away from the zero of $a$. Thus, by monotonicity, $(-G_{nn}(z))^{-1}$ has no zero in $(\beta_j, \alpha_{j+1})$.

If $(a(z))$ has a zero at $\beta_j$, then $(-G_{nn}(\beta_j))^{-1} = \infty$, $(-G_{nn}(\alpha_{j+1})) = 0$, and $(-G)^{-1}$ is finite and monotone in all of $(\beta_j, \alpha_{j+1})$, so always strictly negative. Similarly, if $a(z)$ has a zero at $\alpha_j$, $(-G_{nn}(z))^{-1}$ is strictly positive on $(\beta_j, \alpha_{j+1})$.

In all cases, $(-G_{nn}(z))^{-1}$ is nonvanishing on $(\beta_j, \alpha_{j+1})$, so no $G_{nn}(z)$ has a pole in those intervals, so $\sigma(J) \subset \mathfrak{e}$. By the fact that $G_{nn}(x + i0)$ is pure imaginary, Craig's theorem (Theorem 5.4.19) implies the spectrum is purely a.c. Since

$$\operatorname{Im}(a_n^2 m(x + i0, J_n^+)) = \operatorname{Im}((-m(x + i0, J_n^-))^{-1}) = \tfrac{1}{2} \operatorname{Im}((-G_{nn}(x + i0))^{-1})$$

we see that the a.c. spectrum is of multiplicity 2. □

**Remarks and Historical Notes.** This is the second half of the theory developed by Flaschka–McLaughlin–Krichever–van Moerbeke quoted (with background) in the Notes to the last section.

By the discussion in Example 5.13.4 and the remark after Corollary 5.13.3, if $m$ obeys all the conditions for a function in $\mathcal{M}_{\mathfrak{e}}$, except it is finite and nonzero at $\infty_-$ rather than a pole, then the once-stripped $m_1$ is in $\mathcal{M}_{\mathfrak{e}}$. So every such Jacobi matrix is an almost periodic one with $b_1$ modified.

In the periodic case, the Dirichlet data points are the roots of $p_{p-1}(z)$, which are eigenvalues of the truncated matrix $J_{p-1;F}$, so associated to solutions of $(J - \lambda)u = 0$ with $u_{n=0} = u_{n=p} = 0$, thus Dirichlet eigenvalues, which is the reason for the name. Alternatively, in terms of the operators $J_0^{\pm}$ of the truncated full-line problem, Dirichlet data in the interior of a gap are eigenvalues of $J_0^+$ if in $\mathcal{S}_+$ and of $J_0^-$ if in $\mathcal{S}_-$.

There are basically two ways of thinking of the isospectral torus, $\mathcal{T}_{\mathfrak{e}}$: a set of whole-line Jacobi matrices or as their restrictions to the half-line (which, by almost periodicity, determine the whole-line matrix). The half-line objects are defined as the set of minimal Herglotz functions. The whole-line objects are the set of reflectionless whole-line $J$'s with $\sigma_{\text{ess}}(J) = \Sigma_{\text{ac}}(J) = \mathfrak{e}$. That every such object lies in the isospectral torus, as we have defined it, will be the major theme in Section 7.5, which will also discuss the history of this point of view.

Among all almost periodic Jacobi matrices, the finite gap ones are unusual in that, generically, one expects infinitely many gaps and Cantor spectrum. For results on such generic Cantor spectrum, see [28, 29, 121, 172].

## APPENDIX TO SECTION 5.13: A CHILD'S GARDEN OF ALMOST PERIODIC FUNCTIONS

As we have seen, Jacobi parameters induced by the minimal Herglotz functions associated to a general finite gap set are quasiperiodic, and so almost periodic. In this appendix, we discuss the general definition of quasiperiodic and almost periodic.

Given a function, $f$, on $\mathbb{Z}$ and $n \in \mathbb{Z}$, we define $f_n$ on $\mathbb{Z}$ by

$$f_n(m) = f(n + m) \tag{5.13A.1}$$

Given a bounded function, $f$, on $\mathbb{Z}$, we define

$$\|f\|_\infty = \sup_n |f(n)| \tag{5.13A.2}$$

and let $C(\mathbb{Z})$ be the set of all bounded functions in this norm.

*Definition.* A function, $f$, from $\mathbb{Z}$ to $\mathbb{C}$ is called *almost periodic* (in Bochner sense) if and only if $f$ is bounded and $\{f_n\}_{n\in\mathbb{Z}}$ has compact closure in $\|\cdot\|_\infty$.

*Definition.* A *Bohr almost periodic* function on $\mathbb{Z}$ is a bounded function, $f$, so that for any $\varepsilon$, there is an $L$ so that for all $m \in \mathbb{Z}$, there is an $n$ so that $|n-m| \leq L$ and

$$\|f_n - f\|_\infty < \varepsilon \tag{5.13A.3}$$

Let $\mathbb{T}^1$ be the circle $\partial\mathbb{D} = \{z \mid |z| = 1\}$, $\mathbb{T}^n = \times_{j=1}^n \mathbb{T}^1$, the $n$-dimensional torus, and $\mathbb{T}^\infty$, the countably infinite product. We will think of $\mathbb{T}^n$ as $\partial\mathbb{D}^n$ and use $(z_1, \dots, z_n)$ as coordinates. Notice that we use additive notation for $\mathbb{Z}$ but multiplication for $\mathbb{T}$.

The main theorem at the center of the theory is:

**Theorem 5.13A.1.** *Let $f$ be a bounded function on $\mathbb{Z}$. The following are equivalent:*

(1) *$f$ is (Bochner) almost periodic.*
(2) *$f$ is Bohr almost periodic.*
(3) *$f$ is a uniform limit of finite sums of the form*

$$g_N(n) = \sum_{j=1}^N a_j e^{2\pi i \alpha_j^{(N)} n} \tag{5.13A.4}$$

*for $\alpha_1, \dots, \alpha_N^{(N)} \in \mathbb{R}/\mathbb{Z}$.*

(4) *There exists a continuous function $F$ on $\mathbb{T}^\infty$ and $\{z_j\}_{j=1}^\infty$ in $\mathbb{T}^\infty$ so that*

$$f(n) = F(z^n) \tag{5.13A.5}$$

*where $(z^n)_j = z_j^n$.*

*Remarks.* 1. If $F$ depends on only finitely many variables (equivalently, $F$ can be viewed as a function of a finite-dimensional torus), $f$ is called *quasiperiodic*.

2. In Theorem 5.13.10, we have functions of the form (5.13A.5) on a finite-dimensional torus, but only for $n \geq 0$. So the question comes up how to define almost periodic functions on $n \geq 0$. The answer is as restrictions to $n \geq 0$ of functions almost periodic on $\mathbb{Z}$, there is at most one such extension, for if there were two, their difference would be an almost periodic function vanishing for $n \geq 0$ and, by the Bohr definition, such a function is identically zero.

It is natural to prove this result in the general context of locally compact abelian groups. Let $G$ be such a group, $\mu$ Haar measure, and $\widehat{G}$ the set of characters, that is, continuous homomorphisms of $G$ to $\partial\mathbb{D}$. Besides $\mathbb{Z}$, the example to think about is $\mathbb{R}$.

Let $C(G)$ stand for bounded continuous functions on $G$ with $\|\cdot\|_\infty$. For $f \in C(G)$ and $g \in G$, define $f_g$ by

$$f_g(x) = f(x+g) \tag{5.13A.6}$$

$f$ is (Bochner) almost periodic if $\{f_g\}_{g\in G}$ has compact closure in $\|\cdot\|_\infty$. $f$ is called Bohr almost periodic if and only if for all $\varepsilon$, there is a compact set $K$ so that for all $g$, there is $h$ in $g+K$ so that

$$\|f_n - f\|_\infty \le \varepsilon \tag{5.13A.7}$$

The general form of Theorem 5.13A.1 is:

**Theorem 5.13A.2.** *Let $G$ be a separable compact abelian group. Let $f \in C(G)$. Then the following are equivalent:*
(1) *$f$ is (Bochner) almost periodic.*
(2) *$f$ is Bohr almost periodic.*
(3) *$f$ is a uniform limit of finite sums of the form*

$$g_N(x) = \sum_{j=1}^{N} a_j \chi_j^{(N)}(x) \tag{5.13A.8}$$

*with $\chi_j^{(N)} \in \widehat{G}$.*
(4) *There exists a continuous function $F$ on $\mathbb{T}^\infty$ to $\mathbb{C}$ and a homomorphism $\zeta\colon G \to \mathbb{T}^\infty$ so*

$$f(x) = F(\zeta(x)) \tag{5.13A.9}$$

*Theorem 5.13A.2 $\Rightarrow$ Theorem 5.13A.1.* Only parts (2) and (4) look a little different. For (2), note compact sets in $\mathbb{Z}$ are finite and so contained in intervals. As for (4), note for $G=\mathbb{Z}$, homomorphisms $\zeta\colon G\to\mathbb{T}^\infty$ are given precisely by $\zeta(1)$ since $\zeta(n)=\zeta(1)^n$ (using a product rather than additive notation for $\mathbb{T}$). □

(4) $\Rightarrow$ (3) *in Theorem 5.13A.2.* Let $z_1, z_2, \dots$ be coordinates on $\mathbb{T}^\infty$. Let $\chi_j\colon G\to\partial\mathbb{D}$ be $z_j\circ\varphi$. Then $\chi_j$ is a character on $G$, and thus, so is any finite product of $\chi_j$'s. By the Stone–Weierstrass theorem, polynomials in the $z_j$ are dense in $C(\mathbb{T}^\infty)$, and so $F$ is a uniform limit in polynomials in $z_j$. Thus, $F\circ\varphi$ is a uniform limit of finite linear combinations of characters. □

(3) $\Rightarrow$ (1) *in Theorem 5.13A.2.* A set $Q$ in a complete metric space, $X$, has compact closure if and only if for all $\varepsilon$, there are finitely many $q_1,\dots,q_\ell$ in $X$ so that $\cup_{j=1}^{\ell}\{q \mid \rho(q,q_\ell)<\varepsilon\}$ contains $Q$. If $f$ is a limit of $f_N$'s of the form (5.13A.8), given $\varepsilon$, pick $\varepsilon/2$ so $\|f-f_N\|_\infty<\varepsilon/2$. Since

$$(f_N)_g = \sum_{j=1}^{N} a_j\chi_j(g)\chi_j \tag{5.13A.10}$$

$\{(f_N)_g\}\subset\{\sum_{j=1}^N a_j z_j\chi_j \mid |z_j|=1\}$ is compact, and so covered by finitely many $\varepsilon/2$ balls. Thus, since $\|f_g-(f_N)_g\|_\infty = \|f-f_N\|_\infty$, $\{f_g\}$ is covered by finitely many $\varepsilon$ balls. □

(1) $\Rightarrow$ (2) *in Theorem 5.13A.2.* Given $\varepsilon$, pick $g_1,\dots,g_N$ in $G$ so every $f_g$ is within $\varepsilon$ of some $f_{g_j}$. Let $K=\{-g_1,\dots,-g_N\}$, which is finite, and so compact. If $\|f_g - f_{g_j}\|_\infty<\varepsilon$, then $\|f_{g-g_j}-f\|_\infty<\varepsilon$ and $h=g-g_j\in g+K$. □

*Remark.* Once we have (2) $\Rightarrow$ (1), this implies the compact $K$ in Bohr almost periodic can be taken as a finite set!

**Lemma 5.13A.3.** *If $f$ is Bohr almost periodic, then $f$ is uniformly compact, that is, for any $\varepsilon$, there is a neighborhood $N$ of the identity $e \in G$ so that if $x - y \in N$, then $|f(x) - f(y)| < \varepsilon$.*

*Proof.* Each $f_y$ is continuous at $e$, so given $\varepsilon$, there is $N_y$, a neighborhood of $e$, so that $w \in N_y \Rightarrow |f_y(w) - f_y(e)| < \varepsilon/4$, so if $w, w' \in N_y$, then $|f_y(w) - f_y(w')| < \varepsilon/2$. By continuity of addition, we can find $M_y$, a neighborhood of $e$, so $M_y + M_y \subset N_y$. Thus, if

$$w, w', w'' \in M_y \Rightarrow |f_{y+w''}(w') - f_{y+w''}(w)| < \frac{\varepsilon}{2} \tag{5.13A.11}$$

If $K$ is compact, we have $K \subset \cup_{y \in K}(y + M_y)$, so pick $y_1, \dots, y_\ell$ so $K \subset \cup_{j=1}^{\ell}(y_j + M_{y_j})$ and $M_K = \cap_{j=1}^{\ell} M_{y_j}$. Thus, by (5.13A.11),

$$y \in K,\ w, w' \in M_K \Rightarrow |f_y(w) - f_y(w')| < \frac{\varepsilon}{2} \tag{5.13A.12}$$

Given $\varepsilon$, let $K$ compact be chosen so (5.13A.7) holds for $\varepsilon/4$ and pick $M_K$ as above. Suppose $x - y \in M_K$. By Bohr almost periodicity, there is $h \in K$ so that $\|f_{h-y} - f\|_\infty < \varepsilon/4$. Thus, $\|f_h - f_y\|_\infty < \varepsilon/4$, so by (5.13A.12),

$$w, w' \in M_K \Rightarrow |f_y(w) - f_y(w')| < \varepsilon \tag{5.13A.13}$$

Taking $w = x - y$ and $w' = e$, we see

$$x - y \in M_K \Rightarrow |f(x) - f(y)| < \varepsilon \tag{5.13A.14}$$

which is uniform continuity. $\square$

(2) $\Rightarrow$ (1) *in Theorem 5.13A.2.* By Lemma 5.13A.3, $f$ is uniformly continuous, which implies $x \to f_x$ is continuous as a map of $G$ to $C(G)$. Given $\varepsilon$, let $K$ be the compact set so that (5.13A.7) holds for $\varepsilon/2$. Since $x \to f_x$ is continuous, $\{f_x\}_{x \in K}$ is compact, so we can find $x_1, \dots, x_\ell$ in $K$ whose $\varepsilon/2$ balls cover this set of $f$'s. Given any $y \in G$, there is $x \in K$ so $\|f_{-y+x} - f\|_\infty < \varepsilon/2$, so $\|f_y - f_x\| < \varepsilon/2$ and $f_y$ is within $\varepsilon$ of some $f_{x_j}$. Thus, $\{f_y\}_{y \in G}$ is covered by finitely many $\varepsilon$ balls. Since $\varepsilon$ is arbitrary, $f$ is (Bochner) almost periodic. $\square$

(1) $\Rightarrow$ (4) *in Theorem 5.13A.2.* This final step is the most elaborate and elegant. Let $H \subset C(G)$ be the closure of $\{f_x\}_{x \in G}$. $H$ is called the *hull* of $f$. Define $\varphi_0 \colon G \to H$ by

$$\varphi_0(x) = f_x \tag{5.13A.15}$$

Since (1) $\Rightarrow$ (2) $\Rightarrow$ $f$ is uniformly continuous, $\varphi_0$ is continuous. Since $\|p_x - q_x\|_\infty = \|p - q\|_\infty$, we see that

$$\|f_{x+y} - f_{x'+y'}\|_\infty \le \|f_x - f_{x'}\|_\infty + \|f_y - f_{y'}\|_\infty \tag{5.13A.16}$$

that is,

$$\|\varphi_0(x+y) - \varphi_0(x'+y')\| \le \|\varphi_0(x) - \varphi_0(x')\| + \|\varphi_0(y) - \varphi_0(y')\| \tag{5.13A.17}$$

Let $h, h' \in H$. Picking $x_n, y_n \in G$ so $\varphi(x_n) \to h$, $\varphi(y_n) \to h'$, we see, by (5.13A.17), that $\varphi(x_n + y_n)$ is Cauchy, which allows us to define $h + h'$ ("+" is map of $H \times H$ to $H$, not to be confused with adding the functions!). It is easy to see this turns $H$ into a compact group. Since $H$ is a metric space, compactness implies separability. By definition, $\varphi$ is a homomorphism.

Now we need a fact about compact separable abelian groups (see the Notes): Such groups have characters that separate points, and by separability, there is a countable family, $\{\chi_j\}_{j=1}^{\infty} \subset \widehat{H}$, that separates points. Let $Q\colon H \to \mathbb{T}^\infty$ by $Q(h)_j = \chi_j(h)$ and $\varphi\colon G \to \mathbb{T}^\infty$ by $\varphi = Q \circ \widetilde{\varphi}$. $Q$ is an injective map since $\{\chi_j\}$ separates points. $\varphi$ is a group homomorphism.

Since $H$ is compact, $Q[H]$ is closed in $\mathbb{T}^\infty$. Define $\widetilde{F}\colon H \to \mathbb{C}$ by $\widetilde{F}(h) = h(e)$. Then $F$ is continuous and

$$\widetilde{F}(\varphi(x)) = \widetilde{F}(f_x) = f_x(e) = f(x) \tag{5.13A.18}$$

that is, $\widetilde{F} \circ \varphi = f$. Since $Q$ is one-one, we can define a function $F$ on $Q[H]$ so

$$F \circ Q = \widetilde{F} \tag{5.13A.19}$$

Since $Q[H]$ is closed, $F$ has an extension to $\mathbb{T}^\infty$ by the Tietze extension theorem. We will still use $F$ for this extension. Clearly, (5.13A.19) remains true; $F\colon \mathbb{T}^\infty \to \mathbb{C}$ and

$$F \circ \varphi = F \circ Q \circ \widetilde{\varphi} = \widetilde{F} \circ \widetilde{\varphi} = f \tag{5.13A.20}$$

by (5.13A.18). $\square$

**Remarks and Historical Notes.** The definition of almost periodic functions on $\mathbb{R}$ and their properties is due to Harald Bohr [51, 52], using the definition we gave for Bohr almost periodic on $\mathbb{Z}$ (but for $\mathbb{R}$). The Bochner property (which we codified in the Bochner definition) is due to Bochner [47, 49].

Sometimes what we call "almost periodic" is called "uniformly almost periodic" since there are also Besicovitch almost periodic or $L^2$-almost periodic functions, which we will define below.

For book treatments of the theory, see Besicovitch [44], Bohr [53], Corduneanu [94], and Levitan–Zhikov [279].

We used the fact that any abelian separable compact group, $G$, has enough characters to separate points. This is essentially the Peter–Weyl theorem for such groups (see, e.g., Simon [394]); here is a sketch of the argument explicitly. Let $f$ be a function on $G$ with $f(-x) = \overline{f(x)}$. Define $T\colon L^2(G) \to L^2(G)$ by

$$(Th)(x) = \int f(x-y)h(y)\,d\mu(y)$$

where $d\mu$ is Haar measure. $T$ is Hilbert–Schmidt (so compact) and selfadjoint.

Moreover, if $U_x\colon L^2 \to L^2$ by $(U_x f)(y) = f(y-x)$, then $T$ commutes with $\{U_x\}$. Thus, $\{U_x\}$ leave each eigenspace invariant. If $V$ is such an eigenspace and is finite-dimensional, the $U_x$ are commuting unitaries on $V$, so they have a common eigenvector $\widetilde{\chi}(x)$. Thus,

$$\widetilde{\chi}(x+y) = (U_x\widetilde{\chi})(y) = \lambda_x\widetilde{\chi}(y)$$

and $U_{x+y} = U_x U_y$ implies $\lambda_{x+y} = \lambda_x \lambda_y$. Since $x \to U_x$ is continuous, this shows $\widetilde{\chi}$ is continuous and everywhere nonzero: $\chi(x) = \widetilde{\chi}(x)/\widetilde{\chi}(e)$ is thus a (continuous) character. So the characters span $\mathrm{Ran}(T)$. Since we can find $f_n$ so $T_{f_n} \to \mathbf{1}$, we see the characters $\chi$ span $L^2$, which implies they separate points.

Further developments depend on the notion of the average of an almost periodic function. Given an almost periodic function, $f$, let $H$ be its hull, $\widetilde{F}$ the function in (5.13A.18), and $d\nu$ normalized Haar on $H$. We define

$$\mathrm{Av}(f) = \int_H \widetilde{F}(x)\, d\nu(x) \tag{5.13A.21}$$

For $\mathbb{R}$ or $\mathbb{Z}$, one can prove that

$$\mathrm{Av}(f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\, dx \tag{5.13A.22}$$

(or $\frac{1}{2T+1} \sum_{-T}^{T} f(n)$ for $\mathbb{Z}$).

One defines the Fourier coefficients of $f$ for $\chi \in \widehat{G}$ by

$$\widehat{f}(\chi) = \mathrm{Av}(\bar{\chi} f) \tag{5.13A.23}$$

noting that $\bar{\chi} f$ is also almost periodic. It is not hard to see that $\widehat{f}(\chi)$ is nonzero for only countably many $\chi$'s. Indeed, one has a Plancherel theorem

$$\sum_{\chi \in \widehat{G}} |\widehat{f}(\chi)|^2 = \mathrm{Av}(|f|^2) \tag{5.13A.24}$$

One also has an $L^2$ convergence of Fourier series; if $\{\chi_j\}_{j=1}^{\infty}$ is a numbering of those $\chi$'s with $\widehat{f}(\chi) \neq 0$, then

$$\mathrm{Av}\left( \left| f - \sum_{j=1}^{N} \widehat{f}(\chi_j)\chi_j \right|^2 \right) \to 0 \tag{5.13A.25}$$

These results are all easy to prove by using the fact that if $H$ is the hull, $\widehat{f}(\chi) \neq 0$ implies $\chi \in \widehat{H}$, that is,

$$\chi = \widetilde{\chi} \circ \widetilde{\varphi} \tag{5.13A.26}$$

where $\widetilde{\chi}$ is a character of $H$. (5.13A.24) and (5.13A.25) are then expressions of the fact that characters of $H$ are a basis of $L^2(H, d\nu)$.

For $\mathbb{R}$, one defines Besicovitch almost periodic functions as functions on $\mathbb{R}$, for which there exists, for any $z$, a finite sum $f_N = \sum_{j=1}^{N} a_j^{(N)} e^{i w_j^{(N)} x}$ with

$$\limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f_n - f_N(x)|^2\, dx \leq \varepsilon \tag{5.13A.27}$$

The *frequency module* of $f$, an almost periodic function, is the set of characters of $G$ that comes from $H$, the hull, via (5.13A.26). It is a countable subgroup of $\widehat{G}$. It is generated by $\{\chi \mid \widehat{f}(\chi) \neq 0\}$.

A function is called *limit periodic* if it is a uniform limit of periodic functions. Such functions are obviously almost periodic. A typical example is

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} \cos(2\pi 2^{-n} x) \tag{5.13A.28}$$

We note that the term quasiperiodic is sometimes used for a very different notion from our use and that those quasiperiodic functions are not almost periodic.

The set of all almost periodic functions in $\|\cdot\|$ is Banach algebra. Its Gel'fand spectrum (see [150] for the theory of commutative Banach algebras) is called the Bohr compactification of $G$. It is huge, containing every hull as a subgroup. One can construct it by taking $\widehat{G}$ and putting the discrete topology in it and taking the dual of that.

## 5.14 PERIODIC OPUC

We have discussed OPRL with periodic Jacobi matrices in much of this chapter. The theory of OPUC whose Verblunsky coefficients obey

$$\alpha_{n+p} = \alpha_n \tag{5.14.1}$$

for all $n$ and some fixed $p$ is the subject of Chapter 11 of [400]. Our goal in this section is to sketch some parts of this theory, emphasizing the differences to the OPRL theory.

A major difference is that the transfer matrix for OPRL has determinant 1 since

$$\det\left[\frac{1}{a}\begin{pmatrix} z-b & -1 \\ a^2 & 0 \end{pmatrix}\right] = 1 \tag{5.14.2}$$

while in the OPUC case, the $m$ step transfer matrix has determinant $z^m$ since

$$\det\left[\frac{1}{\rho}\begin{pmatrix} z & -\bar{\alpha} \\ -\alpha z & 1 \end{pmatrix}\right] = z \tag{5.14.3}$$

(see (2.4.3)).

The natural discriminant is thus

$$\Delta(z) = z^{-p/2}\mathrm{Tr}(T_p(z)) \tag{5.14.4}$$

For this reason, it is natural to restrict to the case $p$ even and control $p$ odd by other means (e.g., by viewing it as period $2p$ instead of as period $p$). We shall do this henceforth.

$\Delta(z)$ is thus a Laurent polynomial (i.e., polynomial in $z$ and $z^{-1}$). It is real on $\partial\mathbb{D}$, and one can show the associated measure is purely absolutely continuous on $\mathfrak{e} = \Delta^{-1}([-2, 2]) \subset \partial\mathbb{D}$ with potentially one pure point per gap. The Carathéodory function obeys a quadratic equation and extends to a two-sheeted Riemann surface with branch points at the edges of connected components of $\mathfrak{e}$.

The most significant difference from OPRL comes from the following: If $\mathfrak{e}$ has $\ell + 1$ connected components, in the OPRL case, there are $\ell$ significant gaps—the gap on $\mathbb{C} \setminus \mathfrak{e}$ that goes from $\beta_{\ell+1}$ to $\infty$ and then $-\infty$ to $\alpha_1$ is not considered for