

# BARRY SIMON AND THE JÁNOS BOLYAI INTERNATIONAL MATHEMATICAL PRIZE

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## 1. Introduction

The recipient of the 2015 János Bolyai International Mathematical Prize was Barry Simon (IBM Professor California Institute of Technology). The prize is given every 5 years by the Hungarian Academy of Sciences on the recommendation of a 10 member international committee (this is the only international prize of the Academy). It was established in 1902 for the 100th birth anniversary of the great Hungarian mathematician János Bolyai, one of the founders of non-Euclidean geometry, and the first two awardees were Henry Poincaré (1905) and David Hilbert (1910). Then came World War I and the prize was not given until 2000, when the Academy renewed it. It is commonly accepted that, since there is no Nobel prize in mathematics, part of the original intention was to have a prestigious substitute that honors high quality mathematical work. In the renewed form the prize is given for monographs of high impact written in the preceding 10–15 years. In 2000 Saharon Shelah was the recipient for his book “*Cardinal Arithmetic*”, in 2005 Mikhail Gromov got it for the monograph “*Metric structures for Riemannian and non-Riemannian spaces*”, and the 2010 awardee was Yuri Manin for this work “*Frobenius manifolds, quantum cohomology, and moduli spaces*”.

Barry Simon received the Bolyai Prize for his monumental two-volume treatise “*Orthogonal Polynomials on the Unit Circle*” published by the American Mathematical Society in the Colloquium Publications series in 2005. Simon does not need much introduction: he is one of the most cited

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mathematicians; the author of 21 monographs that has had profound influence on various fields of physics, mathematical physics and mathematics; among others he is the recipient of the Poincaré Prize (2012), the Leroy P. Steele Prize (2016), honorary doctor of Technion (Israel), the University of Wales–Swansea (Great Britain) and the Ludwig-Maximilians-Universität (Germany). His 4-volume treatise “*Methods of Modern Mathematical Physics*” written with Michael C. Reed is the bible of mathematical physics, and his latest, just published 5-volume “*Comprehensive Course in Analysis*” [7] will likely have the same lasting impact. His 400 research papers are on various areas such as quantum field theory, statistical mechanics, quantum mechanics, magnetic fields, just to name a few. He has been a definitive authority on operator theory, Jacobi matrices and spectral theory for a long time. So how did it happen that he wrote a book on orthogonal polynomials and why that book has turned out to be so influential?

## 2. Orthogonal polynomials and Jacobi matrices

The theory of orthogonal polynomials goes back to at least two centuries to the work of Jacobi. Let  $\mu$  be a positive Borel measure on the complex plane with infinite support for which

$$\int |z|^m d\mu(z) < \infty$$

for all  $m \geq 0$ . There are unique polynomials

$$p_n(z) = p_n(\mu, z) = \kappa_n z^n + \cdots, \quad \kappa_n > 0, \quad n = 0, 1, \dots$$

which form an orthonormal system in  $L^2(\mu)$ , i.e.,

$$\int p_m \overline{p_n} d\mu = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases}$$

These  $p_n$ 's are called the orthonormal polynomials corresponding to  $\mu$ .  $\kappa_n$  is the leading coefficient, and  $p_n(z)/\kappa_n = z^n + \cdots$  is called the monic orthogonal polynomial. If  $\mu$  is on the real line then we get real polynomials, while if  $\mu$  is supported on the unit circle, then we get the polynomials with which Simon's book is mainly concerned. In the real case the  $p_n$ 's obey a three-term recurrence formula

$$(1) \quad xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x),$$

where

$$a_n = \frac{\kappa_n}{\kappa_{n+1}} > 0, \quad b_n = \int xp_n^2(x) d\mu(x),$$

and, conversely, any system of polynomials satisfying (1) with real  $a_n > 0$ ,  $b_n$  is an orthonormal system with respect to a (not necessarily unique) measure on the real line (Favard' theorem).

With bounded  $a_n > 0$ ,  $b_n \in \mathbf{R}$  the so-called Jacobi matrix

$$J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \cdots \\ 0 & 0 & a_2 & b_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

defines a self-adjoint bounded linear operator  $J$  on  $l_2$ , a Jacobi operator. Every bounded self adjoint operator with a cyclic vector is a Jacobi operator in an appropriate base (just orthogonalize the orbit of a cyclic vector). Furthermore, any operator when restricted to the closure of the orbit of a non-zero vector is cyclic on that subspace.

To find the eigenvalues of  $J$  one considers the equation  $J\pi = \lambda\pi$ ,  $\pi = (\pi_0(\lambda), \pi_1(\lambda), \dots)$ , which is equivalent to the three-term recurrence

$$(2) \quad a_{n-1}\pi_{n-1} + b_n\pi_n + a_n\pi_{n+1} = \lambda\pi_n, \quad n = 1, 2, \dots$$

$$b_0\pi_0 + a_0\pi_1 = \lambda\pi_0, \quad \pi_0 = 1.$$

Thus,  $\pi_n(\lambda)$  is of degree  $n$  in  $\lambda$ , and  $\lambda$  is an eigenvalue when  $\{\pi_n(\lambda)\} \in l_2$ .

By the spectral theorem  $J$ , as a self-adjoint operator having a cyclic vector  $((1, 0, 0, \dots))$ , is unitarily equivalent to multiplication by  $x$  on some  $L^2_\mu$  space, where  $\mu$  is a positive measure with compact support on the real line. This  $\mu$  is called the spectral measure of  $J$ . It is clear that the support  $S(\mu)$  of  $\mu$  is the set of those  $x$  for which  $xI - J$  is not invertible, so  $S(\mu)$  is the spectrum of  $J$ . Now if  $p_n(\mu) = p_n(\mu, x)$  are the orthonormal polynomials with respect to  $\mu$ , then  $\{p_n(\mu)\}$  is an orthonormal basis in  $L^2_\mu$ . Hence, if  $U$  maps the unit vector  $e_n = (0, \dots, 0, 1, 0, \dots)$  to  $p_n(\mu)$ , then  $U$  can be extended to a unitary operator from  $l_2$  onto  $L^2_\mu$ , and if  $Sf(x) = xf(x)$  is the multiplication operator by  $x$  in  $L^2_\mu$ , then  $J = U^{-1}SU$ . The recurrence coefficients for  $p_n(\mu, x)$  are precisely the  $a_n$ 's and  $b_n$ 's from the Jacobi matrix, i.e.,  $p_n(\mu, x) = c\pi_n(x)$  with some fixed constant  $c$ . Therefore,  $\mu$  is one of the measures for the three-term recurrence (2) in Favard's theorem. Conversely, if we start from a measure  $\mu$  with compact support on the real line, form the orthogonal polynomials and their three-term recurrence and form the Jacobi matrix  $J$  with the recurrence parameters, and  $U$  is the unitary operator mapping  $e_n$  to  $p_n$ , then  $J = U^{-1}SU$ , i.e.,  $J$  is unitarily equivalent to multiplication by  $x$  on  $L^2_\mu$ .

These show that Jacobi operators are equivalent to multiplication by  $x$  in  $L^2_\mu$  spaces if the particular basis  $\{p_n(\mu)\}$  are used. The relation of orthogonal polynomials with Jacobi matrices is very close, for example if we consider the truncated  $n \times n$  matrix

$$J_n = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots & 0 \\ a_0 & b_1 & a_1 & 0 & \cdots & 0 \\ 0 & a_1 & b_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & a_{n-2} \\ 0 & 0 & 0 & \cdots & a_{n-2} & b_{n-1} \end{pmatrix}$$

then it has  $n$  real and distinct eigenvalues, which turn out to be the zeros of  $p_n$ , i.e., the monic polynomial  $p_n(z)/\kappa_n$  is the characteristic polynomial of  $J_n$ :

$$(3) \quad p_n(z)/\kappa_n = \det(zI_n - J_n).$$

Since Simon has been working on Jacobi operators and their spectral properties, even from this short discussion it is evident that he was close to real orthogonal polynomials.

### 3. Orthogonal polynomials on the unit circle

If the orthogonality measure is not real, things change. Indeed, on the real line to have the three-term recurrence formula one expands  $xp_n(x)$  as  $c_{n,n+1}p_{n+1}(x) + c_{n,n}p_n(x) + \cdots + c_{n,0}p_0(x)$ , and notice that, by orthogonality,

$$\begin{aligned} c_{n,j} &= \int xp_n(x)\overline{p_j(x)} d\mu(x) = \int xp_n(x)p_j(x) d\mu(x) \\ &= \int p_n(x)\overline{xp_j(x)} d\mu(x) = 0 \end{aligned}$$

for all  $j < n - 1$ , hence there are only 3 terms in the expansion. If  $\mu$  is not supported on the real line, then we have

$$c_{n,j} = \int zp_n(z)\overline{p_j(z)} d\mu(z) = \int p_n(z)\overline{(\bar{z}p_j(z))} d\mu(z),$$

and we cannot use orthogonality, since  $\bar{z}p_j(z)$  is not a polynomial, and indeed, in general, the coefficients  $c_{n,j}$  will not be zero. Still, on the unit circle  $\mathbf{T}$  there is a substitute, called Szegő recurrence. If  $\mu$  is a nontrivial

probability measure on  $\mathbf{T}$  (that is, not supported on a finite set) the monic orthogonal polynomials  $\Phi_n(z, \mu)$  are uniquely determined by

$$\Phi_n(z) = \prod_{j=1}^n (z - z_{n,j}), \quad \int_{\mathbf{T}} \zeta^{-j} \Phi_n(\zeta) d\mu(\zeta) = 0, \quad j = 0, 1, \dots, n - 1,$$

and the orthonormal polynomials  $\varphi_n$  are  $\varphi_n = \Phi_n / \|\Phi_n\|_{L^2_\mu(\mathbf{T})}$ . However, as opposed to the real case, the orthonormal set  $\{\varphi_n\}_{n \geq 0}$  may not be a basis in  $L^2_\mu(\mathbf{T})$  for the set of polynomials may not be dense in  $L^2_\mu(\mathbf{T})$  (see below).

On  $L^2_\mu(\mathbf{T})$  we consider the  $n$ -\* map  $f^*(\zeta) := \zeta^n \overline{f(\zeta)}$ . Since  $z\bar{z} = 1$  on the unit circle, we get that  $\Phi_{n+1}(z) - z\Phi_n(z)$  is of degree  $n$  and is orthogonal to  $z^j$  for  $j = 1, 2, \dots, n$ . The same is true of  $\Phi_n^*(z)$ , hence

$$\Phi_{n+1}(z) - z\Phi_n(z) = \bar{\alpha}_n \Phi_n^*(z)$$

with some complex numbers  $\alpha_n$ , called the Verblunsky coefficients (this name was coined by Simon and now it is widely accepted, earlier other names like “recurrence coefficients”, “Schur parameters”, “reflection coefficients” were used).

$$(4) \quad \Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n \Phi_n^*(z)$$

is known as Szegő recurrence. At  $z = 0$  it gives  $\alpha_n = -\overline{\Phi_{n+1}(0)}$ . If we apply the  $(n + 1)$ -\* transform to (4), then we obtain

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z \Phi_n(z),$$

which is just another form of the Szegő recurrence.

Since  $\Phi_n^*$  is orthogonal to  $\Phi_{n+1}$  and  $|\Phi_n^*| = |\Phi_n|$ , we obtain from (4)

$$\|\Phi_{n+1}\|_{L^2_\mu(\mathbf{T})}^2 = (1 - |\alpha_n|^2) \|\Phi_n\|_{L^2_\mu(\mathbf{T})}^2, \quad \|\Phi_n\|_{L^2_\mu(\mathbf{T})}^2 = \prod_{j=0}^{n-1} (1 - |\alpha_j|^2),$$

and so  $|\alpha_n| < 1$ . Let  $\Delta^\infty$  be the set of complex sequences  $\{\alpha_j\}_{j=0}^\infty$  with  $|\alpha_j| < 1$ . The map  $V(\mu) = \{\alpha_j(\mu)\}_{j=0}^\infty$  is a well defined map from the set  $\mathcal{P}$  of nontrivial probability measures on  $\mathbf{T}$  to  $\Delta^\infty$ . By a theorem of Verblunsky,  $V$  is a bijection. Furthermore, works of Szegő, Kolmogorov and Krein show that the following are equivalent:

- $\lim_{n \rightarrow \infty} \|\Phi_n\|_{L^2_\mu(\mathbf{T})} = 0$ ,
- $\sum_{n=0}^\infty |\alpha_n|^2 = \infty$ ,
- $\{\varphi_n\}_{n=0}^\infty$  is a basis for  $L^2_\mu(\mathbf{T})$ ,

- $\int_{\mathbf{T}} \log \mu' = -\infty$ , where  $\mu'$  is the Radon-Nikodym derivative of  $\mu$  with respect to arc measure on  $\mathbf{T}$ .

As we can see, for orthogonal polynomials on the unit circle a beautiful theory is emerging. It was originated by Szegő in the late 1910's and early 1920's, and it was first discussed in a compact form in Szegő's book [11]. But is there an analogue of the relation to Jacobi matrices? It turns out that there is, but the corresponding matrix is 5-diagonal and not 3-diagonal (which is not much of a difference for an operator theorist like Simon). To obtain it orthogonalize the sequence  $1, \zeta, \zeta^{-1}, \zeta^2, \zeta^{-2}, \dots$  in  $L^2_{\mu}(\mathbf{T})$  using the Gram-Schmidt procedure to get the so called CMV (Cantero, Moral, and Velázquez) basis (complete orthonormal system)  $\{\chi_n\}_{n=0}^{\infty}$ , and consider the matrix of the operator of multiplication by  $z$  in that basis. We get the so called CMV matrix  $\mathcal{C}(\mu) = (C_{n,m})_{m,n=0}^{\infty}$ , where

$$C_{n,m} = \int \zeta \chi_m(z) \overline{\chi_n(z)} d\mu(z).$$

It turns that it is five-diagonal, and the  $\chi$ 's can be expressed in terms of the  $\varphi$ 's and  $\varphi^*$ 's:

$$\chi_{2n}(z) = z^{-n} \varphi_{2n}^*(z), \quad \chi_{2n+1}(z) = z^{-n} \varphi_{2n+1}(z), \quad n = 0, 1, \dots,$$

and the matrix elements in terms of the  $\alpha$ 's and  $\rho$ 's:  $\mathcal{C} = LM$  where  $L, M$  are block-diagonal matrices

$$L = \text{Diag}(\Theta_0, \Theta_2, \Theta_4, \dots), \quad M = \text{Diag}(1, \Theta_1, \Theta_3, \dots)$$

with

$$\Theta_j = \begin{pmatrix} \bar{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix}, \quad j = 0, 1, \dots$$

(the first block of  $M$  is  $1 \times 1$ ).

The analogy with Jacobi matrices is quite strong, for example, the analogue of (3) in the unit circle case is

$$\Phi_n(z) = \det(zI_n - \mathcal{C}^{(n)}),$$

where  $\mathcal{C}^{(n)}$  is the principal  $n \times n$  block of  $\mathcal{C}$ .

#### 4. OPUC

What follows is part of the personal recollections of Simon told in his acceptance talk at the prize ceremony (see [8]).

In the 1980's and 1990's Simon was working on discretized Schrödinger operators  $\{u_n\} \rightarrow \{u_{n-1} + u_{n+1} + V(n)u_n\}$ . He proved that if  $V$  decays slower than  $n^{-\alpha}$ ,  $\alpha < 1/2$ , then generically the spectrum is singular continuous. On the other hand, it had been known that if  $|V(n)| \leq n^{-\alpha}$ ,  $\alpha > 1$ , then the spectrum is purely absolutely continuous. In the missing range  $1/2 < \alpha \leq 1$  results of Kiselev and Deift showed that absolutely continuous spectrum exists, and Simon raised the question if in that range there can also be a continuous singular spectrum present (mixed spectrum). Often, instead of a power type decay, the condition is in the form  $V \in l_p$ , where the dividing parameter is  $p = 2$  (matching  $\alpha = 1/2$ ). Working with Killip on the problem they realized that if they had an appropriate sum rule relating Jacobi parameters to a spectral quantity (see Szegő's theorem below for an example), they would get the following:

$$\sum_n |a_n - 1|^2 + |b_n|^2 < \infty$$

if and only if the essential spectrum is  $[-2, 2]$ , the spectral measure satisfies

$$\int_{-2}^2 (4 - x^2)^{1/4} \log \mu'(x) dx > -\infty,$$

and if  $\lambda_n$  are the eigenvalues outside  $[-2, 2]$ , then  $\sum_n ||\lambda_n| - 2|^{3/2} < \infty$ . This theorem would prove the existence of Jacobi matrices with  $l_2$  decay and mixed spectrum, for in it there is no hypothesis on the singular part of  $\mu$ , so that can be selected at one's convenience and still get  $l_2$  decay for the potential. While working on the required sum rule (which they eventually found in [5]) Simon came across orthogonal polynomials on the unit circle through lectures given by Dennisov at Caltech on mixed spectrums of Schrödinger operators. He realized that people working on orthogonal polynomials tackled questions very similar to those that were relevant to people in the mathematical physics community in connection with spectral theory. He was drawn to orthogonal polynomials seeing the strong analogy in between the two fields. He observed that the two communities were practically unaware of each other, of the methods and questions in the other field, and even the same theorems were discovered using different language. For example, he discovered that his problem on mixed spectrum had been solved for orthogonal polynomials on the unit circle by Verblunski in 1936. Simon also observed that, while there were many results related to OPUC (Orthogonal Polynomials on the Unit Circle), there was no comprehensive treatment of them in a collected form (Szegő's [11] and Freud's [3] book each had a chapter, and Geronimus had the small book [4], but that was all). He realized that many ideas that were extensively investigated by him and other researchers

in spectral theory had not been studied by the orthogonal polynomial community, so there was a whole new chapter to be developed by applying the techniques and questions from one field to the other. For example, while working on the aforementioned sum rule Killip and Simon proved a conjecture of Nevai on real orthogonal polynomials: if the recurrence coefficients satisfy

$$\sum_n (|a_n - 1| + |b_n|) < \infty,$$

then the measure of orthogonality belongs to the Szegő class (see below). Instead of writing many small papers in this new chapter, around 2001 he decided to write a longer paper (he later admitted he had estimated its length to be about 80 pages) that could serve as an introduction to the other field for researchers in both communities. However, the collection of the results to be put in that paper had a steady grow, and finally his OPUC book emerged with two volumes and with more than a thousand pages.

Volume I discusses the general theory of orthogonal polynomials, while volume II is devoted to spectral theory with various connections and applications. The list of chapter titles is quite illustrative:

Volume I:

- The Basics
- Szegő's Theorem
- Tools for Geronimus Theorem
- Matrix Representations
- Baxters Theorem
- The Strong Szegő Theorem
- Verblunsky Coefficients With Rapid Decay
- The Density of Zeros

Volume II:

- Rakhmanov's Theorem and Related Issues
- Techniques of Spectral Analysis
- Periodic Verblunsky Coefficients
- Spectral Analysis of Specific Classes of Verblunsky Coefficients
- The Connection to Jacobi Matrices

The book discusses many connections/applications of OPUC from stationary stochastic processes through analytic functions, unitary operators, scattering theory up to random matrices. There is also an extended appendix on various topics such as Schur functions, Toeplitz matrices and determinants, Aleksandrov families, transfer matrices etc., and the book closes with conjectures and problems. The review [6] by Nevai contains many more details, historical accounts and personal views of researchers on the monograph.

The book is not an easy reading, but it has had a profound influence on the field of orthogonal polynomials even before its publication (various chapters were available), and it will be the definitive reference work for a long time. It is a worthy follower of Szegő's 1939 classics [11].

Since the Bolyai Prize is a recognition of the Hungarian Academy, we close this paper as an illustration of the many theorems in the book by discussing two results that are related to Hungarian mathematics.

### 5. Szegő's theorem and Simon's higher order Szegő theorem

Szegő's celebrated theorem is a sum-rule: if  $d\mu = \mu' dm + d\mu_s$ ,  $w \in L^1(\mathbf{T})$ , is the decomposition of  $\mu$  into its absolutely continuous and singular part, then

$$\prod_{j=0}^{\infty} (1 - |\alpha_j|^2) = \exp \left( \frac{1}{2\pi} \int_{\mathbf{T}} \log \mu'(\zeta) dm \right).$$

In particular,

$$(5) \quad \sum_{j=0}^{\infty} |\alpha_j|^2 < \infty \iff \log \mu' \in L^1(\mathbf{T}).$$

If either of the conditions in (5) holds, then we say that  $\mu$  belongs to the Szegő class. In this class the Szegő function is defined as

$$D(z) = \exp \left( \frac{1}{4\pi} \int_{\mathbf{T}} \frac{\zeta + z}{\zeta - z} \log \mu'(\zeta) dm(\zeta) \right), \quad |z| < 1.$$

For it  $D(\zeta) = \lim_{r \uparrow 1} D(rz)$  exists almost everywhere on the unit circle and it satisfies  $|D(\zeta)|^2 = w(\zeta)$  a.e. The main asymptotic result of Szegő is the claim that

$$\lim_{n \rightarrow \infty} \varphi_n^*(z) = D^{-1}(z)$$

uniformly on compact subsets of the open unit disk  $\Delta$ .

The following is often called strong Szegő theorem: if  $\mu_s = 0$  and  $\mu$  is in the Szegő class, then

$$\prod_{j=0}^{\infty} (1 - |\alpha_j|^2)^{-j-1} = \exp \left( \sum_{n=0}^{\infty} n |w_n|^2 \right),$$

where  $w_n$  are the Fourier coefficients of  $\log w$ .

Simon came up with the idea to extend Szegő's theorem for the case when  $\log \mu'$  may be infinite. His result from Section 2.8 from his book states that for any  $\zeta_0 \in \mathbf{T}$

$$|\zeta - \zeta_0|^2 \log w \in L^1(\mathbf{T}) \iff \sum_{j=0}^{\infty} |\alpha_{j+1} - \bar{\zeta}_0 \alpha_j|^2 + |\alpha_j|^4 < \infty.$$

There is a generalization to two zeros (see [10]): if  $\zeta_1, \zeta_2 \in \mathbf{T}$ , then for  $\zeta_1 \neq \zeta_2$  we have

$$\begin{aligned} & |\zeta - \zeta_1|^2 |\zeta - \zeta_2|^2 \log w \in L^1(\mathbf{T}) \\ \iff & \sum_{j=0}^{\infty} |\alpha_{j+2} - (\bar{\zeta}_1 + \bar{\zeta}_2) \alpha_{j+1} + \bar{\zeta}_1 \bar{\zeta}_2 \alpha_j|^2 + |\alpha_j|^4 < \infty, \end{aligned}$$

while for  $\zeta_1 = \zeta_2$

$$|\zeta - \zeta_1|^4 \log w \in L^1(\mathbf{T}) \iff \sum_{j=0}^{\infty} |\alpha_{j+2} - 2\bar{\zeta}_1 \alpha_{j+1} + \bar{\zeta}_1^2 \alpha_j|^2 + |\alpha_j|^6 < \infty$$

is true.

## 6. Zeros

It is easy to see that all zeros of the orthogonal polynomials for a measure on the unit circle lie inside the unit disk  $\Delta$ . Paul Turán asked if the zeros can be dense in  $\Delta$ . He did not specify, however, in what sense the density should be considered. The simplest is to ask if the set of all the zeros of all the orthogonal polynomials can be dense in  $\Delta$ . In 1988 Alfaro and Vigil [1] answered this affirmatively. Their result is a consequence of the recurrence formula (4): if  $\{z_n\}$  is given, then one can choose inductively  $\alpha_n \in \Delta$  so that  $z_n$ ,  $n = 1, 2, \dots$  is a zero of  $\Phi_n$ .

In [9] a much stronger statement was proven by Simon and the author. To state it consider the sequence  $\{\nu_n(\mu)\}_{n \geq 1}$  of the normalized counting measures for zeros of  $\Phi_n$ , that is,  $\nu_n = \frac{1}{n} \sum_k \delta_{z_k}$ , where the summation is for all zeros of  $\Phi_n$  counting multiplicity. [9] proves the existence of a universal measure  $\mu$  in the sense that if  $\nu$  is any probability measure on the closed unit disk, then there is a subsequence  $\mathcal{N}$  of the natural numbers such that along  $\mathcal{N}$  the zero counting measures  $\nu_n$  converge to  $\nu$  in the weak\* topology. This is an easy consequence of following theorem of independent interest: if  $\Phi$  is a monic polynomial of degree  $m$  with all its zeros in  $\Delta$  and  $z_1, \dots, z_k$  are arbitrary points in the unit disk, then there is a measure  $\mu$  on the unit circle

such that  $\Phi$  is the  $m$ -th monic orthogonal polynomial with respect to  $\mu$ , i.e.,  $\Phi_m = \Phi$ , and  $z_1, \dots, z_k$  are zeros of the  $(m+k)$ -th orthogonal polynomial  $\Phi_{m+k}$ .

There is a third way to understand Turán's question: can it happen that along the (complete) sequence of the integers  $n$  the set of zeros get dense in  $\Delta$ , i.e., if  $Z_\mu$  is the set of points in  $\overline{\Delta}$  for which there is a sequence  $\{z_n\}$  such that  $z_n$  is a zero of  $\Phi_n$  and  $z_n \rightarrow z$ , then is it possible that  $Z_\mu$  is the whole closed unit disk? That this cannot happen was proven in [2], where the following stronger statement was verified: if  $0 \in Z_\mu$ , then  $Z_\mu$  is a countable set converging to the origin.

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