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Approximation of Feynman Integrals and Markov Fields by Spin Systems

Barry Simon*

In this article I will discuss a similarity in the mathematical structures of two physically quite different classes of systems: the Markov processes associated with quantum mechanical anharmonic oscillators and field theories and the family of lattice models for ferromagnets. In fact, we will see that systems from the first class are limits of systems from the second class. This approximation is on two levels, the first due to Guerra, Rosen, and Simon **[1973]** and the second to Simon and Griffiths **[1973]**. These approximations, their extension and the development of the Ising model methods in constructive quantum field theory made available by them have been a major theme in constructive field theory during the past two and a half years. For a summary of applications up until January, 1974, I would refer you to my Zurich lectures(Simon **[1974]**). More recent work includes that of Glimm and Jaffe **[1974a, 1974b]**, Guerra, Rosen and Simon **[1974]**, Newman **[1974]**, and Spencer **[1974]**.

Here I would like to describe the basic ideas of the approximation and illustrate their application by discussing the proof of the following result which is essentially due to Glimm, Jaffe, Spencer **[1973]**.

THEOREM 1. *Let E_1, E_2, E_3 be the three smallest eigenvalues of the differential operator $H = -\frac{1}{2}(d^2/dx^2) + ax^2 + bx^4$ $(b > 0)$. Then*

(1) $$E_3 - E_2 \geqq E_2 - E_1.$$

Perhaps the most interesting feature of Theorem 1 is that its proof is intimately connected with the fact that the magnetization in a ferromagnet induced by an external magnetic field is a concave function of the inducing field!

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The basic systems. Let us begin by describing the basic systems which we will relate. The free Euclidean quantum field is the Gaussian random process $\phi(\cdot)$ indexed by $\mathscr{S}(\boldsymbol{R}^n)$ with mean zero and covariance

$$\int \phi(f)\,\phi(g)\,d\mu_0 = \langle f, (-\Delta + m^2)^{-1} g\rangle \tag{2}$$

where $\langle\ ,\ \rangle$ is an L^2 inner product. Due to work of Symanzik, Nelson and Osterwalder-Schrader, constructive quantum field theory is now concerned with constructing measures $d\nu = \lim_{\Lambda\to\infty} d\nu_\Lambda$ ($\Lambda \subset \boldsymbol{R}^n$; compact) with

$$d\nu_\Lambda = \exp\left(-\int_\Lambda P(\phi(x))_{\text{ren}}\, d^n x\right) d\mu_0 \Big/ \text{Norm}. \tag{3}$$

where "Norm' represents a normalization factor chosen so that $\int d\nu_\Lambda = 1$. P is a polynomial bounded from below and "ren" indicates that when $n \geqq 2$ certain infinite subtractions are needed. As n increases, the local singularities of $\phi(x)$ become worse and, as a result, the renormalizations more complex. In any event, $d\nu_\Lambda$ has been defined in case $n = 2$ essentially due to work of Nelson and in case $n = 3$, deg $P = 4$, by recent work of Glimm and Jaffe and Feldman. In the physical case $n = 4$ (n is the number of space-time dimensions), there are still rather severe technical problems to be overcome in the definition of (3). (We expect that more details of this subject will be found in Glimm's and Nelson's contributions to these PROCEEDINGS.) When $n = 1$, there is a random process indexed by R, $q(t)$, with $\phi(f) = \int f(t)\,q(t)\,dt$ and this process is connected to the differential operator of Theorem 1 by the Feynman-Kac formula (for a proof, see, e.g., Simon **[1974]**):

THEOREM 2. *Let $n = 1$, $m = 1$ in* (2). *Let $P(x) = bx^4 + (a - \frac{1}{2})x^2$ and let $d\nu$ be the limit of the measures in* (3). *Let Ω_1 be the eigenvector of H normalized by $H\Omega_1 = E_1\,\Omega_1$, $\int \Omega_1(x)^2\,dx = 1$, $H \geqq E_1$. Then*

$$\int f(q(t))\,g(q(0))\,d\nu = \langle f(q)\Omega_1, \exp(-t(H - E_1))\,g(q)\,\Omega_1\rangle. \tag{4}$$

A classical Ising model is a probability measure on $\{-1, 1\}^N$ of the form

$$d\alpha = \exp\left(\sum_{i\neq j} a_{ij}\sigma_i\sigma_j + \sum_i \mu_i\sigma_i\right) \Big/ \text{Norm} \tag{5}$$

where $\sigma_i = \pm 1$ are coordinates for $\{-1, 1\}^N$ thought of as the values of N spins which can point up ($\sigma_i = +1$) or down ($\sigma_i = -1$). The μ_i represent external magnetic fields and the measure α is called ferromagnetic if $a_{ij} \geqq 0$, all $i \neq j$, in which case there is a tendency for the spins to align in parallel. The study of Ising models has been much influenced by certain inequalities involving expectations of σ's and their products. The first of these "correlation inequalities" was proved by Griffiths **[1967]** and the subject has been developed extensively by Ginibre, Griffiths, Hurst, Kelley, Lebowitz, Percus, and Sherman. In particular, Theorem 1 depends on the following inequality of Lebowitz **[1974]** (the proof may be found in the original paper or in Simon **[1974]**):

THEOREM 3. *If $\langle\cdot\rangle$ denotes expectation with respect to a measure $d\alpha$ of the form* (5), *then whenever $a_{ij} \geqq 0$, $\mu_i = 0$,*

$$\langle\sigma_i\sigma_j\sigma_k\sigma_l\rangle \leqq \langle\sigma_i\sigma_j\rangle\langle\sigma_k\sigma_l\rangle + \langle\sigma_i\sigma_k\rangle\langle\sigma_j\sigma_l\rangle + \langle\sigma_i\sigma_l\rangle\langle\sigma_j\sigma_k\rangle. \tag{6}$$

To understand the physical significance of (6), we note that

$$\text{LHS} - \text{RHS} = \partial^3\langle\sigma_i\rangle/\partial\mu_j\partial\mu_k\partial\mu_l. \tag{7}$$

Now $\langle\sigma_i\rangle$ represents the magnetization of spin i in the external field $\{\mu_j\}$. The expected concavity of $\langle\sigma_i\rangle$ as a function of external fields is expressed by the Griffiths-Hurst-Sherman inequalities: If $\mu_i > 0$,

$$\partial^2\langle\sigma_i\rangle/\partial\mu_j\partial\mu_k \leqq 0. \tag{8}$$

Since the derivative in (8) is zero if all $\mu_i = 0$ (by $\mu_i \leftrightarrow -\mu_i$, $\sigma_i \leftrightarrow -\sigma_i$ symmetry), (8) imples that the derivative (7) is negative if all $\mu_i = 0$; this is the inequality (6).

We will also need to discuss generalized Ising models where $d\alpha$ on $\{-1, 1\}^N$ is replaced by a measure $d\beta$ on R^N with

$$d\beta = \exp\left(-\sum b_{ij}\sigma_i\sigma_j + \sum \mu_i\sigma_i\right)\prod_{i=1}^N d\Upsilon_i(\sigma_i) \Big/ \text{Norm} \tag{9}$$

where $d\Upsilon_i$ is a finite measure on R and b_{ij} is a strictly positive definite matrix. The ferromagnetic case is the one with $b_{ij} \leqq 0$ for $i \neq j$. Certain correlation inequalities hold for expectations with respect to β but inequalities as detailed as (6) do not hold without additional restrictions on the $d\Upsilon$.

The approximation theorems. The basic approximation theorem of Guerra, Rosen Simon **[1973]** is:

THEOREM 4. *The measures $d\mu_\Lambda$ of form* (3) *are the limits of generalized ferromagnetic Ising models of form* (9) *in case $n = 1, 2$. The measures $d\Upsilon$ in* (9) *are of the form $d\Upsilon(\sigma) = \exp(-Q(\sigma))d\sigma$ where $Q = (\text{const})P$ + quadratic term.*

Guerra, Rosen, and Simon present a general scheme (the "lattice approximation") which formally approximates $d\mu_\Lambda$ and prove its convergence in case $n \leqq 2$. Recently, Park **[1974]** has proven convergence of this scheme in case $n = 3$, deg $P = 4$. The meaning of Theorem 4 is the following: For each $f \in \mathscr{S}(\boldsymbol{R}^n)$, $\phi(f)$ is a limit of suitable linear combinations of the σ_i in the approximating theory and, if $f \geqq 0$, the coefficients in this linear approximation are positive. Thus multilinear inequalities on expectations (such as (6)) carry over to measures $d\mu_\Lambda$. The basic idea of the proof of Theorem 4 is to replace $\boldsymbol{R}^n$ by a lattice $\boldsymbol{Z}^n\delta$, $\phi(f) \cong \sum_{m\in \boldsymbol{Z}^n} f(m\delta)\sigma_{m,\delta}$ and the Laplacian in (2) by a finite difference approximation. Since the *inverse* covariance matrix appears in the Gaussian for a joint probability distribution, $d\mu_0$ is approximated by general Ising models with $d\Upsilon$ Gaussian and with $b_{ij} \leqq 0$ ($i \neq j$) because $-\Delta$ has finite difference approximations which are negative off-diagonal. $d\nu_\Lambda$ is then approximated by

$$\prod_{m\delta\in\Lambda} \exp(-\delta^n P(\sigma)_{\text{ren}})\,d\mu_0$$

so only the $d\Upsilon$'s are affected by the change from $d\mu_0$ to $d\nu_\Lambda$.

The basic approximation theorem of Griffiths and Simon **[1973]** is

THEOREM 5. *A generalized Ising model of the form* (9) *with $d\Upsilon_i(\sigma) = \exp(-b_i\sigma_i^4 - a\sigma_i^2)d\sigma$ ($b_i > 0$) is a limit of classical Ising models of the form* (5).

To explain the idea of their proof, take $N = mk$ in (5), replace σ_i by $s_{\alpha;r}$, $\alpha = 1, \cdots, m$, $r = 1, \cdots, k$ and let a_{ij} be of the form:

$$a_{\alpha r;\beta t} = -b_{rt}/m \qquad (b_{rr} = 0).$$

Then

$$\sum_{i\neq j} a_{ij}s_is_j = -\sum_{r,t} b_{rt}\left(\sum_\alpha m^{-1/2} s_{\alpha;r}\right)\left(\sum_\beta m^{-1/2} s_{\beta,t}\right) + \text{constant}$$

so by the central limit theorem $\sigma = m^{-1/2}\sum s_{\alpha;r}$ will approach a generalized Ising model with $d\Upsilon$ Gaussian. By adding a term $\text{const}(1/m)\delta_{rt}$ to a we can cancel the Gaussian limit and by rescaling (i.e., taking $\sigma_r = m^{-3/4}\sum s_{\alpha;r}$) get the quartic limit.

The Griffiths-Simon theorem has recently been extended to the approximation of multicomponent fields by multicomponent Ising models (plane rotor and classical Heisenberg models) by Dunlop and Newman [1974].

Proof of Theorem 1. By application of Theorems 3, 4, 5, the path space expectation $d\nu_\Lambda$ with $n = 1$, $P(x) = bx^4 + (a - \frac{1}{2})x^2$ obeys:

$$\langle q(t_1)\, q(t_2)\, q(t_3)\, q(t_4)\rangle \leqq \langle q(t_1)\, q(t_2)\rangle \langle (q(t_3)\, q(t_4)\rangle + 2 \text{ others;}$$

so by taking $\Lambda \to \infty$, letting $t_1 = t_2 = 0$, $t_3 = t_4 = t$ and using Theorem 2:

$$\text{(10)} \quad \langle q^2\Omega_1, \exp(-t(H - E_1))\, q^2\Omega_1\rangle - \langle q^2\Omega_1, \Omega_1\rangle^2 \leqq 2\langle q\Omega_1, e^{-t(H-E_1)}\, q\Omega_1\rangle^2.$$

Letting $\Omega_1, \Omega_2, \cdots$ be the eigenfunction of H, (10) says that

$$\text{(11)} \quad \begin{aligned} &\sum_{m\geqq 2} |\langle q^2\Omega_1, \Omega_m\rangle|^2 \exp(-t(E_m - E_1)) \\ &\leqq 2\left[\sum_{m\geqq 1} |\langle q\Omega_1, \Omega_m\rangle|^2 \exp(-t(E_m - E_1))\right]^2. \end{aligned}$$

Since Ω_m is an even (odd) function of q for m odd (even), $\langle q^2\Omega_1, \Omega_2\rangle = 0$, $\langle q\Omega_1, \Omega_1\rangle = 0$. Moreover, since Ω_m has $m - 1$ nodes, $\langle q\Omega_1, \Omega_2\rangle \neq 0$, $\langle q^2\Omega_1, \Omega_3\rangle \neq 0$. Thus as $t \to \infty$, the leading behavior of the LHS and RHS respectively of (11) is $|\langle q\Omega_1, \Omega_3\rangle|^2 \exp(-t(E_3 - E_1))$ and $|\langle q\Omega_1, \Omega_2\rangle|^4 \exp(-2t(E_2 - E_1))$. It follows that $E_3 - E_1 \geqq 2(E_2 - E_1)$ completing the proof.

References

F. Dunlop and C. Newman (1974), Comm. Math. Phys. (to appear).

J. Glimm and A. Jaffe (1974a), Phys. Rev. **D10** (1974), 536–539.

J. Glimm and A. Jaffe (1974b), Phys. Rev. Lit. **33** (1974), 440–442.

J. Glimm, A. Jaffe and T. Spencer (1973), in *Constructive Quantum Field Theory*, edited by G. Velo and A. S. Wightman, Springer-Verlag, Berlin, 1973, pp. 133–198.

R. Griffiths (1967), J. Mathematical Phys. **8** (1967), 478–489.

F. Guerra, L. Rosen, and B. Simon (1973), Ann. of Math. **101** (1975), 111–259.

F. Guerra, L. Resen and and B. Simon (1974), Comm. Math. Phys. **41** (1975), 19–32.

J. Lebowitz (1974), Comm. Math. Phys. **35** (1974), 87–92.

C. Newman (1974), paper on Hadamard's Factorization Theorem and the Lee-Yang theorem (in preparation).

B. Simon (1974), *The $P(\phi)_2$ Euclidean (quantum) field theory*, Princeton Univ. Press, Princeton, N. J., 1974.

B. Simon and R. Griffiths (1973), Comm. Math. Phys. **11** (1973), 145–164.

Princeton University
Princeton, New Jersey 08540, U.S.A.