Notice that our proof of Theorem 3.7.9 also provides an independent proof of Szegő asymptotics for p_n on $\mathbb{D}$ when (3.7.2) and (3.7.8) hold. It also only needs

$$\limsup (a_1 \dots a_n)^{-1} \geq u(0)$$

which is the easier half of the proof of (3.7.66) (i.e., of (3.6.37)). It then implies the full (3.7.66).

Notice that (3.7.39) expands p_n in terms of $e^{\pm in\theta(x)}$, not $u_n(e^{i\theta(x)})$ and its conjugate. The product of u_n and u_0 is not necessarily L^2, but since $e^{\pm in\theta(x)}$ are in L^∞, their products with u_0 are in L^2.

3.8 THE MOMENT PROBLEM: AN ASIDE

In the next section, we will discuss an application of Szegő's theorem for OPUC to the moment problem on the real line. This section is background but also illustrates the use of OPRL and, in particular, transfer matrices to study the moment problem.

The moment problem in its primeval form is:

Moment Problem: First Form. Given a sequence $\{c_n\}_{n=0}^\infty$ of real numbers, when does there exist a nontrivial measure, $d\mu$, on $\mathbb{R}$ with

$$\int x^n \, d\mu(x) = c_n \tag{3.8.1}$$

When a solution exists, is it unique? If it is not unique, what is the structure of the set of solutions?

Of course, for (3.8.1) to make sense, one needs

$$\int |x|^n \, d\mu < \infty \tag{3.8.2}$$

By structure of the set of solutions, we mean is it closed in the weak topology? (This is not obvious since x^n is not bounded.) Is it of finite or infinite dimension? Among the solutions, are there any that are pure point or singular continuous or purely absolutely continuous?

If there exists a unique solution, we call the moment problem *determinate*, and if there are multiple solutions, *indeterminate*. Since we can replace c_n by c_n/c_0, we can and will always suppose that $c_0 = 1$.

Often the c_n are given by (3.8.1), so existence is trivial. The moment problem then becomes:

Moment Problem: Second Form. Suppose c_n is a sequence given by (3.8.1) for some nontrivial probability measure, $d\mu_0$, on $\mathbb{R}$ obeying (3.8.2). Is $d\mu_0$ the unique measure obeying (3.8.1) for the given c_n, or are there others? If there are others, what is the structure of the solutions?

Example 3.8.1. Fix $0 < \alpha$ real and let c_n be given by

$$c_n = N_\alpha^{-1} \int x^n \exp(-|x|^\alpha) \, dx \tag{3.8.3}$$

where $N_\alpha = \int \exp(-|x|^\alpha)\, dx$ is a normalization constant. Below (see later in this section and then in the next) we will show that this problem is determinate if $\alpha \geq 1$ and indeterminate if $0 < \alpha < 1$. $\square$

There is an obvious necessary condition on the c_n's for there to be any nontrivial measure.

Proposition 3.8.2. *If a solution of the moment problem exists, then for each* $n = 1, 2, \dots,$ *the Hankel determinants*

$$H_m(\{c_n\}_{n=0}^\infty) = \det((c_{j+k-2})_{1\leq j,k\leq m}) \tag{3.8.4}$$

are strictly positive.

Proof. Let $\{\alpha_j\}_{j=1}^m$ lie in $\mathbb{C}^m$. Then

$$\sum_{j,k=1}^m \bar{\alpha}_j \alpha_k c_{j+k-2} = \int \left| \sum_{j=0}^{m-1} \alpha_j x^j \right|^2 d\mu \tag{3.8.5}$$

so H_m is positive as the determinant of a strictly positive matrix. $\square$

We will see later (see Theorem 3.8.4) that, conversely, if $H_m > 0$ for all m, then the moment problem is soluble. For now, we note that it is easy to see that if each H_m is positive, there exists a unique nondegenerate inner product on polynomials with

$$\langle 1, x^m \rangle = c_m \tag{3.8.6}$$

This inner product defines OPs both monic and normalized and Jacobi parameters $\{a_n, b_n\}_{n=1}^\infty \in ((0,\infty)\times\mathbb{R})^\infty$. Thus, we have:

Moment Problem: Third Form. Given a set of Jacobi parameters, $\{a_n, b_n\}_{n=1}^\infty \in ((0,\infty)\times\mathbb{R})^\infty$, when does there exist a measure, $d\mu$, whose Jacobi parameters are $\{a_n, b_n\}_{n=1}^\infty$? If one exists, is it unique? If it is not unique, what is the structure of the set of solutions?

Existence is essentially Favard's theorem discussed in Section 1.3. Jacobi parameters determine moments, so an inner product on polynomials, and (3.8.4) holds. Thus, Problems 1 and 3 are equivalent. We will see (see Theorem 3.8.4) that in this form, the moment problem always has solutions, that is, any set of Jacobi parameters can occur.

Proposition 3.8.3. *Fix* $k \geq 1$. *Let* $\{c_n\}_{n=0}^{2k}$ *be a set of moments with* (3.8.4) *strictly positive for* $m = 1, \dots, k+1$. *Then the set of measures in* $\mathbb{R}$ *obeying* (3.8.1) *for* $n = 0, \dots, 2k-1$ *and*

$$\int x^{2k}\, d\mu \leq c_{2k} \tag{3.8.7}$$

is a nonempty set, compact in the topology of weak-$*$ convergence (i.e., $d\mu_\ell \to d\mu$ *if and only if* $\int f(x)\, d\mu_\ell \to \int f(x)\, d\mu$ *for all bounded continuous functions on* $\mathbb{R}$*).*

Proof. The $\{c_n\}_{n=0}^{2k}$ define an inner product on polynomials of degree up to k, so orthonormal polynomials $\{p_j\}_{j=0}^{k}$, and so Jacobi parameters $\{a_n, b_n\}_{n=1}^{k}$. Choose any value for b_{k+1} and so get a $(k+1) \times (k+1)$ finite Jacobi matrix, $J_{k+1;F}$. Let $d\mu$ be the spectral measure for this matrix and vector δ_1. Then $d\mu$ obeys (3.8.1) for $n = 0, \dots, 2k$, so there is a solution proving the set is nonempty; indeed, we can suppose equality in (3.8.7).

Using the fact that the probability measures on $[-R, R]$ are compact, it is easy to see that the set of probability measures on $\mathbb{R}$ obeying

$$\int_{|x| \geq R} d\mu(x) \leq c_{2k} R^{-2k} \tag{3.8.8}$$

for each R is compact in the topology of weak-$*$ convergence. Here we use $k \geq 1$ to assure weak limits are also probability measures.

(3.8.7) implies (3.8.8). Thus, we need only prove that the set, S, of μ's obeying (3.8.1) for $m \leq 2k-1$ and (3.8.7) is weakly closed.

Let

$$f_{n;R}(x) = \begin{cases} x^n & |x| \leq R \\ R^n & x \geq R \\ (-R)^n & x \leq -R \end{cases} \tag{3.8.9}$$

and suppose $d\mu_\ell \in S$ converges weakly to $d\mu$. Then

$$\int f_{2k;R}\, d\mu_\ell \leq c_{2k} \tag{3.8.10}$$

so, since $f_{2k;R} - R^{2k}$ has compact support,

$$\int f_{2k;R}\, d\mu \leq c_{2k} \tag{3.8.11}$$

and (3.8.7) holds by the monotone convergence theorem.

By dominated convergence, (3.8.7) implies that for any $m = 1, \dots, 2k-1$,

$$\lim_{R \to \infty} \int f_{m;R}\, d\mu = \int x^m\, d\mu \tag{3.8.12}$$

Moreover, for any finite ℓ,

$$\begin{aligned} \int |f_{m;R} - x^m|\, d\mu_\ell &\leq 2 \int_{|x| \geq R} |x|^m\, d\mu_\ell \\ &\leq 2 \int_{|x| \geq R} \left| \frac{x}{R} \right|^{2k-m} |x|^m\, d\mu \\ &\leq 2R^{-(2k-m)} c_{2k} \end{aligned} \tag{3.8.13}$$

so (3.8.12) converges for each ℓ uniformly in ℓ. This plus (3.8.12) plus $\int x^m\, d\mu = c_m$ implies $d\mu$ obeys (3.8.1) for $n = 0, \dots, 2k-1$. $\square$

We thus have existence:

Theorem 3.8.4. *A set, $\{c_n\}_{n=0}^{\infty}$, of real numbers with $c_0 = 1$ has solutions of the moment problem if and only if each $H_m(\{c_n\}_{n=0}^{\infty})$ (given by (3.8.4)) is strictly positive. Any set of Jacobi parameters $\{a_n, b_n\}_{n=1}^{\infty} \in ((0,\infty) \times \mathbb{R})^{\infty}$ is the Jacobi parameter of some measure.*

Remark. The second sentence is essentially Favard's theorem in the general case; see Theorem 1.3.9.

Proof. Let S_k be the set of measures given by Proposition 3.8.3. Since S_k is compact and nonempty, and $S_{k+1} \subset S_k$, we see $\cap_k S_k$ is nonempty. This plus Proposition 3.8.2 proves the first sentence in this proposition. As noted, the first and third forms of the moment problem are equivalent, thus proving the second sentence. $\square$

To go further and analyze uniqueness, we need to briefly study unbounded selfadjoint operators. A *densely defined operator*, A, on a Hilbert space, $\mathcal{H}$, has a domain $D(A) \subset \mathcal{H}$, a dense subspace, and is a linear map of $D(A)$ into $\mathcal{H}$. Associated to A is its *graph*, $\Gamma(A) \subset \mathcal{H} \times \mathcal{H}$, defined by

$$\Gamma(A) = \{(\varphi, A\varphi) \mid \varphi \in D(A)\} \tag{3.8.14}$$

$\Gamma(A)$ is always a subspace of $\mathcal{H} \times \mathcal{H}$. A is called *closed* if and only if $\Gamma(A)$ is closed. B is an *extension* of A if and only if $\Gamma(A) \subset \Gamma(B)$, that is, $D(A) \subset D(B)$ and $B \restriction D(A) = A$.

Given an operator, A, we define $D(A^*)$ to be those $\varphi \in \mathcal{H}$ for which there is an $\eta \in \mathcal{H}$ with

$$\langle \eta, \gamma \rangle = \langle \varphi, A\gamma \rangle \tag{3.8.15}$$

for all $\gamma \in D(A)$. η is uniquely determined if it exists since $D(A)$ is dense. We then set $\eta = A^*\varphi$, so

$$\langle A^*\varphi, \gamma \rangle = \langle \varphi, A\gamma \rangle \tag{3.8.16}$$

for all $\gamma \in D(A)$, $\eta \in D(A^*)$. A^* is called the *adjoint* of A. A^* is thus defined to be the maximal operator so that (3.8.16) holds. If $D(A^*)$ is dense, then it is easy to see that A^* is a closed operator. Note that there is a relation between extension and adjoint:

$$A \subset B \Rightarrow B^* \subset A^*$$

An operator is called

$$\textit{Hermitian} \Leftrightarrow A \subset A^*$$
$$\textit{Selfadjoint} \Leftrightarrow A = A^*$$
$$\textit{Essentially selfadjoint} \Leftrightarrow A \subset A^* = (A^*)^*$$

Notice that if A is Hermitian, then A^* is densely defined and we can define $(A^*)^*$.

Proposition 3.8.5. *Let A be a Hermitian operator and let $z = x + iy \in \mathbb{C} \setminus \mathbb{R}$. Then*

(i) *For all $\varphi \in D(A)$,*

$$\|(A - z)\varphi\|^2 = \|(A - x)\varphi\|^2 + y^2\|\varphi\|^2 \tag{3.8.17}$$

(ii) *A is closed $\Leftrightarrow$ $\mathrm{Ran}(A - z)$ is closed.*

(iii) *A^{**} is the smallest closed extension of A, so we write*

$$\bar{A} = A^{**} \tag{3.8.18}$$

(iv) *$A^* = A^{***}$. Moreover, if A is Hermitian, so is $\bar{A}$.*

(v)

$$\overline{\mathrm{Ran}(A - z)} = \mathrm{Ran}(\bar{A} - z) \tag{3.8.19}$$

(vi)

$$D(A^*) = D(\bar{A}) + \ker(A^* - z) + \ker(A^* - \bar{z}) \tag{3.8.20}$$

(vii) *A is essentially selfadjoint if and only if*

$$\ker(A^* - z) = \ker(A^* - \bar{z}) = \{0\} \tag{3.8.21}$$

Remark. (3.8.20) holds in the sense of algebraic direct sum, that is, any $\psi \in D(A^*)$ is uniquely the sum of three vectors, one in each space.

Proof. (i) (3.8.17) follows from noting that the cross-term

$$\langle (A - x)\varphi, iy\varphi\rangle + \langle iy\varphi, (A - x)\varphi\rangle = 0 \tag{3.8.22}$$

by Hermiticity.

(ii) By (3.8.17),

$$(\varphi, A\varphi) \mapsto (A - z)\varphi \tag{3.8.23}$$

is a metric space equivalence of $\Gamma(A)$ and $\mathrm{Ran}(A - z)$, so one space is complete if and only if the other is.

(iii) Let $J\colon \mathcal{H} \to \mathcal{H}$ by $J\langle\varphi, \psi\rangle = \langle\psi, -\varphi\rangle$. Then

$$\Gamma(A^*) = J[\Gamma(A)^\perp] = [J\Gamma(A)]^\perp \tag{3.8.24}$$

Since $J^2 = -\mathbf{1}$, we see $\Gamma(A^{**}) = [-\Gamma(A)]^{\perp\perp} = \overline{\Gamma(A)}$. Thus, A^{**} is closed and is the smallest closed extension.

(iv) A^* is closed by (3.8.24), so (3.8.18) implies $A^* = A^{***}$. Thus, $A \subset A^*$ implies $A^{**} \subset A^* = (A^{**})^*$.

(v) As noted in the proof of (ii), (3.8.23) is a metric space equivalence, so it takes closures to closures.

(vi) If $\psi \in D(\bar{A})$, $\varphi_+ \in \ker(A^* - z)$, $\varphi_- \in \ker(A^* - \bar{z})$, and

$$\varphi_+ + \varphi_- + \psi = 0 \tag{3.8.25}$$

Then applying $(A^* - z)$ and then $(A^* - \bar{z})$, we see

$$\varphi_- = i(2\,\mathrm{Im}\,z)^{-1}\bar{A}\psi \tag{3.8.26}$$

$$\varphi_+ = -i(2\,\mathrm{Im}\,z)^{-1}\bar{A}\psi \tag{3.8.27}$$

so $\varphi_+ = -\varphi_-$, which implies $\varphi_+ = \varphi_- = 0$, and then $\psi = 0$. This proves uniqueness. If $\eta \in D(A^*)$, since

$$\operatorname{Ran}(\bar{A} - z) + \operatorname{Ran}(\bar{A} - z)^\perp = \mathcal{H}$$

and $\operatorname{Ran}(\bar{A} - z)^\perp = \ker(A^* - \bar{z})$, we can find $\psi \in D(\bar{A})$, $\varphi_- \in \ker(A^* - \bar{z})$ so that

$$(A^* - z)\eta = (\bar{A} - z)\psi + (A^* - z)\varphi_-$$

Thus, $\varphi_+ = \eta - \psi - \varphi_- \in \ker(A^* - z)$.

(vii) By (3.8.20), $D(\bar{A}) = D(A^*)$ if and only if (3.8.21) holds. □

Given *any* sequence $\{u_n\}_{n=1}^\infty$, define $\mathcal{J}u$, a new sequence, by

$$(\mathcal{J}u)_n = a_n u_{n+1} + b_n u_n + a_{n-1} u_{n-1} \tag{3.8.28}$$

where $a_0 = 0$. Define an operator, A, by

$$D(A) = \{u \mid u_n = 0 \text{ for all large } n\} \qquad Au = \mathcal{J}u \tag{3.8.29}$$

Then $A\colon D(A) \to D(A) \subset \ell_2$ is a densely defined operator.

Theorem 3.8.6. (i) *We have that for any $u \in D(A)$ and any sequence v that (both sums are finite)*

$$\sum_{n=1}^\infty \bar{v}_n (Au)_n = \sum_{n=1}^\infty \overline{(\mathcal{J}v)_n}\, u_n \tag{3.8.30}$$

(ii) *We have that*

$$D(A^*) = \{u \in \ell^2 \mid \mathcal{J}(u) \in \ell^2\} \tag{3.8.31}$$

and

$$A^* u = \mathcal{J}(u) \tag{3.8.32}$$

(iii) *If $u, v \in D(A^*)$, then*

$$\langle u, A^* v\rangle - \langle A^* u, v\rangle = -\lim_{n\to\infty} W(\bar{u}, v)(n) \tag{3.8.33}$$

where

$$W_n(f, g) = a_n(f_{n+1} g_n - f_n g_{n+1}) \tag{3.8.34}$$

(iv) *If $u, v \in D(A^*)$ and*

$$\langle u, A^* v\rangle - \langle A^* u, v\rangle \neq 0 \tag{3.8.35}$$

then both

$$u, v \in D(A^*) \setminus D(\bar{A})$$

Remark. (iii) includes the assertion that the limit exists.

Proof. (i) is a simple summation by parts.

(ii) If $u \in \ell^2$ and $\mathcal{J}(u) \in \ell^2$, then (3.8.30) proves $u \in D(A^*)$ and $A^*u = \mathcal{J}(u)$. Conversely, if $u \in D(A^*)$ and $\eta \in A^*$, then by (3.8.30), $\eta - A^*u$ is a sequence with

$$\sum_{n=1}^{\infty} \overline{(\eta - \mathcal{J}(u))_n}\, w_n = 0$$

for all $w \in D(A)$. Picking $w_n = \delta_{kn}$ shows $\eta = \mathcal{J}(u)$, proving (3.8.32) and so $\mathcal{J}(u) \in \ell^2$.

(iii) By a direct calculation,

$$\sum_{n=1}^{N} [\bar{u}_n \mathcal{J}(v)_n - \overline{\mathcal{J}(u)_n}\, v_n] = W(\bar{u}, v)_N \tag{3.8.36}$$

from which (3.8.33) is immediate.

(iv) If $u \in D(\bar{A})$, then

$$\langle A^*u, v\rangle = \langle \bar{A}u, v\rangle = \langle u, A^*v\rangle \tag{3.8.37}$$

so (3.8.35) fails; similarly, if $v \in D(\bar{A})$. □

For each $z \in \mathbb{C}$, we define two sequences, $\pi(z)$, $\xi(z)$, by

$$\begin{aligned} \pi(z)_n &= p_{n-1}(z) \\ \xi(z)_n &= q_{n-1}(z) \end{aligned} \tag{3.8.38}$$

Of course, $W(\pi, \xi)$ is constant and, by (3.2.22),

$$W(\pi, \xi)_n = -1 \tag{3.8.39}$$

Lemma 3.8.7. *If $d\mu$ solves the moment problem and*

$$m_\mu(z) = \int \frac{d\mu(x)}{x - z} \tag{3.8.40}$$

then $\xi(z) + m(z)\pi(z) \in \ell^2$ for any $z \in \mathbb{C} \setminus \mathbb{R}$.

Proof. By (3.2.24),

$$\xi_n(z) + m(z)\pi_n(z) = \langle p_{n-1}, (\cdot - z)^{-1}\rangle \tag{3.8.41}$$

So, by Bessel's inequality,

$$\sum_n |\xi_n(z) + m(z)\pi_n(z)|^2 \le \int \frac{d\mu(x)}{|x - z|^2} \tag{3.8.42}$$

$$= \frac{\operatorname{Im} m_\mu(z)}{\operatorname{Im} z} \tag{3.8.43}$$

□

Note that if $\{p_{n-1}\}_{n=1}^{\infty}$ is an orthonormal basis, we have that equality holds in (3.8.42)/(3.8.43). Here is one of the main results on the moment problem:

Theorem 3.8.8. *The following are equivalent:*
(i) *For one* $z_0 \in \mathbb{C} \setminus \mathbb{R}$, $\pi(z_0) \in \ell^2$.
(ii) *For one* $z_0 \in \mathbb{C} \setminus \mathbb{R}$, $\xi(z_0) \in \ell^2$.
(iii) *A is not essentially selfadjoint.*
(iv) *For all* $z_0 \in \mathbb{C} \setminus \mathbb{R}$, $\pi(z_0) \in \ell^2$ *and* $\xi(z_0) \in \ell^2$.
(v) *The moment problem is indeterminate.*

Remark. We will eventually show (see Theorem 3.8.15) that (iv) can be replaced by all of $\mathbb{C}$.

Proof. We will show that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) so (iii) $\Leftrightarrow$ (iv) will be automatic. We will then prove (v) $\Rightarrow$ (i). We will postpone the proof that (iii) $\Rightarrow$ (v).

<u>(i) $\Leftrightarrow$ (ii).</u> By Theorem 3.8.4, the moment problem has solutions. So for some $m_\mu(z) \neq 0$, $\xi(z_0) + m_\mu(z_0)\pi(z_0) \in \ell^2$. This implies $\pi(z_0) \in \ell^2 \Leftrightarrow \xi(z_0) \in \ell^2$.

<u>(i) $\Leftrightarrow$ (iii).</u> There is a unique sequence solving

$$\mathcal{J}u = z_0 u \tag{3.8.44}$$

and

$$u_{n=1} = 1$$

and no solution with $u_{n=1} = 0$. This is given by $u = \pi$. Thus, by Theorem 3.8.6(ii),

$$\ker(A^* - z_0) \neq \{0\} \Leftrightarrow \pi(z_0) \in \ell^2 \tag{3.8.45}$$

Since $\pi(\bar{z}_0) = \overline{\pi(z_0)}$, we see

$$\ker(A^* - z_0) \neq \{0\} \Leftrightarrow \ker(A^* - \bar{z}_0) \neq \{0\}$$

By Proposition 3.8.5(vii),

$$A \text{ essentially selfadjoint} \Leftrightarrow \pi(z_0) \notin \ell^2 \tag{3.8.46}$$

proving (i) $\Leftrightarrow$ (iii).

<u>(iii) $\Leftrightarrow$ (iv).</u> Obviously, (iv) $\Rightarrow$ (i) $\Rightarrow$ (iii). But since (iii) $\Rightarrow$ (i) for any z_0, it implies it for all z_0.

<u>Not (i) $\Rightarrow$ not (v).</u> Since $\pi(z_0) \notin \ell^2$, there is at most one $m(z_0)$ with $\xi(z_0) + m(z_0)\pi(z_0) \in \ell^2$. So for any two μ's solving the moment problem and all $z_0 \in \mathbb{C} \setminus \mathbb{R}$, $m_{\mu_1}(z_0) = m_{\mu_2}(z_0)$, so $\mu_1 = \mu_2$, that is, we have not (v). $\square$

The following depends only on (v) $\Rightarrow$ (i):

Corollary 3.8.9. *If*

$$\sum_{n=1}^{\infty} a_n^{-1} = \infty \tag{3.8.47}$$

then the moment problem is determinate. In particular, if a moment problem is indeterminate, then

$$\lim_{n\to\infty} a_n = \infty \tag{3.8.48}$$

Proof. If $\pi(z_0) \in \ell^2$, then so is $\xi(z_0)$, and thus,

$$a_n^{-1} = (q_n(z_0)p_{n-1}(z_0) - q_{n-1}(z_0)p_{n-1}(z)) \tag{3.8.49}$$

(by (3.2.22)) lies in ℓ^1. Therefore, (3.8.47) implies not (i) implies not (v). □

Lemma 3.8.10. *For any* $\{a_j\}_{j=1}^n \in \mathbb{R}^n$*, we have*

$$\sum_{j=1}^n (a_1 \dots a_j)^{-1/j} \le 2e \sum_{j=1}^n a_j^{-1} \tag{3.8.50}$$

Proof. We have $1 + x \le e^x$ so $(1 + \frac{1}{n})^n \le e$ and thus, inductively,

$$n^n \le e^n n! \tag{3.8.51}$$

It follows that

$$\begin{aligned}(a_1 \dots a_j)^{-1/j} &= [a_1^{-1}(2a_2^{-1})\dots(ja_j^{-1})]^{1/j}(j!)^{-1/j} \\ &\le ej^{-2}\sum_{k=1}^j ka_k^{-1} \end{aligned}\tag{3.8.52}$$

by the arithmetic-geometric mean inequality.

Thus,

$$\begin{aligned}\sum_{j=1}^n (a_1 \dots a_j)^{-1/j} &\le e\sum_{k=1}^n a_k^{-1}\sum_{j=k}^n \frac{k}{j^2} \\ &\le 2e\sum_{k=1}^n a_k^{-1}\end{aligned}\tag{3.8.53}$$

since

$$\sum_{j=k}^n \frac{k}{j^2} \le 2k\sum_{j=k}^\infty \frac{1}{j(j+1)} = 2 \tag{3.8.54}$$

□

Corollary 3.8.11 (Carleman's criterion)**.** *If*

$$\sum_{n=1}^\infty c_{2n}^{-1/2n} = \infty \tag{3.8.55}$$

then the moment problem is determinate.

Proof. Since $p_n(x) = (a_1 \dots a_n)^{-1}x^n+$ lower order,

$$\langle (a_1 \dots a_n)^{-1} x^n, p_n \rangle = 1 \tag{3.8.56}$$

and thus, by the Schwarz inequality,

$$c_{2n}^{-1/2n} \le (a_1 \dots a_n)^{-1/n} \tag{3.8.57}$$

By (3.8.50), we see (3.8.55) implies (3.8.47). □

Example 3.8.1, revisited. If $\alpha \ge 1$,

$$\begin{aligned} \int x^n \exp(-|x|^\alpha) &\le 2 + \int x^n \exp(-|x|^1) \\ &= 2 + 2n! \le 4n^n \end{aligned} \tag{3.8.58}$$

and

$$c_{2n}^{-1/2n} \ge \frac{1}{8n} \tag{3.8.59}$$

Thus, (3.8.55) holds, and so the moment problem is determinate. □

To get the last step in the proof of Theorem 3.8.8, we need to analyze selfadjoint extensions of A when $\bar{A} \ne A^*$, that is, operators B with $\bar{A} \subset B = B^*$. Since $\bar{A} \subset B$ implies $B^* \subset A^*$, we have

$$\bar{A} \subsetneq B = B^* \subsetneq A^* \tag{3.8.60}$$

where $B \ne \bar{A}$ and $B \ne A^*$ comes from $\bar{A} \ne A^* \ne A^{**}$. In our case where $D(A^*)/D(\bar{A})$ has dimension 2, we must thus have $\dim(D(B)/D(A)) = 1$, which simplifies the analysis.

Theorem 3.8.12. *Suppose $D(A^*)/D(\bar{A})$ has dimension 2. Then*

(i) $D(B) = D(A) + [\varphi]$ *with $\varphi \in D(A^*) \setminus D(A)$ is the domain of a selfadjoint extension (i.e., $A^* \restriction D(B)$ is selfadjoint) if and only if*

$$\langle \varphi, A^*\varphi \rangle \in \mathbb{R} \tag{3.8.61}$$

(ii) *Suppose $\varphi, \psi = D(A^*)$ with $\langle \varphi, A^*\psi \rangle$, $\langle \psi, A^*\varphi \rangle$, $\langle \varphi, A^*\varphi \rangle$, $\langle \psi, A^*\psi \rangle$ all real. Let $t \in \mathbb{R} \cup \{\infty\}$ and let*

$$\varphi_t = \frac{\varphi + t\psi}{1 + |t|} \tag{3.8.62}$$

(*where φ_∞ is interpreted as ψ*). *Then*

$$D(B_t) = D(\bar{A}) + [\varphi_t] \qquad B_t = A^* \restriction D(B_t) \tag{3.8.63}$$

describes all the selfadjoint extensions of A.

Proof. (i) By (3.8.60), $D(B)/D(A)$ is of dimension 1, so for every selfadjoint extension, B, $D(B)$ always has the claimed form. Since $\varphi \in D(B)$,

$$\langle \varphi, A^*\varphi \rangle = \langle \varphi, B\varphi \rangle \tag{3.8.64}$$

is real.

Conversely, if (3.8.61) holds and $\eta \in D(A)$, then

$$\langle \varphi + \eta, A^*(\varphi + \eta)\rangle = \langle \varphi, A^*\varphi\rangle + \langle \eta, A\eta\rangle + \langle \varphi, A\eta\rangle + \langle A\eta, \varphi\rangle \tag{3.8.65}$$

is real, so $A^* \restriction D(A) + [\varphi]$ has real expectation values. By polarization, it is Hermitian. Since $\bar{A} \subsetneqq B \subset B^* \subsetneqq A^*$, we see that $D(B^*)$ must be $D(B)$ since every subspace between $D(B)$ and $D(A^*)$ is either $D(B)$ or $D(A^*)$. Thus, $B = B^*$.

(ii) We have, for all $\eta \in D(\bar{A})$,

$$\operatorname{Im}\langle \varphi + \alpha\psi + \eta, A^*(\varphi + \alpha\psi + \eta)\rangle = (\operatorname{Im}\alpha)[\langle \varphi, A^*\psi\rangle - \langle \psi, A^*\varphi\rangle]$$

Since there is $\alpha, \beta \in \mathbb{C}$ and an $\eta \in D(\bar{A})$ with $A^*(\beta\varphi+\alpha\psi+\eta) = i(\beta\varphi+\alpha\psi+\eta)$, we conclude that $\langle \varphi, A^*\psi\rangle - \langle \psi, A^*\varphi\rangle \neq 0$. It follows that (3.8.61) holds for $\varphi+\alpha\psi$ if and only if $\alpha \in \mathbb{R}$. Given (i), this proves (ii). □

Later (see Theorem 3.8.15), we will prove that if A is not selfadjoint for the concrete Jacobi matrix, then not only is $\pi(z_0), \xi(z_0) \in \ell^2$ for $z_0 \in \mathbb{C} \setminus \mathbb{R}$ but also for $z_0 \in \mathbb{R}$. We use that for now for $z_0 = 0$. We have

$$\mathcal{J}(\pi(0)) = 0 \qquad \mathcal{J}(\xi(0)) = \delta_{\cdot 1} \tag{3.8.66}$$

so if A is the operator of $\mathcal{J}$ restricted to finite sequences, by Theorem 3.8.6(ii), we have

$$\langle \xi(0), A^*(\pi(0))\rangle = \langle \pi(0), A^*(\pi(0))\rangle = \langle \xi(0), A^*(\xi(0))\rangle = 0 \tag{3.8.67}$$

$$\langle \pi(0), A^*(\xi(0))\rangle = 1 \tag{3.8.68}$$

By Theorem 3.8.6(iv), we have $\pi(0), \xi(0) \in D(A^*) \setminus D(A)$ and, by Theorem 3.8.12, there is a one-parameter family, $\{B_t\}_{t\in\mathbb{R}\cup\{\infty\}}$, of selfadjoint extensions with

$$D(B_t) = D(\bar{A}) + [\xi(0) + t\pi(0)] \tag{3.8.69}$$

Proposition 3.8.13. *Suppose $\bar{A}$ is not essentially selfadjoint.*

(i) *For each $z_0 \in \mathbb{C} \setminus \mathbb{R}$, we have $\pi(z_0), \xi(z_0) \in D(A^*) \setminus D(A)$. For each t, there is an $a_t(z_0) \in \mathbb{C}$ so that*

$$\xi(z_0) + a_t(z_0)\pi(z_0) \in D(B_t) \tag{3.8.70}$$

and for every such z_0, all $a_t(z_0)$ are distinct as t varies.

(ii)

$$\langle \delta_1, (B_t - z_0)^{-1}\delta_1\rangle = a_t(z_0) \tag{3.8.71}$$

In particular, if $\bar{A}$ is not selfadjoint, there are multiple solutions to the moment problem.

Remark. The spectral measures for B_t, which solve the moment problem, are called the *von Neumann solutions* of the moment problem.

Proof. As noted in the proof of Theorem 3.8.8,

$$\ker(A^* - z_0) = [\pi(z_0)] \tag{3.8.72}$$

Moreover, as in (3.8.66),

$$(A^* - z_0)\xi(z_0) = \delta_{\cdot 1} \tag{3.8.73}$$

Thus, every solution of $(A^* - z_0)\eta = \delta_0$ has the form

$$\eta = \xi(z_0) + c\pi(z_0) \tag{3.8.74}$$

So for some $a_t(z_0) \in \mathbb{C}$,

$$(B_t - z_0)^{-1}\delta_1 = \xi(z_0) + a_t(z_0)\pi(z_0) \tag{3.8.75}$$

Let η_t be the right side of (3.8.75). By (3.8.33),

$$\langle \pi(\bar{z}_0), A^*\eta_t\rangle - \langle A^*\pi(\bar{z}_0), \eta_t\rangle = 1 \tag{3.8.76}$$

we conclude that $\eta_t \in D(A^*) \setminus D(A)$, so

$$D(B_t) = D(\bar{A}) + [\eta_t] \tag{3.8.77}$$

which implies that the η_t are distinct for distinct t.

Finally, by (3.8.75),

$$\langle \delta_1, (B_t - z_0)^{-1}\delta_1\rangle = a_t(z_0) \tag{3.8.78}$$

proving (3.8.71). $\square$

Next, we turn to the claim that in the indeterminate case, $\pi(z_0), \xi(z_0) \in \ell^2$ also for $z_0 \in \mathbb{R}$. We depend on a useful general perturbation theorem.

Theorem 3.8.14. *Suppose $\{A_j\}_{j=1}^\infty$ and $\{\tilde{A}_j\}_{j=1}^\infty$ are two sequences of bounded operators with bounded inverses, and define*

$$T_n = A_n \dots A_1 \tag{3.8.79}$$

$$\tilde{T}_n = \tilde{A}_n \dots \tilde{A}_1 \tag{3.8.80}$$

$$B_k = T_k^{-1}(\tilde{A}_k - A_k)T_{k-1} \tag{3.8.81}$$

where $T_0 = \tilde{T}_0 = \mathbf{1}$. Then

(i) *We have for each n,*

$$\|T_n^{-1}\tilde{T}_n\| \le \exp\left(\sum_{j=1}^n \|B_j\|\right) \tag{3.8.82}$$

(ii) *If*

$$\sum_{n=1}^\infty \|B_n\| < \infty \tag{3.8.83}$$

then

$$\lim_{n\to\infty} T_n^{-1}\tilde{T}_n \tag{3.8.84}$$

exists and is given by

$$\lim_{n\to\infty} T_n^{-1}\tilde{T}_n = \mathbf{1} + \sum_{j=1}^\infty B_j T_{j-1}^{-1}\tilde{T}_{j-1} \tag{3.8.85}$$

(iii) *If*

$$\sum_{n=1}^{\infty} \|T_n\|^2 < \infty \tag{3.8.86}$$

and (3.8.83) *holds, then*

$$\sum_{n=1}^{\infty} \|\tilde{T}_n\|^2 < \infty \tag{3.8.87}$$

Remark. By (3.8.81) and (3.8.85), we get

$$\lim_{n\to\infty} T_n^{-1}\tilde{T}_n = \mathbf{1} + \sum_{j=1}^{\infty} T_j^{-1}(\tilde{A}_j - A_j)\tilde{T}_{j-1} \tag{3.8.88}$$

Proof. Noticing that

$$T_k^{-1} A_k T_{k-1} = \mathbf{1} \tag{3.8.89}$$

we have

$$T_k^{-1} \tilde{A}_k T_{k-1} = \mathbf{1} + B_k \tag{3.8.90}$$

Therefore,

$$\begin{aligned} T_n^{-1}\tilde{T}_n &= (T_n^{-1}\tilde{A}_n T_{n-1})(T_{n-1}^{-1}\tilde{A}_{n-1}T_{n-2})\dots(T_1^{-1}\tilde{A}_1 T_0) \\ &= (\mathbf{1} + B_n)\dots(\mathbf{1} + B_1) \end{aligned} \tag{3.8.91}$$

(i) Thus,

$$\|T_n^{-1}\tilde{T}_n\| \le \prod_{j=1}^{n}(1 + \|B_j\|) \le \exp\left(\sum_{j=1}^{n} \|B_j\|\right) \tag{3.8.92}$$

(ii) By (3.8.91), we have

$$T_n^{-1}\tilde{T}_n = \mathbf{1} + \sum_{j=1}^{n} B_j(\mathbf{1} + B_{j-1})\dots(\mathbf{1} + B_1) \tag{3.8.93}$$

$$= \mathbf{1} + \sum_{j=1}^{n} B_j T_{j-1}^{-1}\tilde{T}_{j-1} \tag{3.8.94}$$

By (3.8.82),

$$\|B_j T_{j-1}^{-1}\tilde{T}_{j-1}\| \le \|B_j\| \exp\left(\sum_{k=1}^{\infty} \|B_k\|\right) \tag{3.8.95}$$

so the sum is absolutely convergent, implying that the limit exists and is given by (3.8.85).

(iii) By (3.8.82),

$$\|\tilde{T}_n\| \le \|T_n\| \exp\left(\sum_{j=1}^{\infty} \|B_j\|\right) \tag{3.8.96}$$

so (3.8.86) implies (3.8.87). □

To apply this to moment problems, $T_n, A_n, \dots$ will be 2×2 transfer matrices, but we will want to modify from the definition in Section 3.2. There we added an a_n to the lower component of vectors to get a transfer matrix of determinant one. With a_n's bounded from above, this is normally harmless, but here our a_n's are unbounded so we will modify. Given Jacobi parameters $\{a_n, b_n\}_{n=1}^\infty$, we define (with $a_0 \equiv 1$) for this section only,

$$A_n(z) = \begin{pmatrix} \frac{z-b_n}{a_n} & \frac{-a_{n-1}}{a_n} \\ 1 & 0 \end{pmatrix} \tag{3.8.97}$$

so

$$\begin{pmatrix} p_n(z) \\ p_{n-1}(z) \end{pmatrix} = A_n(z) \begin{pmatrix} p_{n-2}(z) \\ p_{n-2}(z) \end{pmatrix} \tag{3.8.98}$$

and

$$T_n(z) = \begin{pmatrix} p_n(z) & -q_n(z) \\ p_{n-1}(z) & -q_{n-1}(z) \end{pmatrix} \tag{3.8.99}$$

to be compared with (3.2.19). Now $\det(T_n) \neq 1$ but rather

$$\det(T_n) = a_n^{-1} \tag{3.8.100}$$

and thus,

$$T_n(z)^{-1} = a_n \begin{pmatrix} -q_{n-1}(z) & q_n(z) \\ -p_{n-1}(z) & p_n(z) \end{pmatrix} \tag{3.8.101}$$

Our perturbation will be to change z to w, so

$$A_n(w) - A_n(z) = \begin{pmatrix} \frac{w-z}{a_n} & 0 \\ 0 & 0 \end{pmatrix} \tag{3.8.102}$$

and

$$B_n \equiv T_n(z)^{-1}(A_n(w) - A_n(z))T_{n-1}(z) \tag{3.8.103}$$

The a_n in (3.8.101) and the a_n^{-1} in (3.8.102) cancel! Thus, with

$$N_n(z) = (|p_n(z)|^2 + |p_{n-1}(z)|^2 + |q_n(z)|^2 + |q_{n-1}(z)|^2)^{1/2} \tag{3.8.104}$$

we obtain

$$\|B_n\| \le |w - z| \, N_n(z) N_{n-1}(z) \tag{3.8.105}$$

and by the Schwarz inequality,

$$\sum_{n=1}^\infty N_n(z)^2 < \infty \Rightarrow \sum_{n=1}^\infty \|B_n\| < \infty \tag{3.8.106}$$

Thus, we can apply Theorem 3.8.14 and find

Theorem 3.8.15. *If $\pi(z), \xi(z)$ are both in ℓ^2 for a single z, then $\pi(w), \xi(w)$ are in ℓ^2 for any $w \in \mathbb{C}$ and*

$$\lim_{n\to\infty} T_n(z)^{-1} T_n(w) \tag{3.8.107}$$

exists.

One defines four functions, $A(z)$, $B(z)$, $C(z)$, and $D(z)$, by

$$\lim_{n\to\infty} T_n^{-1}(z)T_n(w=0) = \begin{pmatrix} -B(z) & -A(z) \\ D(z) & C(z) \end{pmatrix} \tag{3.8.108}$$

and the Nevanlinna matrix by

$$N(z) = \begin{pmatrix} A(z) & C(z) \\ B(z) & D(z) \end{pmatrix} \tag{3.8.109}$$

By (3.8.88), (3.8.99), (3.8.101), and (3.8.102), we get

Proposition 3.8.16. *The Nevanlinna matrix is given by*

$$A(z) = z\sum_{n=0}^{\infty} q_n(0)q_n(z) \tag{3.8.110}$$

$$B(z) = -1 + z\sum_{n=0}^{\infty} q_n(0)p_n(z) \tag{3.8.111}$$

$$C(z) = 1 + z\sum_{n=0}^{\infty} p_n(0)q_n(z) \tag{3.8.112}$$

$$D(z) = z\sum_{n=0}^{\infty} p_n(0)p_n(z) \tag{3.8.113}$$

These functions are entire functions obeying

$$|A(z)| \le C_\varepsilon \exp(\varepsilon|z|) \tag{3.8.114}$$

and similarly for B, C, D. Near $z=0$,

$$B(z) = -1 + O(z) \tag{3.8.115}$$

$$D(z) = D_0 z + O(z^2) \tag{3.8.116}$$

where

$$D_0 > 0 \tag{3.8.117}$$

Proof. The formulae follow from the earlier equations. (3.8.115) is immediate, as is (3.8.116) where

$$D_0 = \sum_{n=0}^{\infty} p_n(0)^2 > 0 \tag{3.8.118}$$

To get (3.8.114), we note that

$$B_k(z) = zb_k \tag{3.8.119}$$

with b_k a constant matrix with $\sum_{k=1}^{\infty}\|b_k\| < \infty$. Thus,

$$\|(1+B_N)\dots(1+B_k)\| \le \prod_{j=1}^{n}(1+|z|\,\|b_j\|)\exp\left(|z|\sum_{j=n+1}^{N}\|b_j\|\right) \tag{3.8.120}$$

from which (3.8.114) follows. □

We can express the resolvent of the selfadjoint extensions, B_t, in terms of the Nevanlinna matrix:

Theorem 3.8.17. *Consider an indeterminate moment problem. For $t \in \mathbb{R} \cup \{\infty\}$ and $z \in \mathbb{C} \setminus \mathbb{R}$, the resolvent of the selfadjoint extensions, B_t, is given by*

$$(\delta_1, (B_t - z)^{-1}\delta_1) \equiv F(z,t) \tag{3.8.121}$$

where for $z, w \in \mathbb{C}$,

$$F(z,w) \equiv -\frac{C(z)w + A(z)}{D(z)w + B(z)} \tag{3.8.122}$$

Proof. Given a sequence, $\{s_n\}_{n=1}^{\infty}$, we let $R_n(s) \in \mathbb{C}^2$ be defined by

$$R_n(s) = (s_{n+1}, s_n) \tag{3.8.123}$$

and we define $w_n : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ by

$$w_n((\alpha, \beta), (\gamma, \delta)) = a_n(\alpha\delta - \beta\gamma) \tag{3.8.124}$$

so that

$$W_n(f, g) = w_n(R_n(f), R_n(g)) \tag{3.8.125}$$

Constancy of the Wronskian for solutions of the same difference equation shows that for any $z \in \mathbb{C}$ and $u, v \in \mathbb{C}^2$,

$$w_n(T_n(z)u, T_n(z)v) = w_0(u, v) \tag{3.8.126}$$

By (3.8.33), if $f, g \in D(B_t)$, then

$$\lim_{n\to\infty} w_n(\overline{R_n(f)}, R_n(g)) = 0 \tag{3.8.127}$$

since $\langle f, B_t g\rangle = \langle B_t f, g\rangle$ by Hermiticity of B_t.

Since

$$R_n(\xi(0) + t\pi(0)) = T_n(0)\begin{pmatrix} t \\ 1 \end{pmatrix} \tag{3.8.128}$$

$$R_n(\xi(z_0) + a_t(z_0)\pi(z_0)) = T_n(z_0)\begin{pmatrix} a_t(z_0) \\ 1 \end{pmatrix} \tag{3.8.129}$$

we see, by (3.8.127), that

$$\lim_{n\to\infty} w_n\left(T_n(0)\begin{pmatrix} t \\ 1 \end{pmatrix}, T_n(z_0)\begin{pmatrix} a_t(z_0) \\ 1 \end{pmatrix}\right) = 0 \tag{3.8.130}$$

So, by (3.8.126),

$$\lim_{n\to\infty} w_0\left(\begin{pmatrix} t \\ 1 \end{pmatrix}, T_n(0)^{-1}T_n(z_0)\begin{pmatrix} a_t(z_0) \\ 1 \end{pmatrix}\right) = 0 \tag{3.8.131}$$

By the existence of the limit, for some constant c,

$$\begin{pmatrix} a_t(z_0) \\ 1 \end{pmatrix} = cT_n(z_0)^{-1}T_n(0)\begin{pmatrix} t \\ 1 \end{pmatrix} \tag{3.8.132}$$

Given (3.8.108), this implies (3.8.121)/(3.8.122). □

Lemma 3.8.18. *For $z \in \mathbb{C}_+$, $\{F(z,t) \mid t \in \mathbb{R} \cup \{\infty\}\}$ is a circle in the upper complex plane. $F(z,\,\cdot\,)$ maps $\mathbb{C}_+$ to the interior of the disk bounded by the circle.*

Proof. By (3.8.121), F maps $\mathbb{R} \cup \{\infty\}$ to $\mathbb{C}_+$ and not to ∞, so the image is a circle in $\mathbb{C}$. Suppose $[F(z,\,\cdot\,)]^{-1}(\infty)$ lies in $\mathbb{C}_-$. Then $F(z,\,\cdot\,)$ maps $\mathbb{C}_-$ to the outside of the circle, and so $\mathbb{C}_+$ to the inside. Since, for $z \in \mathbb{C}_+$, it can never lie in $\mathbb{R}$, by continuity, $[F(z,\,\cdot\,)]^{-1}(\infty)$ is either always in $\mathbb{C}_-$ (or always in $\mathbb{C}_+$), so it suffices to show this for $z = i\varepsilon$, that is, that $\mathrm{Im}(-B(i\varepsilon)/D(i\varepsilon)) < 0$ for ε small and positive. This follows from (3.8.120)/(3.8.121). $\square$

Next, we relate solutions of the moment problem to asymptotics of the Stieltjes transform.

Proposition 3.8.19. *Let μ be a probability measure on $\mathbb{R}$ solving* (3.8.1) *and let*

$$G_\mu(z) = \int \frac{d\mu(x)}{x - z} \tag{3.8.133}$$

Let

$$R_N(\mu; iy) = G_\mu(iy) + \sum_{n=0}^{N} (-i)^{n+1} y^{-n-1} c_n \tag{3.8.134}$$

Then

$$|R_N(\mu; iy)| \le \begin{cases} c_{N+1} y^{-N-2} & N \text{ odd} \\ \frac{1}{2}(c_N + c_{N+2}) y^{-N-2} & N \text{ even} \end{cases} \tag{3.8.135}$$

Conversely, if $G(z)$ is a Herglotz function, so R_N, given by (3.8.134)*, is $O(y^{-N-2})$ for each N, then G is given by* (3.8.133) *for some measure μ solving* (3.8.1).

Proof. If (3.8.133) holds and μ obeys (3.8.1), write

$$(x - iy)^{-1} = \sum_{n=0}^{N} x^n (-i)^n y^{-n-1} + (-i)^{-N-1} x^{N+1} y^{-N-2} \left(1 - \frac{x}{iy}\right)^{-1} \tag{3.8.136}$$

to see that R_N, given by (3.8.134), is given by

$$R_N(\mu; iy) = (-i)^{N+1} y^{-N-2} \int x^{N+1} \left(1 - \frac{x}{iy}\right)^{-1} d\mu(x) \tag{3.8.137}$$

Since $|1 - \frac{x}{iy}| \ge 1$ for x, y real, the N odd case of (3.8.135) is immediate. For N even, use the fact that for such N,

$$|x|^{N+1} \le \tfrac{1}{2} x^N + x^{N+2} \tag{3.8.138}$$

For the converse, start with the Herglotz representation, (2.3.87). Since (3.8.134)/(3.8.135) imply

$$\lim_{y \to \infty} |y|^{-1} |G(iy)| = 0 \tag{3.8.139}$$

we see that $A = 0$. They also imply that

$$yG(iy) \to ic_0 \tag{3.8.140}$$

from which one first sees (with ρ replaced by μ)

$$\int d\mu(x) = c_0 \tag{3.8.141}$$

since

$$\operatorname{Im} yG(iy) = \int \frac{y^2}{x^2+y^2}\, d\mu(x) \tag{3.8.142}$$

and we can use the monotone convergence theorem, and then that there is a cancellation of real parts that implies (3.8.133).

From (3.8.134)/(3.8.135), one sees inductively, using (3.8.136), that

$$\int \frac{(iy)^2 x^{2n-1}}{x-iy}\, d\mu(x) + i\gamma c_{2n-1} \to c_{2n} \tag{3.8.143}$$

which implies, taking real and imaginary parts, that

$$c_{2n} = \lim_{y\to\infty} \int \frac{y^2 x^{2n}}{x^2+y^2}\, d\mu(x) \tag{3.8.144}$$

$$c_{2n-1} = \lim_{y\to\infty} \int \frac{y^2 x^{2n-1}}{x^2+y^2}\, d\mu(x) \tag{3.8.145}$$

Monotone convergence and the first of these implies $\int x^{2n}\, d\mu = c_{2n}$ and then dominated convergence and (3.8.145) implies $\int x^{2n-1}\, d\mu = c_{2n-1}$. $\square$

Corollary 3.8.20. *For $z \in \mathbb{C}_+$, let*

$$\mathbb{D}(z) = \{F(z,w) \mid \operatorname{Im} w > 0\} \tag{3.8.146}$$

be the disk of Lemma 3.8.18. If G has the form (3.8.134) *where μ solves* (3.8.1), *then*

$$G(z) \in \overline{\mathbb{D}(z)} \tag{3.8.147}$$

for all $z \in \mathbb{C}_+$. Conversely, if G is an analytic function on $\mathbb{C}_+$ obeying (3.8.150), *then G has the form* (3.8.133) *for some μ obeying* (3.8.1).

Proof. By Proposition 3.8.19, $G_\mu(iy)$ has an asymptotic series

$$G(iy) \sim -\sum_{n=0}^{\infty} (-i)^{n+1} y^{-n-1} c_n \tag{3.8.148}$$

uniformly in the von Neumann solutions. Since these solutions fill out the circle at the boundary of $\overline{\mathbb{D}(z)}$, the estimates hold in all on $\overline{\mathbb{D}(z)}$, so G solves the moment problem by Proposition 3.8.19.

Conversely, by (3.8.43), if μ solves the moment problem,

$$G_\mu(z) \in \overline{\Delta(z)} \tag{3.8.149}$$

where

$$\overline{\Delta(z_0)} = \left\{ w \,\middle|\, \|\xi(z_0) + w\pi(z_0)\|^2 \le \frac{\operatorname{Im} w}{\operatorname{Im} z_0} \right\}$$

This set is given by a quadratic inequality in $\operatorname{Re} w$, $\operatorname{Im} w$ whose quartic term is $|w|^2\|\pi(z_0)\|^2$. Such a set always describes a disk or the empty set. Since equality holds in (3.8.43) for von Neumann solutions, $\partial\Delta(z) = \partial\mathbb{D}(z)$, so $\Delta(z) = D(z)$ and (3.8.149) is (3.8.147). $\square$

Here is the main result on the description of the solutions of the moment problem in the indeterminate case:

Theorem 3.8.21 (Nevanlinna's Parametrization)**.** *Let $\{c_n\}_{n=1}^\infty$ be the moments of an indeterminate problem, and let A, B, C, D be the elements of the Nevanlinna matrix, and F given by* (3.8.122). *There is a one-one correspondence between $\mathcal{H}$, the set of all analytic functions, φ, of $\mathbb{C}_+$ to $\overline{\mathbb{C}}_+$ so that μ_φ is given by*

$$\int \frac{d\mu_\varphi(x)}{x-z} = F(z,\varphi(z)) \tag{3.8.150}$$

The von Neumann solutions correspond to $\varphi(z) \equiv t$ and all other solutions have $\operatorname{Ran}(\varphi) \subset \mathbb{C}_+$.

Proof. Any function of the form $G(z) \equiv F(z,\varphi(z))$ has G obeying (3.8.147) by Lemma 3.8.18. Conversely, if G obeys (3.8.150), then, because $F(z,\,\cdot\,)$ is a bijection of $\mathbb{C}$ taking $\mathbb{C}_+$ to $\mathbb{D}(z)$, there is a unique φ obeying $G(z) = F(z,\varphi(z))$ with φ analytic or infinite.

By the open mapping theorem, either $\varphi(z) = t \in \mathbb{R}\cup\{\infty\}$ or $\operatorname{Ran}(\varphi) \in \mathbb{C}_+$. Given Corollary 3.8.20, this proves the theorem. $\square$

This allows further analysis of solutions, of which the following is typical:

Theorem 3.8.22. (i) *The von Neumann solutions of an indeterminate moment problem are discrete pure point measures.*

(ii) *If φ is a rational Herglotz function, $d\mu_\varphi$ is pure point.*

(iii) *The positions of the pure points and weights of the von Neumann solutions are real analytic in t. The positions are nonconstant.*

(iv) *There are always purely a.c. and purely s.c. solutions of an indeterminate problem.*

Proof. (i), (ii) In these cases, G_μ has an analytic continuation to an entire meromorphic function.

(iii) This follows from analyticity of A, B, C, D and the form of $F(z,t)$.

(iv) If $d\mu_t$ is the von Neumann solution associated to B_t and $d\nu(t)$ is a probability measure, then

$$d\eta_\nu(x) = \int d\mu_t(x)\,d\nu(t) \tag{3.8.151}$$

is a solution of the moment problem. By (iii), $d\eta_\nu$ is a.c. (resp. s.c.) if $d\nu$ is a.c. (resp. s.c.). $\square$

Remarks and Historical Notes. The critical paper on the moment problem is by Stieltjes [422]. Earlier, Chebyshev had asked about uniqueness for Gaussian measures. The approach via selfadjoint operators was pioneered by Stone [423] and the transfer matrix connection was exploited especially by Simon [395], which we follow in much of this section. For other presentations, see Akhiezer [13] and Shohat–Tamarkin [385]. The name von Neumann solutions comes from Simon [395], after von Neumann's theory of selfadjoint extensions. Such solutions are called N-extremal in Akhiezer [13].

The Nevanlinna parametrization is from [325]. A further result (see [13, 395]) is that the polynomials are dense in $L^2(\mathbb{R}, d\mu)$ if and only if $d\mu$ is a von Neumann solution and their closure has finite codimension if and only if the Nevanlinna function, φ, is rational. All these solutions are extreme points in the convex set of solutions of the moment problem, proving that the extreme points are dense.

Carleman's criterion (Corollary 3.8.11) is due to Carleman [75].

The awkward terminology (at least in English) "determinate" and "indeterminate" comes from the French. While Stieltjes was Dutch, his paper [422] is in French.

There are actually two moment problems discussed in the next section: what we have called "the moment problem" (i.e., solution of (3.8.1) with the measure allowed to be supported anywhere on $\mathbb{R}$) is more properly the *Hamburger moment problem*. The *Stieltjes moment problem* is the problem one gets by restricting to measures supported on $[0, \infty)$.

There is a simple relation between the two problems. Let $d\rho_0$ be a probability measure on $[0, \infty)$ with moments c_n. Define $d\tilde{\rho}_0$ on $\mathbb{R}$ by

$$d\tilde{\rho}_0(x) = \tfrac{1}{2}[\chi_{[0,\infty)}(x)\, d\rho(x^2) + \chi_{(-\infty,0]}(x)\, d\rho(x^2)] \tag{3.8.152}$$

and let

$$\Gamma_n = \int x^n\, d\tilde{\rho}_0(x) = \begin{cases} 0 & n \text{ odd} \\ c_{n/2} & n \text{ even} \end{cases} \tag{3.8.153}$$

(3.8.152) sets up a one-one correspondence between all solutions of the Stieltjes moment problem with moments c_n and all solutions of the Hamburger moment problem with moments Γ_n symmetric under $x \to -x$. It is a basic fact that any indeterminate Hamburger moment problem with vanishing odd moments has multiple solutions that are invariant under $x \to -x$, namely, the von Neumann solutions with $t = 0$ and $t = \infty$. This implies immediately that

Theorem 3.8.23. *Let* $(d\rho_0, c_n)$ *be a measure and set of moments on* $[0, \infty)$. *Let* $(d\tilde{\rho}_0, \Gamma_n)$ *be given by* (3.8.152)/(3.8.153). *Then the Stieltjes moment problem for* $\{c_n\}$ *is determinate* (*resp. indeterminate*) *if and only if the Hamburger moment problem for* $\{\Gamma_n\}$ *is determinate* (*resp. indeterminate*).

Theorem 3.8.23 goes back at least to Chihara [83] and appears also in Berg [42] and Simon [395].