in the non-interacting systems) with pair interactions. In fact, counter examples exist with four-body interactions and spin 1/2 or with pair interactions and spins taking the values ±2,0 (but with 0,±2 having different weightings).

The lattice approximation of Guerra, Rosen, and Simon (1973) discussed already in these lectures by Nelson and by Rosen, approximates $P(\phi)_2$ by general Ising models and thus obtains GKS and FKG inequalities. Here, we wish to discuss a further approximation of Simon and Griffiths (1973) [henceforth SG] which approximates $(\phi^4)_2$ theory by "classical Ising models", i.e. systems with spin-1/2 spins and pair interactions. SG thereby obtain GHS and Lee-Yang theorems for certain $P(\phi)_2$ theories with deg P = 4. In the interests of emphasizing the main ideas we propose only to discuss the Lee-Yang theorem and one of its main applications. We will also not give certain technical details. The reader interested in further details and applications and in the GHS inequalities should consult the original papers of SG and of Simon (1973b) or the lectures of Simon (1974).

In the remainder of this introduction we want to state the Lee-Yang (1952) circle theorem and explain the general strategy of Griffiths (1970) for extending this theorem from spin-1/2 spins to more complicated situations.

<u>Theorem 1</u> Let $a_{ij} \geq 0$ for $1 \leq i < j \leq n$. Let P be the polynomial in $z_1,\ldots,z_n$ of degree 1 in each z_i with 2^n terms given by:

$$P(z_1,\ldots,z_n) = \sum_{\sigma_1=\pm1,\ldots,\sigma_n=\pm1} \exp\Big(\sum_{i<j} a_{ij}\sigma_i\sigma_j\Big) z_1^{\frac{1}{2}(\sigma_1+1)} \cdots z_n^{\frac{1}{2}(\sigma_n+1)}$$

Then if each $z_i \in D \equiv \{z \mid |z| < 1\} \cup \{1\}$, then $P \neq 0$.

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Remarks

1. The connection with Ising ferromagnets is the following

$$e^{-\beta(h_1+\ldots+h_n)} P\left(e^{2\beta h_1},\ldots,e^{2\beta h_n}\right)$$

and represents the partition function of an Ising ferromagnet if spin σ_i is in a magnetic field h_i. Of course, it is not a priori clear why zeros of the partition function are important. This idea of Yang-Lee (1952) is further discussed in §3 below.

2. Since $P(z_1,\ldots,z_n) = (z_1\ldots z_n)P(z_1^{-1},\ldots,z_n^{-1})$ by spin-flip symmetry $(\sigma_i \to -\sigma_i)$, P is also non-zero if each $z_i \in D^{-1}$ and in particular $P(z,z,\ldots,z) = 0$ can only happen if $|z| = 1$, i.e. $P(z,\ldots z)$ has its roots on the unit circle, hence the name "circle theorem".

3. There are various proofs of this theorem: the original Lee-Yang (1952) proof found also in Ruelle (1969); an unpublished proof of S. Sherman found in Simon (1974); a proof of Asano (1970) described also in Simon (1974); and a proof of Newman (1973).

By combining Remarks 1 and 2, Theorem 1 is easily seen to be equivalent to:

<u>Theorem 1'</u> Let $a_{ij} \geqslant 0$ be fixed for $1 \leqslant i < j \leqslant n$ and let

$$Z(h_1,\ldots,h_n) = \sum_{\sigma_1=\pm1,\ldots,\sigma_n=\pm1} \exp(\sum a_{ij}\sigma_i\sigma_j + \sum h_i\sigma_i) \tag{1}$$

Then $Z \neq 0$ if each $h_i \in \tilde{D} = \{h | \mathrm{Re}\, h > 0\} \cup \{h = 0\}$.

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Griffiths (1970) proposed a very simple and beautiful way of extending Theorem 1' to more complex situations. As a typical case, consider a spin 1 ferromagnet, i.e. each spin s can take the values $0,\pm2$ with equal probability. We thus seek a zero theorem for the function

$$\tilde{Z}(h_1,\ldots,h_n) = \sum_{s_i=\pm2,0} \exp(\sum a_{ij}s_is_j + \sum h_is_i) \tag{2}$$

Griffiths suggests first looking at a two spin, spin 1/2 ferromagnet with $a_{12} = 1/2\,\ln z$. Thus:

$$\text{prob } (s \equiv \sigma_1 + \sigma_2 = +2) = \sqrt{2}/\text{Normalization} = \text{prob } (s = -2)$$

$$\text{prob } (s = 0) = \binom{2}{0}\left(1/\sqrt{2}\right)/\text{Normalization} = \sqrt{2}/\text{Normalization}$$

That is, s looks like a spin-1 spin. In particular

$$2^{n/2}\tilde{Z}(h_1,\ldots,h_n) = \sum_{\substack{\sigma_{i1}=\pm1\\ \sigma_{i2}=\pm1}} \exp\left(\sum_{i,j} a_{ij}(\sigma_{i1}+\sigma_{i2})(\sigma_{j1}+\sigma_{j2}) + \sum_i 1/2\,\ln z\,\sigma_{i1}\sigma_{j1}\right) \exp\left(\sum_i h_i(\sigma_{i1}+\sigma_{i2})\right) \tag{3}$$

by replacing the sum over $\sigma_{i1} = \pm 1$, $\sigma_{i2} = \pm 1$ by a sum over $\sigma_{i1} + \sigma_{i2} = \pm 2, 0$ doing the sum over the other degrees explicitly. On account of (3), $Z \neq 0$ if $h_i \in \tilde{D}$.

§2. The Improved DeMoivre-Laplace Limit Theorem

It is now clear how to go about trying to prove a Lee-Yang theorem for $P(\phi)_2$. First approximate $P(\phi)_2$ by the lattice approximation, i.e. by Ising ferromagnets with pair interactions and single spin distributions $e^{-Q(q)}dq$ where $\deg Q = \deg P$ and Q is even if P is even (which we will suppose). Thus we need only obtain $e^{-Q(q)}dq$ as the output probability distribution for the total spin of an Ising ferromagnet with spin-1/2 spins and pair distributions. More accurately, we need only obtain it as the limit of suitably rescaled output distributions. This is because the Lee-Yang theorem in the form of Theorem 1' is preserved under limits on account of the following consequence of the argument principle: If $f_n(z)$ is a sequence of functions analytic and non-zero in a connected region $D \subset \mathbb{C}$ and if $f_n \to f$ uniformly on compacts of D, then f is either identically zero or non-vanishing in D. To apply this limit theorem, one needs uniform bounds on the approximating distributions as well as pointwise convergence. Below we will only prove pointwise convergence; the extra bounds (which require higher order terms in Stirling's formula) can be found in SG.

Consider first the case $\deg Q = 2$. It is easy to handle this case, for take N uncoupled spin-1/2 spins. Then the probability that $s = \Sigma s_i$ is μ is just $2^{-N}\binom{N}{(N+\mu)/2}$ where $\binom{a}{b}$ is a binomial coefficient. The DeMoivre-Laplace limit theorem asserts that the binomial distribution for large N looks like a Gaussian, explicitly:

$$\binom{N}{\frac{N+\mu}{2}} \sim D_N \exp(-\mu^2/2N) \tag{4}$$

for a suitable constant D_N. What (4) means is that if s is fixed then

$$D_N^{-1}\binom{N}{\frac{N+\mu_N(s)}{2}} \to \exp(-s^2/2)$$

as $N \to \infty$ where $\mu_N(s)$ is defined by

$$\frac{N+\mu_N(s)}{2} = \left[\frac{N+s\sqrt{N}}{2}\right]$$

and $[x]$ = greatest integer less than x. To prove (4), one needs Stirling's formula:

$$\log n! \sim n \log n$$

from which

$$\log\binom{N}{\frac{N+\mu}{2}} \sim C_N - Nh(\mu/N)$$

with C_N a suitable constant and

$$h(x) = 1/2[(1+x)\log(1+x) + (1-x)\log(1-x)] \quad .$$

For x small, $h(x) = 1/2\ x^2 + 1/12\ x^4 + O(x^6)$. Thus for $s = \mu/\sqrt{N}$ fixed,

$$\log \binom{N}{\frac{N+\mu}{2}} \sim C_N - N(1/2(\mu/N)^2 + O(\mu/N)^4) = C_N - 1/2\ s^2 + O(1/N)$$

from which (4) follows.

Next consider $\deg Q = 4$; in fact suppose $Q(q) = q^4$. It is clear how to modify the Gaussian behavior above to get out q^4 ; just cancel the Gaussian and re-scale; i.e.

$$\binom{N}{\frac{N+\mu}{2}} e^{\mu^2/2N} \sim D_N\ e^{-\mu^4/12N^3} \tag{5}$$

For if we fix $s = \mu/N^{3/4}$:

$$\log \binom{N}{\frac{N+\mu}{2}} e^{\mu^2/2N} \sim C_N - N\,[1/12(\mu/N)^4 + O(\mu/N)^6]$$

$$= C_N - 1/12\ s^4 + O(1/N^{\frac{1}{2}})$$

But $\binom{N}{\frac{N+\mu}{2}} e^{\mu^2/2N}$ is the <u>unnormalized</u> probability distribution for an N spin-1/2 Ising magnet with energy $H = -\ 1/2N(\Sigma s_i)^2$ which is ferromagnetic. A similar argument works for any $Q(q) = aq^4 + bq^2$ with $a > 0$.

We are thus able to conclude the following basic theorem from SG:

<u>Theorem 2</u> (Lee-Yang theorem for $(\phi^4)_2$) Let $\langle\cdot\rangle$ denote a spatially cutoff expectation value with free, Dirichlet or half-Dirichlet boundary conditions (see Guerra etal. (1973)) and $P(x) = ax^4+bx^2$; $a > 0$. Let $h \geq 0$ be in $L^\infty \cap L^1(\mathbb{R}^2)$. Then

$$F(z) = \langle\exp(z\phi(h))\rangle$$

is an entire analytic function whose zeros lie on the axis $\mathrm{Re}\,z = 0$.

For $\deg Q \geq 6$, we have the following negative situation (SG): There are definitely sixth degree Q's which are not the limit of spin-1/2 pair-interacting(*) ferromagnets and for which the Lee-Yang theorem fails. Thus, in the lattice approximation, the Lee-Yang theorem fails for certain Q's . This suggests, but certainly does not prove, that the Lee-Yang theorem is false for some $P(\phi)_2$ theories with $\deg P = 6$.

(*) Of course if four-body interacting is allowed, there is no problem in approximating sixth degree Q's as a $1/12(\Sigma s_i)^4/N^3$ term in the energy plus re-scaling leads to $\exp(-1/30\ s^6)$.

§3. Clustering of the Schwinger Functions of $P = ax^4 + bx^2 - \mu x$ ($\mu \neq 0$)

Theorem 2 is a striking looking but, at first sight, apparently not very powerful theorem. That it is intimately connected with analyticity of the pressure is a discovery of Yang-Lee (1952) translated to $(\phi^4)_2$ by SG. That it implies strong bounds and falloff is a discovery of Lebowitz-Penrose (1968), developed in $(\phi^4)_2$ by Simon (1973b). We wish to indicate these ideas in this section. We discuss the case of Dirichlet boundary conditions although similar results hold for half-Dirichlet B.C.

Fix a,b and for μ real and $\Lambda \subset \mathbb{R}^2$, bounded, let

$$\alpha_\Lambda(\mu) = \frac{1}{|\Lambda|} \ln \int [\exp(-\int_\Lambda :a\phi^4 + b\phi^2 - \mu\phi:_D)] d\mu^D_{0,\Lambda}$$

Then, by a result of Guerra et al. (1973), for any real μ $\alpha_\Lambda(\mu) \to \alpha_\infty(\mu)$ as $\Lambda \to \infty$ (Fisher). The main point of the Lee-Yang theorem (Theorem 2) is that $\alpha_\Lambda(\mu)$ has an analytic continuation to the right half-plane, $\mathrm{Re}\,\mu > 0$. Moreover, it is clear that

$$\mathrm{Re}\,\alpha_\Lambda(\mu) \leq \alpha_\Lambda(\mathrm{Re}\,\mu) .$$

Let $f_\Lambda(\mu) = \exp(\alpha_\Lambda(\mu))$. We thus see that $|f_\Lambda(\mu)|$ is uniformly bounded on compacts of $\{\mu | \mathrm{Re}\,\mu > 0\}$ and converging for $\mu \in \mathbb{R}$ and thus, by the Vitali convergence theorem, convergent on $\{\mu | \mathrm{Re}\,\mu > 0\}$. Since the $f_\Lambda(\mu)$ are non-vanishing there and f_∞ is not identically zero, it is never zero, so $\alpha_\Lambda(\mu) \to \alpha_\infty(\mu)$ for all μ with $\mathrm{Re}\,\mu > 0$. We summarize by:

<u>Theorem 3</u> $\alpha_\infty(\mu)$ has an analytic continuation to the entire right half-plane and $\alpha_\Lambda(\mu) \to \alpha_\infty(\mu)$ uniformly on compacts of the right half-plane.

In particular, $\sup_\Lambda d^2\alpha_\Lambda/d\mu^2 < \infty$, for any fixed $\mu > 0$. In terms of the Dirichlet state expectation value $\langle\ \rangle_\Lambda$ for the $ax^4 + bx^2 - \mu x$ theory:

$$\frac{d^2\alpha_\Lambda}{d\mu^2} = \frac{1}{|\Lambda|} [\langle\phi(\chi_\Lambda)\phi(\chi_\Lambda)\rangle_\Lambda - \langle\phi(\chi_\Lambda)\rangle_\Lambda\langle\phi(\chi_\Lambda)\rangle_\Lambda]$$

so by the Lee-Yang result $\sup_\Lambda d^2\alpha_\Lambda/d\mu^2 < \infty$, there is a D with

$$[\langle\phi(\chi_\Lambda)\phi(\chi_\Lambda)\rangle_\Lambda - \langle\phi(\chi_\Lambda)\rangle_\Lambda\langle\phi(\chi_\Lambda)\rangle_\Lambda] \leq D|\Lambda| \tag{6}$$

Now the two-point truncated Schwinger function

$$S_2^T(x-y) = \langle\phi(x)\phi(y)\rangle_\infty - \langle\phi(x)\rangle_\infty\langle\phi(y)\rangle_\infty$$

is positive and monotone decreasing as $|x-y| \to \infty$. If the limit is some $C > 0$, then

$$[\langle\phi(\chi_\Lambda)\phi(\chi_\Lambda)\rangle_\infty - \langle\phi(\chi_\Lambda)_\infty \langle\phi(\chi_\Lambda)\rangle_\infty] \geq C|\Lambda|^2 . \tag{7}$$

(6) and (7) are not directly contradictory although they clearly almost are and a further argument [Simon (1973b)] show they are: we conclude that $S_2^T \to 0$ as

$|x-y| \to \infty$. But then, by a result of Simon (1973a) (described in Rosen's lectures), all the truncated Schwinger functions go to 0 . We summarize:

Theorem 4 In the infinite volume $P(\phi)_2$ Dirichlet state with $P(x) = ax^4 + bx^2 - \mu x$ $(\mu \neq 0)$, the Schwinger functions obey clustering, i.e.
$\int \phi(f_1 \otimes \delta_t)\ldots\phi(f_k \otimes \delta_t)\phi(f_{k+1} \otimes \delta_0)\ldots\phi(f_n \otimes \delta_0)d\nu \to [\int d\nu\ \phi(f_1 \otimes \delta_0)\ldots\phi(f_k \otimes \delta_0)]$
$[\int d\nu\ \phi(f_{k+1} \otimes \delta_0)\ldots\phi(f_n \otimes \delta_0)]$ as $t \to \infty$.

§4 The Wightman Axioms

The status of the basic Euclidean objects for Dirichlet and half-Dirichlet states when $P(x) = ax^4 + bx^2 - \mu x$ $(\mu \neq 0)$ are given by the following table:

TABLE

	Dirichlet	Half-Dirichlet
$(\Lambda \to \infty)$ Schwinger Functions	Yes$^{(2)}$	Yes$^{(1)}$
$(\Lambda \to \infty)$ Pressure	Yes$^{(2)}$	Yes$^{(3)}$
$(\ell < \infty)$ Transfer Matrix	?	Yes$^{(3)}$
$(\Lambda = \infty)$ $S_2^T \to 0$	Yes$^{(4)}$	Yes$^{(4)}$
$(\Lambda = \infty)$ OS Axioms	Yes	Yes

where (1) = Nelson (these lectures), (2) = Guerra et al. (1973) (3) = Guerra et al. (1974), (4) = Simon (1973a). If the Osterwalder-Schrader (1973) reconstruction theorem is valid (there is presently a gap in the proof), all the Wightman axioms hold for the infinite volume D and HD theories. And in any event, on account of the existence of a transfer matrix, the Wightman axioms do hold for the HD theory [see Simon (1974)].

§5. α_∞ and Dymanical Instability

The field theoretic analog of a phase transition is the notion of dynmaical instability (Wightman (1969)), i.e. the existence of more than one infinite volume theory associated to a fixed interaction by some mechanism for associating infinite volume theories to interactions, e.g. the DLR equations described in Guerra's lectures. The expected picture for $\phi^4 + b\phi^2 - \mu\phi$ theories has been described in Jaffe's lecture. In this section we want to supplement the picture given by Jaffe explaining the connection between dynamical instability and the Fock space energy per unit volume, α_∞ , of Guerra (1972). As Guerra explained in his lectures, α_∞ is just the pressure. There is an old idea in field theory associated with the name of Bogoliubov that dynamical instability is present precisely when $\langle\phi(0)\rangle_{\phi^4+b\phi^2-\mu\phi}$ is discontinuous in μ. In statistical mechanical language, Bogoliubov is saying that the phase transition is first order and has the field as long range order parameter. This picture is supported by the following which combines results from SG and Simon (1973b):

Theorem 5 Fix b and let $\alpha_\infty(\mu)$ denote the pressure for the $\phi^4 + b\phi^2 - \mu\phi$ and let $\langle\ \rangle_\mu$ denote the infinite volume (Dirichlet) state for this theory. Consider the statements:

(A) There is a mass gap in the $\langle\ \rangle_{\mu=0}$ theory.

(B) $\alpha_\infty(\mu)$ is differentiable at $\mu = 0$.

(C) The "magnetization" $\langle\phi(0)\rangle_\mu$ is continuous at $\mu = 0$.

(D) There is a unique vacuum in the $\langle\ \rangle_{\mu=0}$ theory.

Then (A) $\Rightarrow$ (B) $\Leftrightarrow$ (C) $\Rightarrow$ (D).

Remarks

1. We emphasize that (A) is a statement about the $\langle\ \rangle_{\mu=0}$ theory and not its decomposition into unique vacuums.

2. Suppose the picture described in Jaffe's lectures holds. Then there is a critical value b_c. When $b > b_c$ we expect (A) to hold so (B),(C) hold. When $b < b_c$, we expect (D) to fail for the following reason: The Wightman theories for $\mu = 0$ with unique vacuum (there should be two such theories!) have $\langle\phi(0)\rangle \neq 0$. But by $\phi \to -\phi$ symmetry in the Dirichlet B.C. theories the value of $\langle\phi(0)\rangle_\mu = 0$. Thus the $\langle\cdot\rangle_{\mu=0}$ theory should not have a unique vacuum. Since (D) fails so do (A),(B). Thus away from the critical point: differentiality of the pressure should be a sensitive test of dynamical instability. At the critical point one expects (B)-(D) to hold on the basis of most stat. mech. models although we emphasize that there are stat. mech. models where (B),(C) fail at the critical point.

3. By general arguments $\alpha_\infty(\mu)$ is convex in μ and so continuous. By Lee-Yang it is analytic away from $\mu = 0$.

Sketch of proof (A)$\Rightarrow$(B). We need only a bound on

$$\frac{d^2\alpha_\Lambda}{d\mu^2} = \frac{1}{|\Lambda|}\langle\phi(\chi_\Lambda)\phi(\chi_\Lambda)\rangle_{T,\mu,\Lambda}$$

uniform in Λ and $|\mu| \leq 1$. By using the GHS and GI,II inequalities one obtains a uniform bound on the falloff of $\langle\phi(x)\phi(y)\rangle_{T,\mu,\Lambda}$ as $|x-y| \to \infty$ and this yields a bound on $d^2\alpha_\Lambda/d\mu^2$. See SG.

(B)$\Leftrightarrow$(C). $$\alpha_\infty(\mu) - \alpha_\infty(0) = \lim_{\Lambda\to\infty}\int_0^\mu \frac{1}{|\Lambda|}\langle\phi(\chi_\Lambda)\rangle_\mu\, d\mu$$

By using the GI,II inequalities one shows that

$$\alpha_\infty(\mu) - \alpha(0) = \int_0^\mu m(\mu)\, d\mu$$

with $m(\mu) = \langle\phi(0)\rangle_\mu$ and that $m(0+) = \lim_{\mu\downarrow 0} m(\mu)$ exists. One then proves that $m(0+)$ is the right derivative of $\alpha_\infty(\mu)$ and $m(0-)$, so differentiability of $\alpha_\infty(\mu)$ at $\mu = 0$ is equivalent to continuity of m . See SG for details.

(C)=>(D). Suppose (D) fails. Then $S^T_{2,\mu=0}(x-y) \to c^2 > 0$ as $x-y \to \infty$. But by symmetry $\phi \Longleftrightarrow -\phi$, $S^T_{2,\mu=0} = S_{2,\mu=0}$. By GII, $S_{2,\mu>0} \geq S_{2,\mu=0}$ so $S_{2,\mu>0} \to d^2 \geq c^2$ as $x-y \to 0$. But by Theorem 4, $S^T_{2,\mu>0} \to 0$ so $\langle\phi(0)\rangle_{\mu>0} \geq c$. By symmetry $\langle\phi(0)\rangle_{\mu>0} \leq c$ so $\langle\phi(0)\rangle_\mu$ is not continuous at $\mu = 0$. For details see Simon (1973b).

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