$H^n(M) = H^n(N) = \mathbb{Z}$ for the homology groups. Any continuous $f\colon M \to N$ induces a map $H^n(f)\colon H^n(M) \to H^n(N)$, which is a group homomorphism, and so of the form $k \to Dk$ for some $D \in \mathbb{Z}$, called the *degree*, $\deg(f)$, of f.

Now let f be a C^∞ map. A point $m \in M$ is called a *regular point* if df_m, the derivative of f at m, is nonsingular. A point $n \in N$ is called a *regular value* if each point in $f^{-1}(n)$ is a regular point. In particular, if $f^{-1}(n)$ is empty, n is regular. By compactness and the inverse function theorem, each regular value has $f^{-1}(n)$, a finite set. Sard's theorem asserts the set of regular values is the complement of a set of measure zero.

If m is a regular value, the signature of f at m, $S_m(f)$, is the sign of $\det(f)$. (In general, this requires one to pick orientations on M and N as does determining the sign of $\deg(f)$; if $M = N$, making the two orientations the same fixes signs.) The fundamental theorem of degree theory says that for any regular value, n,

$$\sum_{m\in f^{-1}(n)} S_m(f) = \deg(f) \tag{5.12.75}$$

In particular, if $f^{-1}(n)$ is empty, $\deg(f) = 0$, and then regular points with $f^{-1}(n) \neq \emptyset$ must have an even number of points to get the sum of ± 1 to be 0. So if f is one-one, the degree is ± 1, and so f is onto, as claimed.

In the case studied in this section for f meromorphic on $\mathcal{S}$, f maps $\mathcal{S}$ to $\mathcal{S}_{\mathrm{R}}$, the Riemann sphere, and the topological degree is the degree as we have defined it. Analytic functions, f, where nonsingular, are conformal and so have signature $+1$ and (5.12.75) and (5.12.20) agree at points, a, for which $n(f; z, a) = 0$ or 1 for all z.

For expositions of degree theory for smooth maps, see Fonseca–Gangbo [137], Guillemin–Pollack [189], Krawcewicz–Wu [247], Lloyd [283], Milnor [305], and Spivak [416].

5.13 MINIMAL HERGLOTZ FUNCTIONS AND ISOSPECTRAL TORI

In Section 5.2, we saw the m-function, $m(z)$, for a periodic Jacobi matrix, J, with essential spectrum an ℓ-gap set, $\mathfrak{e}$, has a meromorphic continuation to $\mathcal{S}_\mathfrak{e}$. From the point of view of the last section, we will see m has some simple properties. And it will turn out that the study of all J's that lead to a fixed $\mathfrak{e}$ is related to the study of functions with these properties.

Theorem 5.13.1. *m is a meromorphic function on $\mathcal{S}_\mathfrak{e}$ with the following properties:*

(i) *m is Herglotz in the sense that if* $\operatorname{Im} z > 0$,

$$\operatorname{Im} m(z_+) > 0 \tag{5.13.1}$$

that is, $\operatorname{Im} m > 0$ *on* $\mathcal{S}_+ \cap \mathbb{C}_+$.

(ii) *On $\mathcal{S}_+$ near ∞_+,*

$$m(z) = -\frac{1}{z} + O\left(\frac{1}{z^2}\right) \tag{5.13.2}$$

(iii) *m has degree* $\ell + 1$.
(iv) *m has one zero and one pole on each set* $\{G_j\}_{j=1}^{\ell}$ *and, moreover, a zero at* ∞_+ *and a pole at* ∞_-.

Proof. (i) and (ii) hold for any m-function; see Example 2.3.1 and (2.3.10).

By Theorem 5.2.1, $m(z)$ obeys the quadratic equation

$$\alpha(z)m(z)^2 + \beta(z)m(z) + \gamma(z) = 0 \tag{5.13.3}$$

where

$$\alpha(z) = a_p p_{p-1}(z) \tag{5.13.4}$$

and the discriminant is $\Delta(z)^2 - 4$. Thus,

$$m(z) = -\frac{\beta(z) \pm \sqrt{\Delta^2(z) - 4}}{2\alpha(z)} \tag{5.13.5}$$

$m(z)$ clearly has a meromorphic continuation to all of $\mathcal{S}$ since $\sqrt{\Delta^2 - 4}$ has branch points precisely at the edges of open gaps (the double zeros of $\Delta^2 - 4$ at closed gaps are not branch points) with the only possible poles at $\infty_\pm$ and at the zeros of $p_{p-1}(x)$.

These zeros are analyzed in Theorem 5.4.16: one occurs in each gap. If the gap is closed, $\Delta^2 - 4$ has a double zero, and since that means $\beta^2 - \alpha\gamma = 0$ and $\alpha = 0$, we have $\beta = 0$. So, in (5.13.5), α has a simple zero and the numerator is also zero. So (as also remarked in Proposition 5.10.2), m has neither zero nor pole at the closed gaps.

If a gap is not closed and the zero is at the interior point of the gap, z_0, then $\alpha(z)$ has a simple zero at z_0. Since $\Delta(z_0) \neq \pm 2$, $\beta^2 - \alpha\gamma \neq 0$, so $\beta(z_0) \neq 0$. Thus,

$$-\beta(z) \pm \sqrt{\beta^2(z) - \alpha(z)\gamma(z)}$$

vanishes at one of $(z_0)_\pm$ and is nonzero (indeed, $-2\beta(z_0)$) at the other point. So m has a single pole on one sheet or the other, but not both.

If the pole is at a resonance, that is, at an edge, z_0, of a closed gap, $\Delta^2 - 4$ has a simple zero at z_0 and $\beta(z_0)^2 = \alpha(z_0)\gamma(z_0) + (\Delta^2(z_0) - 4) = 0$. Thus, $\beta(z_0) = c(z - z_0) + O((z - z_0)^2)$ while $\sqrt{\Delta^2 - 4} = c(z - z_0)^{1/2} + O((z - z_0)^{3/2})$ and $m(z) = c(z - z_0)^{-1/2} + O(1)$, so by the way poles are counted at branch points, m has a simple pole at z_0. We have thus proven $m(z)$ has exactly one pole in each G_j, $j = 1, \dots, \ell$.

By coefficient stripping (see (3.2.28)),

$$m(z)^{-1} = b_1 - z - a_1^2 m_1(z) \tag{5.13.6}$$

Since m_1 is also the m-function of a periodic Jacobi matrix, m_1 has one pole in each gap, and so m has exactly one zero in each two-sheeted gap.

Besides zeros of α, the only other possible poles of $m(z)$ are at $\infty_\pm$. At ∞_+, m is zero by (5.13.2). Thus, since $\alpha(z) \sim c_1 z^{p-1}$, $\beta(z) \sim c_2 z^p$, and $\Delta^2(z) \sim z^{2p}$, we must have $\beta(z)$ cancelling the z^p growth of $\sqrt{\Delta^2 - 4}$ at ∞_+. That means at ∞_-, the numerator is $-2c_2 z^p + O(z^{p-1})$ and so, $m(z)$ has a simple pole at ∞_-.

We have thus proven m has exactly $\ell + 1$ simple poles, so m has degree $\ell + 1$. Since we have accounted for $\ell + 1$ zeros of m, we have them all. □

This leads to a natural definition in the context of general finite gap sets, not just those that are periodic spectra.

Definition. Let $\mathfrak{e}$ be a finite gap subset of $\mathbb{R}$ and let $\mathcal{S}_\mathfrak{e}$ be the associated Riemann surface. A *minimal Herglotz function* on $\mathcal{S}_\mathfrak{e}$ is a meromorphic function m on $\mathcal{S}_\mathfrak{e}$ obeying:

(i) m is Herglotz in the sense that (5.13.1) holds for $z \in \mathcal{S}_+ \cap \mathbb{C}_+$ and $\operatorname{Im} m(x_+ + i0)$ has compact support.

(ii) m obeys (5.13.2) (so m is a discrete m-function in the sense of Section 2.3).

(iii) $\deg(m) = \ell + 1$.

(iv) m has a pole at ∞_-.

Remark. The word minimal is used because m has minimal degree among non-square root-free functions.

The set of all minimal Herglotz functions on $\mathcal{S}_\mathfrak{e}$ will be denoted by $\mathcal{M}_\mathfrak{e}$. We will show first that $\mathcal{M}_\mathfrak{e}$ is a torus of dimension ℓ; indeed, naturally associated to the torus $\mathbb{T}_\mathfrak{e}$ of (5.12.57). We will then study the Jacobi matrix associated to an m in $\mathcal{M}_\mathfrak{e}$ and prove, for general $\mathfrak{e}$, it is almost periodic, and if $\mathfrak{e}$ comes from one periodic Jacobi matrix, then all the minimal Herglotz functions associated to $\mathfrak{e}$ have associated periodic Jacobi matrices and have the same Δ. This will provide the promised proof that the set of periodic J's with a given Δ is a torus.

Here is the general structure of minimal Herglotz functions:

Theorem 5.13.2. *Every minimal Herglotz function, m, in $\mathcal{M}_\mathfrak{e}$ has the form*

$$m(z) = \frac{p(z) \pm \sqrt{R(z)}}{a(z)} \tag{5.13.7}$$

where

$$\operatorname{Deg}(a) = \ell \tag{5.13.8}$$

$$\operatorname{Deg}(p) = \ell + 1 \tag{5.13.9}$$

and $-p$ is monic. Moreover,

(i) *p and a are real polynomials.*

(ii) *a has one simple zero in each gap.*

(iii) *m has exactly one simple pole in each gap plus the pole at ∞_-.*

(iv) *m has exactly one simple zero in each gap plus the zero at ∞_+.*

Remarks. 1. A polynomial is called *real* if all its coefficients are real.

2. In the periodic case with closed gaps, a is not the 2α of (5.13.5) but it has zeros at closed gaps that occur in the numerator removed. In addition, even if all gaps are open and $\Delta^2 - 4$ has simple zeros, it is not R, but rather $(a_1 \dots a_p)^{-2} R$.

Proof. As a rational function on $\mathcal{S}$, m has the form

$$m(z) = \frac{p(z) \pm q(z)\sqrt{R(z)}}{a(z)} \tag{5.13.10}$$

By (5.12.23) and $\deg(m) = \ell + 1$, we see $\deg(q) = 0$, so we can take $q = 1$. Also by (5.12.23), $\deg(a) \le \ell + 1$. Since (5.13.2) holds, and on $\mathcal{S}_+$,

$$+\sqrt{R(z)} = z^{\ell+1} + O(z^\ell) \tag{5.13.11}$$

near ∞_+, we must have that

$$p(z) = -z^{\ell+1} + O(z^\ell) \tag{5.13.12}$$

(since $\deg(a) \le \ell + 1$ means the $z^{\ell+1}$ term in the numerator must cancel). Thus, $-p$ is monic and (5.13.9) holds.

Since $-\sqrt{R(z)}$ (i.e., $\sqrt{R(z)}$ on $\mathcal{S}_-$) has the opposite sign, near ∞_-,

$$p(z) \pm \sqrt{R(z)} = -2z^{\ell+1} \tag{5.13.13}$$

so to have a pole at ∞_-, we must have

$$\mathrm{Deg}(a) \le \ell \tag{5.13.14}$$

Since $m(z)$ is real on $(\beta_{\ell+1}, \infty)$ and $\sqrt{R(z)}$ is real there, $p(z)/a(z)$ is real there. So, by analyticity, all its zeros and poles come in conjugate pairs or lie on $\mathbb{R}$. Since $-p$ is monic, we see p and then a is real.

On each band, p/a is real, so

$$\mathrm{Im}\, m(x_+ + i0) = \frac{\mathrm{Im}\sqrt{R(x_+ + i0)}}{a(x + i0)} \tag{5.13.15}$$

Since $\sqrt{R(x)}$ changes sign from one band to the next, a must change sign to keep $\mathrm{Im}\, m(x_+ + i0) \ge 0$. Thus, a has an odd number of zeros in each gap.

Since there are ℓ gaps and, by (5.13.14), at most ℓ zeros, we conclude each gap has precisely one zero and (5.13.8) holds.

As in the analysis in the proof of Theorem 5.13.1, if a has a zero at a point, z_0, in the interior of a gap where $R(z_0) \ne 0$, m must have a pole at either $(z_0)_+$ or $(z_0)_-$ (or both), and if a has a zero at a band edge, z_0, $p(z) \pm \sqrt{R(z)}$ vanishes at $(z - z_0)^{1/2}$ or approaches a constant. Thus, in that case also, m has a pole at z_0. Thus, m has at least one pole in each gap, and so since ∞_- is a pole and there are only $\ell + 1$ poles, we see each gap has exactly one simple pole.

Define m_1 by (5.13.6) where b_1, a_1 are picked so $m_1(z)$ obeys (5.13.2). By coefficient stripping, m_1 is a Herglotz function and clearly, m_1 is meromorphic on $\mathcal{S}$. m_1 has a pole at each finite zero of m and, by $\deg(m) = \ell + 1$ and the fact that ∞_- is not a zero, and by (5.13.2), ∞_+ is a simple zero, we know m has an ℓ finite zeros. Thus, m_1 has ℓ poles in $\mathcal{S} \setminus \{\infty_\pm\}$. At ∞_+, m_1 has a zero and, by (5.13.6) and $m(z)^{-1} \to 0$ at ∞_-, we see m_1 has a simple pole at ∞_-. Thus, $\deg(m_1) = \ell + 1$ and ∞_- is a pole, so m_1 is also in $\mathcal{M}_{\mathfrak{e}}$.

By the analysis above, m_1 has exactly one simple pole in each gap so, by (5.13.6), $m(z)$ has exactly one simple zero in each gap. □

Along the way, we have also proven:

Corollary 5.13.3. *If $m \in \mathcal{M}_{\mathfrak{e}}$, the coefficient stripped m_1 defined by* (5.13.16) *also lies in $\mathcal{M}_{\mathfrak{e}}$.*

Remark. The proof of this corollary did not use that m had a pole at ∞_-, only that m did not have a zero at ∞_-.

Example 5.13.4. This example shows that property (iv) in the definition of minimal Herglotz functions is not automatic. Let J be a periodic Jacobi matrix, and for $y \in \mathbb{R}$, let J_y be the matrix where only b_1 is changed from b_1 to $b_1 + y$. Let $m_y(z)$ be the associated m-function. By (5.13.6) and the fact that J_y and J once-stripped are the same, we see

$$m_y(z)^{-1} = y + m(z)^{-1} \tag{5.13.16}$$

Thus, m_y is also a meromorphic function of degree $\ell + 1$ and so obeys (i)–(iii) of the definition of $\mathcal{M}_{\mathfrak{e}}$. But, by (5.13.16),

$$m_y(\infty_-) = y^{-1} \tag{5.13.17}$$

so m_y fails to obey condition (iv) of the definition.

m_y still has a pole in each gap, but instead of a pole at ∞_-, there is one additional pole on $(-\infty, \alpha_1] \cup [\beta_{\ell+1}, \infty)$ whose location and sheet depend on the sign and magnitude of y. Also, now $\deg(a) = \ell + 1$ rather than $\deg(a) = \ell$.

Changing a_1 from the periodic value changes the degree of m. □

There is a natural map, $\mathcal{D}$, from $\mathcal{M}_{\mathfrak{e}}$ to $\mathbb{T}_{\mathfrak{e}}$, the torus described in (5.12.57). Namely, each $f \in \mathcal{M}_{\mathfrak{e}}$ has ℓ poles other than at ∞_-, one each in $G_1, G_2, \dots, G_\ell$. The set of these poles describes a point $(z_1, \dots, z_\ell) \in \mathbb{T}_{\mathfrak{e}}$. This is called the *Dirichlet data* for f. $\mathcal{D}$ is called the *Dirichlet map*. The reason for this name will be explained in the Notes.

Theorem 5.13.5. *$\mathcal{D}$ is a one-one continuous map of $\mathcal{M}_{\mathfrak{e}}$ onto $\mathbb{T}_{\mathfrak{e}}$. In particular, $\mathcal{M}_{\mathfrak{e}}$ is topologically a torus.*

Remark. Here $\mathcal{M}_{\mathfrak{e}}$ is topologized using the topology of uniform convergence (uniform as $\mathcal{S}_{\mathrm{R}}$-valued functions).

Proof. We will describe a point in $\mathbb{T}_{\mathfrak{e}}$ with coordinates

$$\mathcal{D}(f) = (z_1, \delta_1; z_2, \delta_2; \dots) \tag{5.13.18}$$

where $z_j \in [\beta_j, \alpha_{j+1}]$ and δ_j is ± 1, with the convention that we take $\delta_j = -1$ if z_j is at a band edge.

Any $f \in \mathcal{M}_{\mathfrak{e}}$ has the form

$$f(z) = \int_{\mathfrak{e}} \frac{g(x)\,dx}{x - z} + \sum_{\{j \mid \delta_j = 1\}} \frac{w_j}{x - z_j} \tag{5.13.19}$$

where

$$g(x) = \frac{1}{\pi} \operatorname{Im} f(x_+ + i0) \tag{5.13.20}$$

$$w_j = \lim_{\varepsilon \downarrow 0} (i\varepsilon) f((x_j)_+ + i\varepsilon) \tag{5.13.21}$$

This is just (2.3.7), (2.3.41), (2.3.54), and (2.3.58) where only the poles on $\mathcal{S}_+$ are relevant, since the measure is $\lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \operatorname{Im} f(x_+ + i\varepsilon)\,dx$. Poles at branch points do not enter the sum because they only have $|x - z_j|^{-1/2}$ singularities. (They will affect g; at nonresonant gap edges, g vanishes as $(x - z_0)^{1/2}$, while at resonance edges, g diverges as $(x - z_0)^{-1/2}$.)

We know f has the form

$$f(z) = \frac{p(z) + \sqrt{R(z)}}{a(z)} \tag{5.13.22}$$

a has zeros at precisely the points $\{z_j\}_{j=1}^{\ell}$, so

$$a(z) = A \prod_{j=1}^{\ell} (z - z_j) \tag{5.13.23}$$

Since all $z_j < \alpha_{\ell+1}$ and $\operatorname{Im}(\sqrt{R(x_+ + i0)}) > 0$ on $[\alpha_{\ell+1}, \beta_{\ell+1}]$ (from $\sqrt{R(x_+ + i0)} > 0$ on $(\beta_{\ell+1}, \infty)$ and the branch of $(z - \beta_{\ell+1})^{1/2}$, which is positive on $(\beta_{\ell+1}, \infty) + i0$ has positive imaginary part on $(-\infty, \beta_{\ell+1}) + i0$), we have $A > 0$.

Thus, by (5.13.20), in (5.13.19) for $x \in \mathfrak{e}$,

$$g(x) = \frac{1}{\pi} \frac{\sqrt{|R(x)|}}{A \prod_{j=1}^{\ell} |x - z_j|} \tag{5.13.24}$$

while, by (5.13.21),

$$w_j = \frac{2\sqrt{|R(z_j)|}}{A \prod_{k \neq j} |z_k - z_j|} \tag{5.13.25}$$

for to avoid a pole on $\mathcal{S}_-$, we must have $p(z_j) - \sqrt{R(z_j)} = 0$, which yields to 2 in the numerator.

The normalization condition $f(z) = -z^{-1} + O(z^{-2})$ is equivalent to

$$\int_{\mathfrak{e}} g(x)\, dx + \sum_{\{j | \delta_j = 1\}} w_j = 1 \tag{5.13.26}$$

which determines A. Thus, knowing $\mathcal{D}(f)$ determines A and then g and w_j, and then f, which proves the map is one-one.

Conversely, given a set of Dirichlet data (i.e., a point in $\mathbb{T}_{\mathfrak{e}}$), define $a(z)$ by (5.13.23) where A is determined by (5.13.26), determine $p(z)$ by (since $(p(z) + \sqrt{R(z)})/a(z)$ is $O(z^{-1})$)

$$p(z) + \sqrt{R(z)} = O(z^{\ell-1}) \tag{5.13.27}$$

near ∞_+ (which determines the top two coefficients of $p(z)$) and the conditions (since m has no pole at $(z_j; \delta_j)$)

$$p(z_j) \mp \delta_j \sqrt{R(z_j)} = 0 \tag{5.13.28}$$

This defines f by (5.12.7). Tracking signs of a proves $\operatorname{Im} f(x_+ + i0) \geq 0$ on $\mathfrak{e}$ and that the residues of poles on $\mathcal{S}_+$ are positive. Thus, the Cauchy integral formula proves in $\mathbb{C}_+ \cap \mathcal{S}_+$

$$f(z) = \int_{\Gamma_+} \frac{f(w)}{w - z}\, dw \tag{5.13.29}$$

and then (5.13.19), which shows $\operatorname{Im} f > 0$ on $\mathcal{S}_+ \cap \mathbb{C}_+$.

In (5.13.29), Γ_+ is the contour in the proof of Theorem 5.12.3 and the fact that constructed f has $O(|z|^{-1})$ at ∞_+ means the contour at ∞_+ in the full Cauchy integral formula vanishes. This proves existence. $\square$

Each $f \in \mathcal{M}_\mathfrak{e}$ is an m-function, so the m-function of a unique Jacobi matrix, J_f, which is determined either from the spectral measure $g(x)\,dx + \sum_{\{j \mid \delta_j = 1\}} w_j \delta_{z_j}$ or from the continued fraction expansion at ∞_+. The topology on $\mathcal{M}_\mathfrak{e}$ is equivalent to the topology of pointwise convergence on the parameters in J_f (once we prove J_f is periodic or almost periodic, this will be the same as uniform convergence in n). Note that f determines a_1, b_1 directly by

$$f(z)^{-1} = -z + b_1 + a_1^2 z^{-1} + O(z^{-2}) \tag{5.13.30}$$

at ∞_+.

We will study the n-dependence of the Jacobi parameters by studying the impact of coefficient stripping. We proved in Corollary 5.13.3 that $f \to f_1$, coefficient stripping given by (5.13.30) and (5.13.6) is a map of $\mathcal{M}_\mathfrak{e}$ to $\mathcal{M}_\mathfrak{e}$. We will also need a map of

$$\widetilde{\mathfrak{A}} \colon \mathcal{M}_\mathfrak{e} \to \mathbb{T}^\ell$$

the canonical ℓ-torus, $\mathbb{R}^\ell / \mathbb{Z}^\ell$, by mapping $\mathbb{T}_\mathfrak{e}$ to $\mathbb{T}^\ell$ by Corollary 5.12.11, and composing this with $\mathcal{D}$, that is, if

$$\mathcal{D}(f) = (z_1, \dots, z_\ell) \qquad (z_j \in G_j)$$

then

$$\widetilde{\mathfrak{A}}(f) = \sum_{j=1}^{\ell} \mathfrak{A}(z_j) - \mathfrak{A}(z_j^{(0)}) \tag{5.13.31}$$

where $z_j^{(0)}$ is some convenient point, say $z_j^{(0)} = \alpha_j$.

We can prove uniform (over the isospectral torus) bounds on the weight.

Theorem 5.13.6. *There are positive constants C, D so that uniformly over $\mathbb{T}_\mathfrak{e}$, one has for all $x \in \mathfrak{e}$,*

$$D R(x)^{1/2} \le g(x) \le C R(x)^{-1/2} \tag{5.13.32}$$

Proof. We have

$$\operatorname{dist}(x, \mathbb{R} \setminus \mathfrak{e}) \min_{j=1,\dots,\ell+1} \left(\tfrac{1}{2}\, |\beta_j - \alpha_j|\right)^{\ell-1} \le \prod_{j=1}^{\ell} |x - z_j| \le |\beta_{\ell+2} - \alpha_1|^\ell \tag{5.13.33}$$

so, by (5.13.24), for some C_1, D_1,

$$D_1 A^{-1} R(x)^{1/2} \le g(x) \le C_1 A^{-1} R(x)^{-1/2} \tag{5.13.34}$$

Also, we have, by (5.13.25),

$$0 \leq w_j \leq A^{-1} C_2 \tag{5.13.35}$$

where

$$C_2 = 2|\beta_{\ell+1} - \alpha_1|^{\ell+1} (\min|\beta_j - \alpha_j|)^{-\ell+1} \tag{5.13.36}$$

(5.13.26) and these bounds provide uniform (in $\mathcal{T}_{\mathfrak{e}}$) upper and strictly positive lower bounds on A and then (5.13.34) implies (5.13.32). □

Theorem 5.13.7. (a) $\widetilde{\mathfrak{A}}$ *is a bijection of* $\mathcal{M}_{\mathfrak{e}}$ *to* $\mathbb{T}^{\ell}$.
(b) *Coefficient stripping* $f \to f_1$ *obeys*

$$\widetilde{\mathfrak{A}}(f_1) - \widetilde{\mathfrak{A}}(f) = \mathfrak{A}(\infty_-) - \mathfrak{A}(\infty) \tag{5.13.37}$$

Proof. (a) $\widetilde{\mathfrak{A}}$ is the composition of $\mathcal{D}$ and the map of Corollary 5.12.11, each of which is a continuous bijection.

(b) f has poles at the points in $\mathcal{D}(f)$ plus at ∞_- and, by (5.13.6) (other than at $\infty_\pm$), zeros of f are precisely poles of f_1 plus the zeros at ∞_+. Thus, by the first half of Abel's theorem (Theorem 5.12.7),

$$\widetilde{\mathfrak{A}}(f) + \mathfrak{A}(\infty_-) = \widetilde{\mathfrak{A}}(f_1) + \mathfrak{A}(\infty_+)$$

which is (5.13.37). □

This is truly a remarkable theorem: $f \to f_1$ is a map of a torus to itself. In general, iterating maps on a torus is complicated, but if the map is just addition by a fixed group element, iteration n times is just adding n times that element! $x \to x + nx_0$ is an affine map (on $\mathbb{R}^{\ell}$), so (5.13.37) is sometimes summarized by the phrase: "Abel's map linearizes coefficient stripping." With this in place, we get some immediate consequences (they are corollaries, but so significant that we call them theorems!):

Theorem 5.13.8. *Let* $\mathfrak{e} \subset \mathbb{R}$ *be a finite gap set. Let* $p \in \{1, 2, \dots\}$. *The following are equivalent:*

(i) *One Jacobi matrix,* J_f, *associated to one* $f \in \mathcal{M}_{\mathfrak{e}}$ *is periodic of period* p.
(ii) *All Jacobi matrices,* J_f, *associated to all* $f \in \mathcal{M}_{\mathfrak{e}}$ *are of period* p.
(iii) *Each harmonic measure,* $\rho_{\mathfrak{e}}(\mathfrak{e}_j)$ (*where* $\mathfrak{e}_j = [\alpha_j, \beta_j]$) *is rational with*

$$p\rho_{\mathfrak{e}}(\mathfrak{e}_j) \in \mathbb{Z} \tag{5.13.38}$$

(iv) *There is a polynomial of degree* p *with*

$$\Delta^{-1}([-2, 2]) = \mathfrak{e} \tag{5.13.39}$$

(*inverse as a map from* $\mathbb{C}$).

Proof. Consider the statement

$$p(\mathfrak{A}(\infty_-) - \mathfrak{A}(\infty_+)) = 0 \tag{5.13.40}$$

that is, p times the element of the torus is the identity. By (5.13.37), if $f_1, f_2, \dots$ are what we get by coefficient stripping, (5.13.40) is equivalent to

$$\widetilde{\mathfrak{A}}(f_p) - \widetilde{\mathfrak{A}}(f) = 0 \tag{5.13.41}$$

for one f or for all f! Since $\widetilde{\mathfrak{A}}$ is a bijection, this is equivalent to $f_p = f$, that is, J is itself after stripping p times, that is, J is periodic!

By (5.12.71), (5.13.40) holds if and only if

$$p \sum_{j=1}^{k} \rho_{\mathfrak{e}}(\mathfrak{e}_j) \in \mathbb{Z}$$

for $k = 1, 2, \dots, \ell$, which is equivalent to (5.13.38).

Finally, we note that (i) $\Rightarrow$ (iv); just take Δ to be the discriminant. Conversely, (iv) implies (5.13.40). For let

$$F(z) = -\Delta(z) \pm \sqrt{\Delta^2(z) - 4}$$

Since $\Delta^{-1}([-2, 2]) = \mathfrak{e}$, $\Delta^2 - 4$ has double roots at internal points of $\mathfrak{e}$ and single roots at edges of $\mathfrak{e}$, so F is meromorphic on $\mathcal{S}_{\mathfrak{e}}$. Since

$$\pm\sqrt{\Delta^2 - 4} = \pm(\Delta(z) + O(\Delta(z)^{-1})) \tag{5.13.42}$$

we see at ∞_+, F has a zero of order p and at ∞_- a pole of order p. It thus has degree p (since there are no other poles) and so no other zeros (as can also be seen by noting that $F(z)^{-1} = \frac{1}{4}(-\Delta(z) \mp \sqrt{\Delta^2 - 4})$). Thus, (5.13.40) is just the first part of Abel's theorem for F. $\square$

Notice that Theorem 5.13.8 implies Theorem 5.5.25 (given Proposition 5.5.26) and provides a proof of that theorem. Our proof of Aptekarev's theorem (i.e., (ii) $\Rightarrow$ (iii) in Theorem 5.5.25) is indirect: Rational harmonic measure implies (5.13.40) by the calculation in (5.12.71) and that implies there is a periodic J and then Δ is its discriminant. Peherstorfer's proof [338] is via a direct construction—its OPUC analog appears as Theorem 11.4.8 in [400].

The following generalizes the Borg–Hochstadt theorem (Theorem 5.4.21):

Corollary 5.13.9. *Let $\{a_n, b_n\}_{n=1}^{\infty}$ be a set of Jacobi parameters obeying*

$$a_{n+p} = a_n \qquad b_{n+p} = b_n \tag{5.13.43}$$

where $p = kq$ with k and q integral. Suppose all the gaps G_j are closed for $j \neq k, 2k, \dots, (q-1)k$. Then, a, b are periodic at period q, that is,

$$a_{n+q} = a_n \qquad b_{n+q} = b_n \tag{5.13.44}$$

Remark. The Borg–Hochstadt theorem is the case $q = 1$.

Proof. Each band has harmonic measure m/q. $\square$

For general finite gap sets, the Jacobi matrices are quasiperiodic:

Theorem 5.13.10. *Let $\mathfrak{e}$ be a finite gap set and J_f a Jacobi matrix whose m-function is a minimal Herglotz function in $\mathcal{M}_{\mathfrak{e}}$. Then its Jacobi parameters are almost periodic. To be totally explicit, there are real analytic functions $A_{\mathfrak{e}}$ and $B_{\mathfrak{e}}$ on $\mathbb{T}^{\ell}$, the standard ℓ torus with values in $(0, \infty)$ and $\mathbb{R}$, respectively, so that for every such J_f, we have $t_0 \in \mathbb{T}^{\ell}$ so that*

$$a_n = A_{\mathfrak{e}}(t_0 - n\omega) \qquad b_n = B_{\mathfrak{e}}(t_0 - n\omega) \tag{5.13.45}$$

where ω is given in terms of the harmonic measures of $\mathfrak{e}$ by (5.12.71).

Proof. Define $\tilde{A}_{\mathfrak{e}}$ and $\widetilde{B}_{\mathfrak{e}}$ on $\mathcal{M}_{\mathfrak{e}}$ by

$$f(z)^{-1} = -z + \widetilde{B}_{\mathfrak{e}}(f) + \tilde{A}_{\mathfrak{e}}(f)^2 z^{-1} + O(z^{-2}) \tag{5.13.46}$$

which are clearly real analytic on $\mathcal{M}_{\mathfrak{e}}$. Define

$$A_{\mathfrak{e}} = \tilde{A}_{\mathfrak{e}} \circ \widetilde{\mathfrak{A}}^{-1} \qquad B_{\mathfrak{e}} = \widetilde{B}_{\mathfrak{e}} \circ \widetilde{\mathfrak{A}}^{-1}$$

where $\widetilde{\mathfrak{A}}$ is the bijection of $\mathcal{M}_{\mathfrak{e}}$ to $\mathbb{T}^\ell$ of Theorem 5.13.7. Then (5.13.45) is just (5.13.37) iterated. □

One can naturally use (5.13.45) to define (a_n, b_n) for all $n \in \mathbb{Z}$ and so get natural two-sided Jacobi matrices for any $\mathfrak{e}$. The set of such two-sided matrices is called the *isospectral torus*, $\mathcal{T}_{\mathfrak{e}}$, for $\mathfrak{e}$. In the periodic case, it is precisely the set of periodic J's with a given Δ. Just as Chapter 3 is the theory of special classes of perturbations of $\mathcal{T}_{\mathfrak{e}}$ for $\mathfrak{e} = [-2, 2]$, we want to understand the analogous perturbations for general $\mathfrak{e}$. For the rational harmonic measure case, this will be the subject of Chapter 8 and for general $\mathfrak{e}$'s, of Chapter 9.

Finally, we use these ideas to find another proof of (5.2.11) and show that for the general finite gap situation, the whole-line Jacobi matrices are reflectionless (i.e., have purely imaginary Green's functions).

Theorem 5.13.11. *Let $\mathfrak{e}$ be a finite gap set, m a minimal Herglotz function on $\mathcal{S}_{\mathfrak{e}}$, and J the two-sided Jacobi matrix given by* (5.13.45) *for $n \in \mathbb{Z}$, so that*

$$m(z) = m(z; J_0^+) \tag{5.13.47}$$

Then

$$m(z; J_0^-) = (a_0^2 m(\tau(z)))^{-1} \tag{5.13.48}$$

that is, one can recover $m(z; J_0^{-1})$ from the second sheet values of m.

Remark. In the periodic case, this provides another proof of (5.2.11).

Proof. By the fact that $m(z)$ has a pole at ∞_- and by (5.13.7), we see that $m_1(z) - (-a_1^{-2}z + a_1^{-2}b_1)$ has a zero at ∞_-, so near ∞_-,

$$m_1(z) = -a_1^{-2}z + a_1^{-2}b_1 + O(z^{-1}) \tag{5.13.49}$$

In particular, near ∞_- on $\mathbb{C}_+ \cap \mathcal{S}_+$, $\operatorname{Im} m_1(\tau(z)) \leq 0$. On the other hand, on $\mathfrak{e}$, $m_1(\tau(x + i0)) = \overline{m(x + i0)}$ also has a negative imaginary part. Finally, the same argument that showed poles on $\mathcal{S}_+$ have positive residues shows they have negative residues on $\mathcal{S}_-$ (for on $\mathcal{S}_-$, $p(z) + \sqrt{R(z)} = 0$ and $-2\sqrt{R(z)}/a(z)$ has positive sign). Thus, by the maximum principle for harmonic functions, $\operatorname{Im} m_1(\tau(z)) \leq 0$ on $\mathcal{S}_+ \cap \mathbb{C}_+$.

It follows that $(a_1^2 m_1(\tau(z)))^{-1}$ is a discrete m-function. Similarly, if we let

$$m_{+,n}(z) = m(z; J_n^+) \tag{5.13.50}$$

then

$$m_{-,n}(z) \equiv (a_n^2 m_{+,n}(\tau(z))^{-1} \tag{5.13.51}$$

is a discrete m-function.

With this definition, the recursion relation

$$m_{+,n}(z)^{-1} = b_{n+1} - z - a_{n+1}^2 m_{+,n+1}(z) \tag{5.13.52}$$

which initially holds on $\mathcal{S}_+ \cap \mathbb{C}_+$ extends by analytic continuation, and since $\tau(z) = z$ implies

$$a_n^2 m_{-,n}(z) = b_{n+1} - z - (m_{-,n+1}(z))^{-1} \tag{5.13.53}$$

which shows inductively that the Jacobi parameters associated to $m_{-,n}$ are $\{a_{j-2+n}, b_{j-1+n}\}_{j=1}^\infty$, that is, J_n^-. Thus,

$$m_{-,n}(z) = m(z; J_n^-) \tag{5.13.54}$$

which for $n = 0$ is (5.13.48). □

Theorem 5.13.12. *Let J be a two-sided Jacobi matrix in $\mathcal{T}_{\mathfrak{e}}$ where $\mathfrak{e}$ is a finite gap set. Then,*

(i) *The diagonal Green's function, $G_{nn}(z)$, is pure imaginary for $z = x + i0$ with $x \in \mathfrak{e}$. Thus, J is reflectionless on $\mathfrak{e}$.*

(ii) *$\sigma(J) = \mathfrak{e}$ and the spectrum is purely absolutely continuous of uniform multiplicity 2.*

Proof. (i) By (5.4.45),

$$G_{nn}(z) = -\frac{1}{a_n^2 m(z; J_n^+) - m(z; J_n^-)^{-1}} \tag{5.13.55}$$

On $\mathfrak{e}$,

$$\begin{aligned} m(x + i0, J_n^-) &= m(\tau(x - i0), J_n^-) \\ &= \overline{m(x + i0, J_n^-)} \end{aligned} \tag{5.13.56}$$

so, by translates of (5.13.48),

$$m(x + i0, J_n^-)^{-1} = \overline{a_n^2 m(x + i0, J_n)} \tag{5.13.57}$$

and, by (5.13.55), G_{nn} is pure imaginary.

(ii) By (5.13.55) and (5.13.48),

$$(-G_{nn}(z))^{-1} = a_n^2[m(z; J_n^+) - m(\tau(z); J_n^+)] \tag{5.13.58}$$

for all $z \in \mathbb{C} \setminus \mathfrak{e}$.

Consider a gap $[\beta_j, \alpha_{j+1}]$. Writing m in the form $(p \pm \sqrt{R})/a$, we see

$$(-G_{nn}(z))^{-1} = \frac{2a_n^2\sqrt{R(z)}}{a(z)}$$

where $a(z)$ has a single zero in $[\beta_j, \alpha_{j+1}]$.

Suppose first that zero is in (β_j, α_{j+1}). Then $(-G_{nn}(z))^{-1}$ vanishes at β_j and α_{j+1}. Moreover, on $\mathbb{R} \setminus \sigma(J)$,

$$\frac{d}{dx} G_{nn}(x) > 0 \Rightarrow \frac{d}{dx} (-G_{nn}(x))^{-1} > 0$$

away from the zero of a. Thus, by monotonicity, $(-G_{nn}(z))^{-1}$ has no zero in (β_j, α_{j+1}).

If $(a(z))$ has a zero at β_j, then $(-G_{nn}(\beta_j))^{-1} = \infty$, $(-G_{nn}(\alpha_{j+1})) = 0$, and $(-G)^{-1}$ is finite and monotone in all of (β_j, α_{j+1}), so always strictly negative. Similarly, if $a(z)$ has a zero at α_j, $(-G_{nn}(z))^{-1}$ is strictly positive on (β_j, α_{j+1}).

In all cases, $(-G_{nn}(z))^{-1}$ is nonvanishing on (β_j, α_{j+1}), so no $G_{nn}(z)$ has a pole in those intervals, so $\sigma(J) \subset \mathfrak{e}$. By the fact that $G_{nn}(x + i0)$ is pure imaginary, Craig's theorem (Theorem 5.4.19) implies the spectrum is purely a.c. Since

$$\operatorname{Im}(a_n^2 m(x + i0, J_n^+)) = \operatorname{Im}((-m(x + i0, J_n^-))^{-1}) = \tfrac{1}{2} \operatorname{Im}((-G_{nn}(x + i0))^{-1})$$

we see that the a.c. spectrum is of multiplicity 2. □

Remarks and Historical Notes. This is the second half of the theory developed by Flaschka–McLaughlin–Krichever–van Moerbeke quoted (with background) in the Notes to the last section.

By the discussion in Example 5.13.4 and the remark after Corollary 5.13.3, if m obeys all the conditions for a function in $\mathcal{M}_{\mathfrak{e}}$, except it is finite and nonzero at ∞_- rather than a pole, then the once-stripped m_1 is in $\mathcal{M}_{\mathfrak{e}}$. So every such Jacobi matrix is an almost periodic one with b_1 modified.

In the periodic case, the Dirichlet data points are the roots of $p_{p-1}(z)$, which are eigenvalues of the truncated matrix $J_{p-1;F}$, so associated to solutions of $(J - \lambda)u = 0$ with $u_{n=0} = u_{n=p} = 0$, thus Dirichlet eigenvalues, which is the reason for the name. Alternatively, in terms of the operators $J_0^\pm$ of the truncated full-line problem, Dirichlet data in the interior of a gap are eigenvalues of J_0^+ if in $\mathcal{S}_+$ and of J_0^- if in $\mathcal{S}_-$.

There are basically two ways of thinking of the isospectral torus, $\mathcal{T}_{\mathfrak{e}}$: a set of whole-line Jacobi matrices or as their restrictions to the half-line (which, by almost periodicity, determine the whole-line matrix). The half-line objects are defined as the set of minimal Herglotz functions. The whole-line objects are the set of reflectionless whole-line J's with $\sigma_{\text{ess}}(J) = \Sigma_{\text{ac}}(J) = \mathfrak{e}$. That every such object lies in the isospectral torus, as we have defined it, will be the major theme in Section 7.5, which will also discuss the history of this point of view.

Among all almost periodic Jacobi matrices, the finite gap ones are unusual in that, generically, one expects infinitely many gaps and Cantor spectrum. For results on such generic Cantor spectrum, see [28, 29, 121, 172].

APPENDIX TO SECTION 5.13:
A CHILD'S GARDEN OF ALMOST PERIODIC FUNCTIONS

As we have seen, Jacobi parameters induced by the minimal Herglotz functions associated to a general finite gap set are quasiperiodic, and so almost periodic. In this appendix, we discuss the general definition of quasiperiodic and almost periodic.

Given a function, f, on $\mathbb{Z}$ and $n \in \mathbb{Z}$, we define f_n on $\mathbb{Z}$ by

$$f_n(m) = f(n + m) \tag{5.13A.1}$$