But

$$\frac{(e^{i\theta}-e^{i\varphi})(e^{i\theta}-e^{-i\varphi})}{e^{i\theta}} = (e^{i\theta}+e^{-i\theta}) - (e^{i\varphi}+e^{-i\varphi})$$

so

$$\frac{z}{(z-e^{i\varphi})(z-e^{-i\varphi})}\bigg|_{z=e^{i\theta}} = \frac{1}{\cos\theta - \cos\varphi} > 0$$

for $|\theta| < \varphi$, that is, on $\text{supp}(d\mu)$. Thus, (2.14.30) cannot hold. By rotation covariance, any pair of zeros in a gap can be rotated to this case. □

We can make the consequences of (iv) explicit:

Corollary 2.14.5. *Let* $(z_0, \dots, z_{n-1})$ *and* $(w_0, \dots, w_n)$ *be the zeros of* $\Phi_n(z;\beta)$ *and* $\Phi_{n+1}(z;\tilde{\beta})$*, respectively, counted counterclockwise. Then one of the following happens:*

(a) Φ_n *and* Φ_{n+1} *have a single zero in common, which, by cyclic relabeling, we can suppose is* $z_0 = w_0$*. In that case, each of the n intervals* (z_0, z_1), $(z_1, z_2), \dots, (z_{n-2}, z_{n-1})$, (z_{n-1}, z_0) *has exactly one* w.

(b) Φ_n *and* Φ_{n+1} *have no zeros in common, in which case among the n intervals,* $(z_0, z_1), \dots, (z_{n-1}, z_0)$*, one has exactly two* w*'s and each of the others has exactly one* w.

Proof. Follows from the fact that each of the n intervals $(z_0, z_1), \dots, (z_{n-1}, z_0)$ must contain at least one w. There is only one other w left. □

Remarks and Historical Notes. For properties (iii)–(v) for OPRL, see Section 1.2 of [399]. The gap property (property (iii)) comes as follows: If x_0, x_1 are two zeros of P_n in (a, b), which is disjoint from $\text{supp}(d\mu)$, then $P_n/(x-x_0)(x_1-x)$ is of degree $n-2$, so orthogonal to P_n, so

$$\int |P_n|^2 (x-x_0)^{-1}(x_1-x)^{-1}\, d\mu(x) = 0$$

But $(x-x_0)^{-1}(x_1-x)^{-1}$ is positive on $\text{supp}(d\mu)$. This classical argument motivated the final proof in the section.

For purposes of Gaussian quadrature on $\partial\mathbb{D}$, POPUC were introduced by Jones, Njåstad, and Thron [210]. Their zeros and other properties have been studied by Golinskii [174], Cantero–Moral–Velázquez [69], Wong [462], and Simon [405]. Our discussion here using CD kernels is influenced by Wong [462]. Most of Theorem 2.14.4 is from [69, 174] with parts from [405].

The use of b_n and of the recursion (2.14.29) is due to Khrushchev [219].

2.15 ASYMPTOTICS OF THE CD KERNEL: WEAK LIMITS

This is the first of three sections on the asymptotics of the CD kernel for OPUC, $K_n(w, z)$, especially when $|w| = |z| = 1$ and $w = z$ or $|w - z|$ is small. In this section, we will say something about limits of $\frac{1}{n+1} K_n(e^{i\theta}, e^{i\theta})\, d\mu(\theta)$ as a measure.

We start by relating it to limits of the zero counting measure for paraorthogonal polynomials.

Given a measure $d\mu$ on $\partial\mathbb{D}$, we let $d\nu_n$ be the zero counting measure for Φ_n, that is, ν_n is a pure point measure with

$$\nu_n(\{w\}) = n^{-1} \times (\text{multiplicity of } w \text{ as a zero of } \Phi_n) \tag{2.15.1}$$

Similarly, for any $\beta \in \partial\mathbb{D}$, we let $\nu_n^{(\beta)}$ be the zero counting measure for the POPUC $\Phi_n(z;\beta)$ (all multiplicities are one). Finally, we define

$$d\mu^{(N)}(\theta) = \frac{1}{N+1} K_N(e^{i\theta}, e^{i\theta})\, d\mu(\theta) \tag{2.15.2}$$

which is a probability measure on $\partial\mathbb{D}$, since $\int |\varphi_j|^2\, d\mu = 1$. $\nu_n^{(\beta)}$ is a probability measure on $\partial\mathbb{D}$ and ν_n on $\overline{\mathbb{D}}$. Here is a result that says they have the same weak limits:

Theorem 2.15.1. *For any $\ell = 1, 2, \dots$ and any β,*

$$\left| \int z^\ell\, d\mu^{(N)} - \int z^\ell\, d\nu_{N+1}^{(\beta)} \right| \le \frac{2\ell}{N+1} \tag{2.15.3}$$

$$\left| \int z^\ell\, d\nu_{N+1} - \int z^\ell\, d\nu_{N+1}^{(\beta)} \right| \le \frac{2\ell}{N+1} \tag{2.15.4}$$

In particular, for a subsequence, $N(1) < N(2) < \dots$, $d\nu_{N(j)+1}^{(\beta_j)} \xrightarrow{w} d\nu_\infty$ if and only if $d\mu^{(N(j))} \xrightarrow{w} d\nu_\infty$ (for one, and then for all choices of $\{\beta_j\}$), and in that case, for any $\ell = 1, 2, \dots$,

$$\lim_{j\to\infty} \int z^\ell\, d\nu_{N(j)+1} = \int z^\ell\, d\nu_\infty(z) \tag{2.15.5}$$

Conversely, if (2.15.5) *holds for some $d\nu_\infty$ on $\partial\mathbb{D}$, then $d\mu^{(N(j))} \xrightarrow{w} d\nu_\infty$.*

Proof. $\varphi_0, \dots, \varphi_N$ are a basis for $\operatorname{Ran}(\pi_{N+1})$, so with $A = \pi_{N+1} M_z \pi_{N+1}$,

$$\int z^\ell\, d\nu_{N+1} = \frac{1}{N+1} \operatorname{Tr}(A^\ell) = \frac{1}{N+1} \sum_{j=0}^{N} \langle \varphi_j, (A_j)^\ell \varphi_j \rangle \tag{2.15.6}$$

and similarly,

$$\int z^\ell\, d\nu_{N+1}^{(\beta)} = \frac{1}{N+1} \sum_{j=0}^{N} \langle \varphi_j, (U_\beta^{(N+1)})^\ell \varphi_j \rangle \tag{2.15.7}$$

By definition of K_N,

$$\int z^\ell\, d\mu^{(N)} = \frac{1}{N+1} \sum_{j=0}^{N} \langle \varphi_j, z^\ell \varphi_j \rangle \tag{2.15.8}$$

If $j \le N - \ell$, $(A_j)^\ell \varphi_j = (U_\beta^{(N+1)})^\ell \varphi_j = z^\ell \varphi_j$, so the terms in the sum cancel for such j's. Since $|\langle \varphi_j, z^\ell \varphi_j \rangle| \le 1$ and similarly for A and $U_\beta^{(N+1)}$ for any j, the remaining terms contribute at most $2\ell/(N+1)$ to the difference of the sums. This proves (2.15.3) and (2.15.4).

For $d\mu^{(N)}$ and $d\nu_{N+1}^{(\beta)}$, we have measures on $\partial\mathbb{D}$ so $\int z^{-\ell}\,d\eta = \overline{\int z^{\ell}\,d\eta}$. Polynomials in z and z^{-1} are dense in the continuous functions on $\partial\mathbb{D}$, so weak convergence is equivalent to convergence of $\int z^{\ell}\,d\eta$ (for all $\ell \geq 0$), which happens for one of $d\mu^{(N(j))}$ and $d\nu_{N(j)+1}^{(\beta_j)}$ if and only if it happens for both (by (2.15.3)). And convergence then implies (2.15.5). For the converse, note that (2.15.5) implies convergence of the moments of $d\nu_{N(j)+1}^{(\beta)}$ by (2.15.4). □

This is especially useful since there is a class of measures $d\mu$ for which w-lim $d\nu_n^{(\beta)}$ can be seen to be $\frac{d\theta}{2\pi}$.

Proposition 2.15.2. *Consider the conditions*

$$\text{(a)} \qquad \lim_{n\to\infty} (\rho_0 \dots \rho_{n-1})^{1/n} = 1 \tag{2.15.9}$$

$$\text{(b)} \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} |\alpha_j|^2 = 0 \tag{2.15.10}$$

$$\text{(c)} \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} |\alpha_j| = 0 \tag{2.15.11}$$

Then (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c). *If*

$$\sup_n |\alpha_n| = R < 1 \tag{2.15.12}$$

then (b) $\Rightarrow$ (a) *also.*

Proof. (b) $\Leftrightarrow$ (c). Since $|\alpha_j| < 1$, we have that $|\alpha_j|^2 < |\alpha_j|$. This and the Schwarz inequality imply

$$\left(\frac{1}{n}\sum_{j=0}^{n-1}|\alpha_j|\right)^2 \leq \frac{1}{n}\sum_{j=0}^{n-1}|\alpha_j|^2 \leq \frac{1}{n}\sum_{j=0}^{n-1}|\alpha_j| \tag{2.15.13}$$

(a) $\Rightarrow$ (b). We have that

$$-\log|\rho_j|^2 = |\alpha_j|^2 + \sum_{k=2}^{\infty} \frac{1}{k}|\alpha_j|^{2k} \geq |\alpha_j|^2 \tag{2.15.14}$$

so

$$\frac{1}{n}\sum_{j=0}^{n-1}|\alpha_j|^2 \leq -\log[(\rho_0 \dots \rho_{n-1})^{2/n}] \tag{2.15.15}$$

Thus, (a) $\Rightarrow \lim(-\log(\rho_0 \dots \rho_{n-1})^{2/n}) = 0 \Rightarrow$ (b).

(b) $\Rightarrow$ (a) if (2.15.12) holds. If (2.15.12) holds, then for some K (can be taken $-R^{-1}\log(1-R)$),

$$-\log|\rho_j|^2 \leq K|\alpha_j|^2$$

so

$$\frac{K}{n}\sum_{j=0}^{n-1}|\alpha_j|^2 \geq -\log[(\rho_0\dots\rho_{n-1})^{2/n}] \tag{2.15.16}$$

so (b) plus the fact that $\rho_j < 1$ implies (a). □

Definition. Let μ be a measure on $\partial\mathbb{D}$. If

$$\lim_{n\to\infty}(\rho_0\dots\rho_{n-1})^{1/n} = 1$$

we say μ is *regular*.

Regularity has two important consequences:

Theorem 2.15.3. *Let μ be a measure on $\partial\mathbb{D}$, which is regular. Then for any $z \in \mathbb{C}\setminus\mathbb{D}$, we have*

$$\lim_{n\to\infty}|\Phi_n(z)|^{1/n} = \lim_{n\to\infty}|\varphi_n(z)|^{1/n} = |z| \tag{2.15.17}$$

Remark. The proof shows the convergence is uniform on compact subsets of $\mathbb{C}\setminus\overline{\mathbb{D}}$.

Proof. Since $(\rho_1\dots\rho_n)^{1/n} \to 1$, we need only prove the result for Φ_n. Suppose $|z| > 1$. By Szegő recursion and $|\Phi_n(z)| \geq |\Phi_n^*(z)|$ if $|z| > 1$ (see (2.9.11)), we have

$$(|z| - |\alpha_n|)|\Phi_n(z)| \leq |\Phi_{n+1}(z)| \leq (|z| + |\alpha_n|)|\Phi_n(z)| \tag{2.15.18}$$

Since $|z| > 1$ holds, there is a $K(|z|)$ so that for all n,

$$1 - |\alpha_n|\,|z|^{-1} \geq \exp(-K|\alpha_n|) \tag{2.15.19}$$

Moreover, if $|z| > 1$,

$$1 + |\alpha_n|\,|z|^{-1} \leq \exp(|\alpha_n|) \tag{2.15.20}$$

Thus, (2.15.18) plus induction implies

$$\exp\left(-K\left(\sum_{j=0}^{n-1}|\alpha_j|\right)\right) \leq \frac{|\Phi_n(z)|}{|z|^n} \leq \exp\left(\sum_{j=0}^{n-1}|\alpha_j|\right) \tag{2.15.21}$$

(2.15.11) thus implies (2.15.17) for Φ_n.

This proves (2.15.17) for $|z| > 1$ and the limit is uniform in θ, for $z = re^{i\theta}$ with $r > 1$ fixed. By the maximum principle, for any $r > 1$,

$$|\Phi_n(e^{i\theta})| \leq \sup_{\varphi}|\Phi_n(re^{i\varphi})| \tag{2.15.22}$$

This plus the uniformity implies for any $r > 1$,

$$\limsup\Big[\sup_{\theta}|\Phi_n(e^{i\theta})|^{1/n}\Big] \leq r$$

Since r is arbitrary, the lim sup is at most 1.

Since the ρ's for the second kind polynomials are the same, we have

$$\limsup|\psi_n(e^{i\theta})|^{1/n} \leq 1 \tag{2.15.23}$$

But by (2.4.57),

$$|\varphi_n(e^{i\theta})|\,|\psi_n(e^{i\theta})| \geq 1 \tag{2.15.24}$$

This plus (2.15.22) implies

$$\liminf|\varphi_n(e^{i\theta})|^{1/n} \geq 1 \tag{2.15.25}$$

and so (2.15.17) for $|z| = 1$. □

Theorem 2.15.4. *Let μ be a measure on $\partial\mathbb{D}$, which is regular. Then*

$$\operatorname*{w-lim}_{n\to\infty} d\mu^{(n)} = \frac{d\theta}{2\pi} \tag{2.15.26}$$

and for any $\{\beta_j\} \in \partial\mathbb{D}$,

$$\operatorname*{w-lim}_{n\to\infty} d\nu_n^{(\beta_n)} = \frac{d\theta}{2\pi} \tag{2.15.27}$$

Proof. By Theorem 2.15.1, it suffices to prove for $\ell \geq 1$,

$$\int z^\ell \, d\nu_n(z) \to 0 \tag{2.15.28}$$

since $\frac{d\theta}{2\pi}$ is the unique measure on $\partial\mathbb{D}$ with $\int e^{i\ell\theta}\,d\eta(\theta) = 0$ for $\ell > 0$.

Let $d\nu_\infty$ be an arbitrary weak limit point of $d\nu_n$. For $|z| > 1$, $\log|z-w|$ is continuous for $w \in \overline{\mathbb{D}}$, so

$$\int \log|z-w|\,d\nu_n(w) \to \int \log|z-w|\,d\nu_\infty(w) \tag{2.15.29}$$

Since

$$\frac{1}{n}\log|\Phi_n(z)| = \int \log|z-w|\,d\nu_n(w) \tag{2.15.30}$$

(2.15.17) implies for $|z| > 1$,

$$\int \log\left|1 - \frac{w}{z}\right| d\nu_\infty(w) = 0 \tag{2.15.31}$$

In the region $|z| > 1$, uniformly in $|w| \leq 1$, $\log|1 - \frac{w}{z}|$ is the real part of an analytic function, so

$$\int \log\left(1 - \frac{w}{z}\right) d\nu_\infty(w) = 0 \tag{2.15.32}$$

since we first see it is an imaginary constant and then, by taking $|z| \to \infty$, we see the constant is zero. Now

$$\log\left(1 - \frac{w}{z}\right) = -\sum_{j=1}^\infty \frac{1}{j}\left(\frac{w}{z}\right)^j \tag{2.15.33}$$

uniformly in $|w| \leq 1$ and $|z| \geq 2$, so interchanging the sum and integral, we see

$$\int w^j \, d\nu_\infty(w) = 0 \tag{2.15.34}$$

for $j \geq 1$, proving (2.15.28). □

We have thus proven that if $d\mu$ is regular, then

$$\frac{1}{n+1} K_n(e^{i\theta}, e^{i\theta})\left[w(\theta)\, \frac{d\theta}{2\pi} + d\mu_{\rm s}\right] \xrightarrow{w} \frac{d\theta}{2\pi} \tag{2.15.35}$$

When the Szegő condition holds, (2.9.30) says $\frac{1}{n+1} K_n\, d\mu_{\rm s} \xrightarrow{w} 0$, and one might hope that this is true more generally (indeed, see Theorem 2.17.7), which leads us to a natural guess that under suitable hypotheses, pointwise in θ,

$$\frac{1}{n+1} K_n(e^{i\theta}, e^{i\theta}) w(\theta) \to 1 \tag{2.15.36}$$

It is precisely this surmise that we explore in the next two sections. Of course, it cannot hold at points with $w(\theta) = 0$. Note, however, if $d\mu_{\rm s} = 0$, (2.15.35) implies that if the left side of (2.15.36) converges uniformly, the limit must be 1.

Remarks and Historical Notes. Theorem 2.15.1 is from Simon [409]. Regularity will be discussed more extensively in Section 5.9, mainly in the context of OPRL. In particular, its history is discussed in the Notes to that section. That regularity implies zeros are distributed according to an "equilibrium" measure (which is $\frac{d\theta}{2\pi}$ for $\partial\mathbb{D}$) is a major theme of that section. The proof of (2.15.28) is essentially potential theoretic—this is discussed in Section 5.5.

2.16 ASYMPTOTICS OF THE CD KERNEL: CONTINUOUS WEIGHTS

In this section, we will study the asymptotics of the CD kernel for continuous nonvanishing weights and apply this to obtain a refined estimate on the zeros of POPUC. We will call a function, f, on $\partial\mathbb{D}$ "continuous" on an interval $I = [\alpha, \beta]$ (i.e., $\alpha, \beta \in \partial\mathbb{D}$ and I is the set of points between α and β going counterclockwise from α to β) if, *as a function on* $\partial\mathbb{D}$, it is continuous at each $z \in [\alpha, \beta]$. This is stronger than saying the restriction of f to I is continuous on I; in particular, it says something if $\alpha = \beta$ and I is a single point. Here is the main theorem of this section:

Theorem 2.16.1 (Levin–Lubinsky [275]). *Let $d\mu$ be a regular probability measure on $\partial\mathbb{D}$ of the form*

$$d\mu = w(\theta)\, \frac{d\theta}{2\pi} + d\mu_{\rm s} \tag{2.16.1}$$

Suppose for an interval $I = [\alpha, \beta] \subset \partial\mathbb{D}$,

(a) $\operatorname{supp}(d\mu_{\rm s}) \cap I = \emptyset$

(b) *w is "continuous" on I and nonvanishing there.*

Then

(1) (Diagonal Asymptotics) *For any $A < \infty$, uniformly in $z_\infty \in I$, and sequences $z_n \in \partial\mathbb{D}$ with $n|z_n - z_\infty| \leq A$ for all n,*

$$\frac{1}{n+1} K_n(z_n, z_n) \to w(z_\infty)^{-1} \tag{2.16.2}$$