BOSE QUANTUM FIELD THEORY AS AN ISING FERROMAGNET: RECENT DEVELOPMENTS

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One of the main new tools made available in constructive field theory by the Euclidean revolution of 1971-73 is the approximation of scalar field theory by Ising models and the resulting correlation inequalities. There has been quite active development of this line in the past year and our goal here is to review some of these developments. While we will have something to say about the basic ideas of the Ising approximations, we will suppose that the reader has some previous exposure to them, either from the original papers [Guerra, Rosen, Simon (1973) and Simon, Griffiths (1973)] or from the pedagogic reviews [Simon (1974 b,c) or the contributions of Guerra, Rosen, and Simon to Velo-Wightman (1973)].

§1. The Basic Framework

The basis of the Ising methodology is two approximations: the lattice approximation [Guerra, Rosen, Simon (1973)] and the classical Ising approximation [Simon-Griffiths (1973)]. Rather than state precise technical results, we paraphrase the original results in two metatheorems below. By a "generalized Ising model" we mean a finite collection of random variables ("spins") whose joint probability distribution has the form

$$\exp(+\sum_{i<j} a_{ij}x_i x_j)d\nu_1(x_1)\ldots d\nu_n(x_1)$$

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for measures ν_i on R. By a "classical Ising model", we mean a model where each "spin", x_i can only take the values ±1. If each $a_{ij} \geq 0$, the model is called "ferromagnetic".

Metatheorem 1 (the "lattice approximation") Any $P(\phi)_2$ Euclidean field theory is a limit of generalized ferromagnetic Ising models and if P is even, each ν_i is even.

Metatheorem 2 (the "classical Ising approximation") Any $P(\phi)_2$ Euclidean field theory with $P(X) = aX^4 + bX^2 - \mu X$ is a limit of ferromagnetic classical Ising models with $d\nu_i(x_i) = D_i \exp(+c_i \mu x_i)[\delta(x_i+1) + \delta(x_i-1)]$ for suitable positive constants D_i, c_i.

Remarks 1. As noted in the original papers, these theorems extend formally to three and four space-time dimensions; (for three dimensions, see below).

2. The classical Ising approximation proceeds in two steps. One first passes to the lattice approximation and then approximates the spins of the lattice approximation by classical Ising spins. The latter part is O.K. in three or four dimensions so the removal of "formal" in Remark 1 is only a question of convergence of the lattice approximation.

3. We have not been too explicit about what we mean by "limit". What is critical is that the Euclidean fields are limits of positive linear combinations of spins in a way that allows one to immediately extend any multilinear inequality ("correlation inequality") from the spin systems to the field theories.

There have been several extensions of the general formalism in the past year:

Theorem 1.1 [Park (1974)] The lattice approximation converges for the finite volume, free boundary condition $(\phi^4)_3$ field theory.

Theorem 1.2 [Dunlop-Newman (1975)] The $(\phi_1^2 + \ldots + \phi_n^2)^2$ field theory in two dimensions is a limit of ferromagnetic magnetic "spin" systems where the spins take values on the n-1 dimensional sphere (i.e. plane rotor if n=2 ; classical Heisenberg if n=3).

Theorem 1.3 [Guerra-Rosen-Simon (1975)] The $P(\phi)_2$ lattice approximation in a box with periodic or Neumann boundary conditions is convergent.

Remarks 1. Thus far, Park's result has found no applications. This situation should change with the control of the infinite volume $(\phi^4)_3$ theory (see Osterwalder's contribution to these proceedings). It would be useful to have Park's result extended to Dirichlet boundary conditions.

2. Dunlop-Newman have also proven Lee-Yang theorems for $n=2,3$ rotors [the $n \geq 0$ case is under active investigation (H. Mahler (private communication))] and thus one has analyticity of the pressure in certain multicomponent field theories.

3. The box lattice approximation with periodic B.C. of course has spins coupled at opposite boundary points. Neumann B.C. involves dropping a term $\frac{1}{2}(q-q')^2$ from the action for any pair of nearest neighbors with q in the region and q' out (for comparison, with Dirichlet B.C., one only drops the $-qq'$ term).

4. Baker (1974a) has also discussed the Neumann lattice theory (which he unfortunately calls "free boundary conditions").

5. Correlation inequalities between Dirichlet and Periodic states have been exploited by Guerra, Rosen, and Simon (1974) [see §5, below].

§2. Some New Correlation Inequalities

Many of the recent applications of correlation inequalities in field theory have made use of inequalities proven in the past year and a half and, in turn, many of these inequalities have been proven by authors interested in field theory applications. The three new classes of inequalities are:

Theorem 2.1 [Lebowitz (1974)] In a classical ferromagnetic Ising model with positive external field (i.e. $d\nu_i = D_i e^{+\mu_i x_i}[\delta(x_i+1) + \delta(x_i-1)]$ for any finite sets A and B:

$$\sum_{C\subset A;\, D\subset B} (-1)^{|C|+|D|}\; T(C,L;A\setminus C,B\setminus D) \geq 0$$

$$\sum_{C\subset A;\, D\subset B} (-1)^{|C|}\; T(C,D;A\setminus C,B\setminus D) \leq 0$$

where

$$T(C,D;E,F) = \langle S^{C\Delta D}\rangle \langle S^{E\Delta F}\rangle - \langle S^C\rangle \langle S^D\rangle \langle S^E\rangle \langle S^F\rangle$$

with $S^C = \prod_{i \in C} S_i$.

Theorem 2.2 [Newman (1975a)] Let Λ be any set. Let $\mathcal{P}$ be a family of partitions of Λ into two disjoint subsets Λ_1, Λ_2 so that any partition of Λ into pairs (or if $\#(\Lambda)$ is odd into pairs plus one one-element set) is a refinement of some $\{\Lambda_1, \Lambda_2\} \in \mathcal{P}$. Then, in any classical ferromagnetic Ising model with positive external field:

$$\langle \sigma^{\Lambda} \rangle \leq \sum_{\{\Lambda_1, \Lambda_2\} \in \mathcal{P}} \langle \sigma^{\Lambda_1} \rangle \langle \sigma^{\Lambda_2} \rangle$$

Theorem 2.3 [Cartier (1974), Percus (1974), Sylvester (1974)] Define u_n by:

$$u_n(i_1, \ldots, i_n) = \frac{\partial^n}{\partial \mu_1 \ldots \partial \mu_n} \langle \exp(\mu_1 \sigma_{i_1} + \ldots + \mu_n \sigma_{i_n}) \rangle \Big|_{\mu_i = 0}$$

Then, in any zero field ferromagnetic classical Ising model:

$$u_6(i_1, \ldots, i_6) \geq 0$$

Remarks 1. Lebowitz inequalities are usually written in terms of "Percus variables" and look simpler in that form.

2. The rather complex condition on $\mathcal{P}$ in Newman's inequality is not only sufficient for the inequalities in question to hold but also necessary in that should $\mathcal{P}$ not obey that condition, there are Ising ferromagnets where the inequality fails.

3. By the "dummy spin trick", one need only prove Newman's inequality in zero field with $\#(\Lambda)$ even. Newman does this by a graphical expansion.

4. If $\#(\Lambda)$ is even, Newman's inequalities imply by induction [Newman (1975a)]:

$$\langle \sigma^{\Lambda} \rangle \leq \sum_{\text{pairings}} \langle \sigma^{i_1 j_1} \rangle \ldots \langle \sigma^{i_n j_n} \rangle$$

where the sum runs over all ways of writing Λ as n disjoint pairs (and a similar inequality by the dummy spin method if $\#(\Lambda)$ is odd). Earlier Glimm-Jaffe (1974b) have proven an inequality of this form from Lebowitz inequalities but with a constant of order $n!$ in front of the sum.

5. The GKS inequalities assert that in positive field $u_1 \geq 0$,

$u_2 \geq 0$ and GHS imply that $u_3 \leq 0$; Lebowitz' inequalities imply $u_4 \leq 0$ at zero field. It is conjectured that $u_{2n}(-1)^{n+1} \geq 0$ at zero field.

6. As of yet, no application of $u_6 \geq 0$ is known. Feldman (1974) has remarked that the general inequality $u_{2n}(-1)^n \geq 0$ at zero field would yield a proof of a mass gap result that has now been proven by Spencer using alternate means (see §5).

7. These inequalities all extend to ϕ^4 Euclidean field theories (and the first two to $\phi^4 - \mu\phi$ theories) if σ_i is replaced by $\phi(x_i)$.

§3. Construction of States

Most of the recent applications of the Ising approximation in field theory have extended trends set by the very earliest applications (Guerra, Rosen, Simon (1973), Simon, Griffiths (1973), Nelson's contribution to Velo-Wightman (1973), and Simon (1973, 1974a)) which in turn followed trends set by the applications of correlation inequalities in statistical mechanics.

One of Griffiths' original applications of his inequalities was to prove the existence of the infinite volume limit for the correlation functions of a spin system. Nelson extended this idea to construct an infinite volume $P(\phi)_2$ theory for P = even poly − μX (the half-Dirichlet state). Glimm and Jaffe (1974 b) have recently combined correlation inequalities with the cluster expansion (see the contribution of Glimm, Jaffe, and Spencer to Velo-Wightman (1973)) to construct an infinite theory for these P ("weak coupling boundary conditions") which they then show is Nelson's state. One advantage of this construction is that it provides a proof of $:\phi^j:$ $(j \leq \deg P)$ bounds for Nelson's state. These bounds have been used by Glimm-Jaffe (1973) and by Fröhlich (1974b).

§4. Domination by the Two-Point Function

One of the earliest results obtained using Ising methods in field theory [Simon (1973)] asserts that in any $P(\phi)_2$ theory the mass gap is determined by the falloff of the truncated two-point Schwinger function. Newman (1975a) has found a new proof of this for even ϕ^4 theories.

For as a special case of his inequalities (Theorem 2.2), he considers $\Lambda = \{1,\dots,n;\ 1',\dots,m'\}$ with $n+m$ even, his inequalities include:

$$(1\dots n\ 1'\dots m') \leq (1\dots n)\ (1'\dots m') + \sum_{i,j} (ij')\ (1\dots\hat{i}\dots n\ 1'\dots\hat{j}\dots m') \qquad (*)$$

which implies the stated falloff.

Glimm and Jaffe (1974b) have shown that the two-point function dominates more than just rate of falloff. Using Lebowitz' inequality inductively they prove bounds on $S_n \equiv \langle\phi(x_1)\dots\phi(x_n)\rangle$ which include:

Theorem 4.1 [Glimm-Jaffe (1974b)] In any $\phi^4 - \mu\phi$ field theory, for any positive $f_1,\dots,f_n$

$$0 \leq S_n(f_1,\dots,f_n) \leq 2^{n-1}(n-1)!\ \|f_1\| \dots \|f_n\|$$

where $\|f\| = S_2(f,f)^{1/2}$.

Remarks 1. Letting $|||f||| = S_2(|f|,|f|)^{1/2}$ one has that

$$|S_n(f_1,\dots,f_n)| < 2^{n-1}(n-1)!\ |||f_1||| \dots |||f_n|||$$

2. The $2^n(n-1)!$ in these bounds can be replaced by $\left[\frac{n+1}{2}\right]!$ (see theorem 4.3 below).

3. The bound in Remark 1 is precisely of the form needed to be able to recover the Minkowski region according to the (revised) Osterwalder-Schrader axioms [Osterwalder-Schrader (1975)].

4. Formally, these bounds hold in four dimensions. To be of use there, S_2 must be a distribution at coincident (i.e. an L^1 function). It is order by order in perturbation theory which is suggestive that it will be also in the actual theory.

Newman (1975 a,b) has extended Theorem 4.1 in two different ways:

Theorem 4.2 (Newman (1975b)) In any $\phi^4 + a\phi^2 - \mu\phi$ $(\mu \geq 0)$, field theory, for any positive test function, f:

$$\langle e^{\phi(f)}\rangle \leq \exp[S_1(f) + \tfrac{1}{2}\, S_2^T(f,f)]$$

where

$$S_2^T(x,y) = S_2(x,y) - S_1(x)S_1(y)$$

Theorem 4.3 (Newman (1975a)) In any even ϕ^4 theory:

$$0 \le S_{2n}(x_1,\ldots,x_{2n}) \le \sum_{pairs} S_2(x_{i_1},x_{j_1})\ldots S_2(x_{i_n},x_{j_n})$$

Remarks 1. The proof of Theorem 4.2 is surprisingly short and simple considering its detailed information. Let $F(\mu) = \ln\langle e^{\mu\phi(f)}\rangle$. Then GHS directly implies that $d^3F/d\mu^3 \le 0$ for all $\mu > 0$! Since $F(0) = 0$, $F'(0) = S_1(f)$ and $F''(0) = S_2^T(f,f)$ we immediately find that

$$F(\mu) \le \mu S_1(f) + \tfrac{1}{2}\mu^2 S_2^T(f,f)$$

2. Theorem 4.3 is just the translation of one of Newman's inequalities to field theory (see Remark 4 in §2). There are also inequalities for $\phi^4 - \mu\phi$ theories.

3. Theorem 4.3. has a rather dramatic sounding restatement: the Schwinger functions of any ϕ^4 theory are dominated by those of the generalized free field with the same two-point function.

4. Theorem 4.3 is quite directly an extension of Theorem 4.1 improving the constant in front of the product of norms. Theorem 4.2 implies bounds on $S_n(f)$ via two remarks. First, by GKS and $|\langle A\rangle| \le \langle |A|\rangle$, $G(\mu) \equiv \langle e^{\mu\phi(f)}\rangle$ obeys $|G(\mu)| \le G(|\mu|)$. Next, Cauchy estimates on the entire function $G(\mu)$ implies bounds on $G^{(m)}(0)$.

5. (This remark is due independently to the author and J. Fröhlich) In one sense, Theorem 4.2 is a very important improvement of Theorem 4.1. For, if S_2^T has very weak falloff properties (and such falloff is to be expected if $\mu > 0$; see §5 below; also note that ϕ bounds for some $\mu > 0$ implies ϕ bounds for $\mu = 0$ by GKS) then Theorem 4.2 implies that

$$\langle \exp(\phi(h \otimes \chi_{(0,T)}))\rangle \le \exp(cT)$$

at least under some weak regularity assumption on S_2^T at the coincidence singularity. Modulo technical details, it is a basic result of Fröhlich (1974a) (see Simon (1974b) for additional discussion) that such bounds imply ϕ-bounds in the sense of Glimm-Jaffe (1972).

§5. Mass Gap and Spectral Results

Cluster expansions and Bethe-Salpeter equations have proved a powerful tool for studying the mass spectrum in weakly coupled (and large fugacity) theories (see Glimm's contribution to these Proceedings). Thus far, correlation inequalities have provided the only tool for studying these questions in the strong coupling regime where they provide much less information than is available in the weak coupling regime. Some new information on these questions has been obtained in the past year:

Theorem 5.1 [Guerra, Rosen, and Simon (1974)] There is a mass gap (i.e. 0 is an isolated simple eigenvalue of the Hamiltonian) in any $(a\phi^4 + b\phi^2 - \mu\phi)_2$ field theory with $\mu \neq 0$.

Theorem 5.2 [Spencer (1974)] In any even $(\phi^4)_2$ theory, the first excited even state has an energy at least twice as large as the first excited state.

Remarks 1. These extend earlier results which used correlation inequalities, namely those of Simon (1974a) and of Glimm, Jaffe, and Spencer (their contribution to Velo-Wightman (1973)) respectively.

2. Theorem 5.1 follows Ising model arguments of Lebowitz and Penrose (1974). The main difficulties are technical ones involving boundary conditions.

3. By even state, we mean one obtained by applying an even number of fields to the vacuum. In more physical terms, Theorem 5.2 asserts that in a $(\phi^4)_2$ theory there are no even "G-parity" bound states (at least, below the two-particle threshold).

4. Spencer's proof of Theorem 5.2 uses Lebowitz inequalities and a very clever trick. Newman (1975a) has a simple proof using his inequality (*) quoted in 4. For, if n and m are even, the sum in (*) has $\exp(-2mt)$ falloff if the prime and unprimed indices are separated by Euclidean time, t.

§6. "Coupling Constants" Variation

Another subject of considerable interest involves monotonicity and smoothness information about physical parameters (mass, vertex functions at special points, etc.) as functions of bare "coupling constants" (bare mass, bare coupling constant, etc.). The earliest applications of correlation inequalities by Guerra, Rosen, and Simon (1973) include a statement of this type, which in one form says that in a $P(\phi)_2$ theory with $P(X) = a_{2m}X^{2m} + \dots + a_2X^2$ (even P), the physical mass is monotone increasing as a_2 increases (with $a_{2m},\dots,a_4$, m_0 fixed). Recently, Glimm and Jaffe, in a series of papers [Glimm, Jaffe (1974a,c,d,e); see Guerra, Rosen, Simon (1974) for an additional critical index result] have proven a large number of such bounds and bounds also on critical exponents. We quote an example:

<u>Theorem 6.1</u> (Glimm-Jaffe (1974a)) For fixed m_0 and λ, let $m(\sigma)$ denote the physical mass of the $(\lambda\phi^4 + \frac{1}{2}\sigma\phi^2)_2$ theory. Then for $\sigma > \sigma_c$ (the critical value where the even theory stops possessing a mass gap) $m(\sigma)$ is Lipschitz continuous and for $\sigma > \sigma' > \sigma_c$:

$$m(\sigma)^2 - m(\sigma')^2 \leq (\sigma - \sigma')$$

<u>Remarks</u> 1. If $m(\sigma)$ were differentiable, the GRS result quoted above says that $dm^2/d\sigma \geq 0$ and the Glimm-Jaffe results say that $dm^2/d\sigma \leq 1$.

2. If $m(\sigma) \to 0$ as $\sigma \to \sigma_c$ [Baker (1974b) has announced a closed related result], then one has that $m(\sigma) \leq (\sigma-\sigma_c)^{1/2}$.

§7. The $:\cos\phi:_2$ Theory

I would like to briefly describe some recent work of Fröhlich (1975) which doesn't fit into either the general topic of my talk or of Glimm's talk but which I feel should be mentioned at this conference. Fröhlich has constructed a $:\cos\phi:_2$ theory (quantized (massive) sine-Gordon equation) or more accurately the $\int_0^\alpha d\nu(\alpha)[:\cos(\alpha\phi+\eta(\alpha)):]$ theory where η is a function, ν a signed measure and one needs $\alpha < \sqrt{4\pi}$ for the finite volume and small coupling theories and $\alpha < 4/\sqrt{\pi}$ for the strongly coupled theories. The introduction of a new beast to the zoo of two-dimensional field theories (and a beast which is non-renormalizable in three or more dimensions at that!) does not, in itself, seem cause for excitement. But this theory has one extremely striking property: <u>for sufficiently small coupling constant, the Feynman series for all the</u>

Schwinger functions are convergent! (This is definitely false for $(\phi^4)_2$ - Jaffe (1965)).

Remarks 1. The intuitive reason the series are convergent once the theory is proven to exist is quite simple. $:\cos\phi:_2$ is equivalent to $:\sin\phi:_2$ by translation of the field. But $\lambda:\sin\phi:_2$ and $-\lambda:\sin\phi:_2$ are equally good theories via $\phi \to -\phi$ covariance.

2. The key requirement is the control of $\langle\exp(-\lambda U_\Lambda)\rangle$ for finite Λ and $\lambda \in R$, for then the cluster expansion machine can be turned to prove analyticity of the Schwinger functions.

3. Fröhlich uses the idea of Albeverio-Hoegh Krohn (1973) that these theories are equivalent to certain statistical mechanical theories. By some clever use of Guerra, Rosen, Simon (1973), he reduces control of $\langle\exp(-\lambda U_\Lambda)\rangle$ to control of a purely Coulomb system in two dimensions (in finite volume and with image charges) and then appeals to methods of Deutsch-Lavand (1974).

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