10.4. The Group $\mathbb{U}(1,1)$

Let $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. $\mathbb{U}(1,1)$ is the group of all 2×2 matrices obeying

$$A^* J A = J \tag{10.4.1}$$

As we saw in (1.5.36), each of the basic matrices

$$A(\alpha, z) = (1 - |\alpha|^2)^{-1/2} \begin{pmatrix} z & -\bar{\alpha} \\ -\alpha z & 1 \end{pmatrix} \tag{10.4.2}$$

lies in $\mathbb{U}(1,1)$, and we will eventually prove $\mathbb{U}(1,1)$ is the smallest group containing all the $A(\alpha, z)$. Since the matrix $T_n(z)$ of (3.2.28) is a product of $A(\alpha_j, z)$'s, it lies in $\mathbb{U}(1,1)$, and as the $T_n(z)$ becomes a key tool in the sections below, general features of $\mathbb{U}(1,1)$ will be important. So our goal in this section is to study these features.

We will also define $\mathbb{SU}(1,1) = \{A \in \mathbb{U}(1,1) \mid \det(A) = 1\}$. By (10.4.1), $|\det(A)|^2 = 1$ if $A \in \mathbb{U}(1,1)$, so $\det(A) = e^{i\theta}$. Then $e^{-i\theta/2} A \in \mathbb{SU}(1,1)$, so $\partial\mathbb{D} \times \mathbb{SU}(1,1) \to \mathbb{U}(1,1)$ by $(\omega, A) \mapsto \omega A$ is a two-fold cover of $\mathbb{U}(1,1)$. We will define $(\det(A))^{1/2}$ to be that square root with argument in $[0, \pi)$ and, given $A \in \mathbb{U}(1,1)$, define $A_{\mathrm{s}} \in \mathbb{SU}(1,1)$ by

$$\begin{aligned} A_{\mathrm{s}} &= \det(A)^{-1/2} A \\ A &= \det(A)^{1/2} A_{\mathrm{s}} \end{aligned} \tag{10.4.3}$$

Let $\tilde{J}$ be any matrix with $\tilde{J}^* = \tilde{J} = \tilde{J}^{-1}$ and $\mathrm{Tr}(\tilde{J}) = 0$. Then there is a unitary W with $W \tilde{J} W^{-1} = J$. It follows if we define

$$\mathbb{U}(1,1;\tilde{J}) = \{A \mid A^* \tilde{J} A = \tilde{J}\}$$

and similarly for $\mathbb{SU}(1,1;\tilde{J})$, we have

$$W \mathbb{U}(1,1;\tilde{J}) W^{-1} = \mathbb{U}(1,1) \tag{10.4.4}$$

so the groups are unitarily equivalent. We mention this because

PROPOSITION 10.4.1. *Let* $J_r = \left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)$. *Then* $\mathbb{SU}(1,1;J_r)$ *is equal to* $\mathbb{SL}(2,\mathbb{R})$, *the set of* 2×2 *real matrices of determinant* 1.

PROOF. If $A = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ and $\det(A) = 1$, then $A^{-1} = \begin{pmatrix} \beta & -\gamma \\ -\delta & \alpha \end{pmatrix}$ so (if $\det(A) = 1$), by a direct calculation,

$$A^t = J_r A^{-1} J_r^{-1} \tag{10.4.5}$$

and thus (under $\det(A) = 1$!)

$$A^* J_r A = J_r \Leftrightarrow A^* = A^t \Leftrightarrow A = \bar{A}$$

That means $\mathbb{SU}(1,1;J_r) = \{A \mid A \text{ is real}\} \cap \{A \mid \det(A) = 1\}$, which is $\mathbb{SL}(2,\mathbb{R})$, as claimed. $\square$

This shows a close connection between Jacobi matrices and OPUC. Transfer matrices for Jacobi matrices and Schrödinger operators at energy $E \in \mathbb{R}$ lie in $\mathbb{SL}(2,\mathbb{R}) = \mathbb{SU}(1,1;J_r)$. Those for OPUC lie in $\mathbb{U}(1,1;J)$. j_r, the bilinear form induced by J_r, is given by

$$j_r(x,x) = i(\bar{x}_1 x_2 - \bar{x}_2 x_1)$$

so J-invariance for OPUC is an analog of constancy of the Wronskian in the OPRL case.

COROLLARY 10.4.2. *If $A \in \mathbb{SU}(1,1)$, then* $\mathrm{Tr}(A)$ *is real.*

PROOF. By unitary equivalence, we need only prove this for one $\tilde{J}$. Since $\mathbb{SU}(1,1;J_r)$ consists of real matrices, clearly $\mathrm{Tr}(A) \in \mathbb{R}$ for that realization. □

We can use the reality of trace to identify the eigenvalues of an $A \in \mathbb{U}(1,1)$:

THEOREM 10.4.3. *Let $A \in \mathbb{SU}(1,1)$ be different from $\pm\mathbf{1}$. Then one of the following holds:*

(a) (Elliptic Case) $\mathrm{Tr}(A) \in (-2,2)$. *Then A has two distinct eigenvalues $\lambda_1(A), \lambda_2(A)$ with $\lambda_2(A) = \lambda_1^*(A)$ and $|\lambda_1(A)| = |\lambda_2(A)| = 1$.*
(b) (Parabolic Case) $\mathrm{Tr}(A) = \pm 2$. *Then A has a single eigenvalue at $+1$ or -1 with algebraic multiplicity 2 and geometric multiplicity 1.*
(c) (Hyperbolic Case) $\mathrm{Tr}(A) \in \mathbb{R}\backslash[-2,2]$. *Then A has two distinct eigenvalues $\lambda_1(A), \lambda_2(A)$ with $\lambda_i(A)$ real, $|\lambda_1(A)| > 1 > |\lambda_2(A)|$ and $\lambda_2(A) = \lambda_1(A)^{-1}$.*

If $A \in \mathbb{U}(1,1)$ is nonconstant, then one of the following holds:

(d) (Elliptic Case) $|\mathrm{Tr}(A)| < 2$. *A has two distinct eigenvalues $\lambda_1(A), \lambda_2(A)$ with $|\lambda_1(A)| = |\lambda_2(A)| = 1$.*
(e) (Parabolic Case) $|\mathrm{Tr}(A)| = 2$. *Then A has a single eigenvalue $\lambda_1(A)$ with $|\lambda_1(A)| = 1$. It has algebraic multiplicity 2 and geometric multiplicity 1.*
(f) (Hyperbolic Case) $|\mathrm{Tr}(A)| > 2$. *Then A has distinct eigenvalues $\lambda_1(A), \lambda_2(A)$ with $|\lambda_1(A)| > 1 > |\lambda_2(A)|$ and $\lambda_2(A) = \overline{\lambda_1(A)^{-1}}$ (i.e., $|\lambda_2| = |\lambda_1|^{-1}$ and* $\arg(\lambda_2) = \arg(\lambda_1)$*).*

Moreover, if $Au = \lambda u$, then $A^*(Ju) = \lambda^{-1}(Ju)$. Finally,

(g) *In the elliptic case, if u_1, u_2 are the two eigenvectors $\langle u_2, Ju_1\rangle = 0$, $\langle u_1, Ju_1\rangle > 0 > \langle u_2, Ju_2\rangle$.*
(h) *In the hyperbolic or parabolic case, if $Au = \lambda u$, then $\langle u, Ju\rangle = 0$.*

PROOF. Writing $A = (\det(A))^{1/2} A_s$, the results for $\mathbb{SU}(1,1)$ imply those for $\mathbb{U}(1,1)$, so we will suppose $A \in \mathbb{SU}(1,1)$. The eigenvalues of A obey

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$$

so

$$\lambda^2 - 2x\lambda + 1 = 0 \tag{10.4.6}$$

with $x \in \mathbb{R}$. If $|x| < 1$, then $x = \cos(\theta)$, $\theta \neq 0, \pi$, for some θ and the solutions of (10.4.6) are $e^{\pm i\theta}$, proving (a). If $x = \pm 1$, the solutions are ± 1. Since $A \neq \pm\mathbf{1}$, ± 1 cannot be eigenvalues of geometric multiplicity 2. This proves (b). If $|x| > 1$, $x = \pm\cosh(\alpha)$ for some $\alpha > 0$ and roots are $\pm e^{\alpha}$ and $\pm e^{-\alpha}$, proving (c).

To prove that

$$Au = \lambda u \Rightarrow A^*(Ju) = \lambda^{-1}(Ju) \tag{10.4.7}$$

we need only note that (10.4.1) is equivalent to

$$A^*J = JA^{-1} \tag{10.4.8}$$

To prove (g), we note that if $Au_j = \lambda_j u_j$, then

$$\begin{aligned}\lambda_1^{-1}\langle u_2, Ju_1\rangle &= \langle u_2, A^*Ju_1\rangle \qquad \text{(by (10.4.7))}\\ &= \langle Au_2, Ju_1\rangle \\ &= \bar{\lambda}_2\langle u_2, Ju_1\rangle\end{aligned}$$

Since $\lambda_1^{-1} = \lambda_2$ and $\lambda_2 \neq \bar{\lambda}_2$, this implies $\langle u_2, Ju_1\rangle = 0$. Since J is nondegenerate and nondefinite, one $\langle u_j, Ju_j\rangle$ is positive and one negative.

To prove (h) in the hyperbolic case, suppose $Au = \lambda u$. Then

$$\begin{aligned} \lambda^{-1}\langle u, Ju\rangle &= \langle u, A^* Ju\rangle \\ &= \langle Au, Ju\rangle \\ &= \bar\lambda\langle u, Ju\rangle \end{aligned}$$

Since $\bar\lambda \neq \lambda^{-1}$ in this case, $\langle u, Ju\rangle = 0$.

In the parabolic case, if $Au = \pm u$, there is a w with $Aw = \pm w + u$. Thus

$$\begin{aligned} \langle u, Ju\rangle &= \langle Aw, Ju\rangle \mp \langle w, Ju\rangle \\ &= \langle w, A^* Ju\rangle \mp \langle w, Ju\rangle \\ &= \langle w, JA^{-1}u\rangle \mp \langle w, Ju\rangle \\ &= \pm\langle w, Ju\rangle \mp \langle w, Ju\rangle \\ &= 0 \end{aligned}$$

□

Remark. One can also use (10.4.8) and $\sigma(A^*) = \overline{\sigma(A)}$ to prove (a)–(c).

A key consequence of this theorem is:

Corollary 10.4.4. *$\{A \in \mathbb{U}(1,1) \mid$ the eigenvalues of A are unequal and of magnitude $1\}$ is open.*

Proof. It is given by $|\mathrm{Tr}(A)| < 2$. □

Example 10.4.5. Let $\alpha \in \mathbb{D}$ and $A(\alpha, z)$ given by (10.4.2). Then, if $z = e^{i\theta}$, $|\mathrm{Tr}(A(\alpha, z))| = \rho^{-1} 2\cos(\frac{\theta}{2})$. Thus $\sup_n \|A^n\| < \infty$ if and only if $|\cos(\frac{\theta}{2})| < \rho$ or, equivalently, $|\sin(\frac{\theta}{2})| > |\alpha|$ or $\theta \in (\theta_{|\alpha|}, 2\pi - \theta_{|\alpha|})$ where $\theta_{|\alpha|} = 2\arcsin(|\alpha|)$. This is precisely the region of the essential spectrum of Geronimus polynomials (Example 1.6.12). We will see later (Theorem 11.1.2) that this is no coincidence. □

Any invertible matrix A can be decomposed $A = U|A|$ with $|A| = \sqrt{A^*A}$ positive and U unitary. We will show that $|A| \in \mathbb{U}(1,1)$ if A is, and explicitly find all such $|A|$ and U, and so parametrize $\mathbb{U}(1,1)$.

Theorem 10.4.6. (i) *Any $A \in \mathbb{U}(1,1)$ can be written uniquely as*

$$A = U|A| \tag{10.4.9}$$

with $U, |A| \in \mathbb{U}(1,1)$, $|A| \geq 0$, and U unitary.

(ii) *Any $A \in \mathbb{U}(1,1)$ with $A > 0$ has the form*

$$A = \begin{pmatrix} \cosh(x) & e^{i\varphi}\sinh(x) \\ e^{-i\varphi}\sinh(x) & \cosh(x) \end{pmatrix} \tag{10.4.10}$$

for some $x \geq 0$ and $\varphi \in [0, 2\pi)$ or equivalently,

$$A = \rho^{-1}\begin{pmatrix} 1 & -\bar\alpha \\ -\alpha & 1 \end{pmatrix} \tag{10.4.11}$$

with $\rho = (1 - |\alpha|^2)^{1/2}$ and $\alpha \in \mathbb{D}$.

(iii) *Any $U \in \mathbb{U}(1,1)$ with U unitary has the form*

$$U = \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix} \tag{10.4.12}$$

(iv) *Any* $A \in \mathbb{U}(1,1)$ *has the form* $A = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ *with*

$$|\alpha| = |\beta| \qquad |\gamma| = |\delta| \qquad |\alpha|^2 - |\gamma|^2 = 1 \tag{10.4.13}$$

$$\arg(\alpha) + \arg(\beta) = \arg(\gamma) + \arg(\delta) \tag{10.4.14}$$

and conversely. Elements of $\mathbb{SU}(1,1)$ *are described by* (10.4.13) *and*

$$\arg(\alpha) + \arg(\beta) = 0 = \arg(\gamma) + \arg(\delta) \tag{10.4.15}$$

Equivalently,

$$A = \begin{pmatrix} \alpha & \gamma \\ \bar\gamma & \bar\alpha \end{pmatrix} \tag{10.4.16}$$

with $|\alpha|^2 - |\gamma|^2 = 1$.

(v) *As topological spaces,* $\mathbb{U}(1,1)$ *is homeomorphic to* $\mathbb{D} \times \partial\mathbb{D} \times \partial\mathbb{D}$ *and* $\mathbb{SU}(1,1)$ *to* $\mathbb{D} \times \partial\mathbb{D}$*. In particular, both* $\mathbb{U}(1,1)$ *and* $\mathbb{SU}(1,1)$ *are connected.*

PROOF. It is useful to prove (ii) before (i).

(ii). If $A > 0$, $\det(A) > 0$, so $|\det(A)| = 1$ implies $A \in \mathbb{SU}(1,1)$. Thus, if $A = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$, $A^{-1} = \left(\begin{smallmatrix} \beta & -\gamma \\ -\delta & \alpha \end{smallmatrix}\right)$ and $JA^{-1}J^{-1} = \left(\begin{smallmatrix} \beta & \gamma \\ \delta & \alpha \end{smallmatrix}\right)$. Thus $A = JA^{-1}J$ if and only if $\alpha = \beta$. $A \geq 0$ implies $\alpha > 0$, $\delta = \bar\gamma$, and $\alpha^2 - |\gamma|^2 = 1$. Thus $\alpha = \cosh(x)$ and $\gamma = \sinh(x)e^{i\varphi}$, proving (10.4.10). Moreover, if

$$P(x,\varphi) = \begin{pmatrix} \cosh(x) & e^{i\varphi}\sinh(x) \\ e^{-i\varphi}\sinh(x) & \cosh(x) \end{pmatrix} \tag{10.4.17}$$

an easy calculation shows

$$P(x,\varphi)P(y,\varphi) = P(x+y,\varphi) \tag{10.4.18}$$

and, in particular,

$$P(x,\varphi)^{1/2} = P(\tfrac{1}{2}\,x,\varphi) \tag{10.4.19}$$

(i). $A^*A \in \mathbb{SU}(1,1)$ and $A^*A \geq 0$ so, by (ii) and (10.4.19), $|A| \in S\mathbb{U}(1,1)$ and so $U = A|A|^{-1} \in \mathbb{U}(1,1)$. Uniqueness is a general feature of the polar decomposition $A = U|A|$.

(iii). $U^* = U^{-1}$ and (10.4.1) imply that $UJ = JU$, so U is diagonal.

(iv). By (i)–(iii),

$$\begin{aligned} A &= \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix} \begin{pmatrix} \cosh(x) & e^{i\varphi_3}\sinh(x) \\ e^{-i\varphi_3}\sinh(x) & \cosh(x) \end{pmatrix} \\ &= \begin{pmatrix} e^{i\varphi_1}\cosh(x) & e^{i(\varphi_1+\varphi_3)}\sinh(x) \\ e^{i(\varphi_2-\varphi_3)}\sinh(x) & e^{i\varphi_2}\cosh(x) \end{pmatrix} \end{aligned}$$

from which (10.4.13) and (10.4.14) are immediate. $A \in \mathbb{SU}(1,1)$ if and only if $\varphi_1 = -\varphi_2$, from which (10.4.15) is immediate.

(v). Let $\mathbb{U}_d(1)$ be the diagonal unitary matrices and $\mathbb{SU}_d(1)$ those $U \in \mathbb{U}_d(1)$ with $\det(U) = 1$. Let P_J be the positive definite matrix of the form (10.4.11). $(U, |A|) \to U|A|$ maps $\mathbb{U}_d(1) \times P_J$ (resp. $\mathbb{SU}_d(1) \times P_J$) bijectively and bihomeomorphically onto $\mathbb{U}(1,1)$ (resp. $\mathbb{SU}(1,1)$). Since P_J is homeomorphic to $\mathbb{D}$ by (10.4.11) and $\mathbb{U}_d(1)$ (resp. $\mathbb{SU}(1)$) to $\partial\mathbb{D} \times \partial\mathbb{D}$ (resp. $\partial\mathbb{D}$), the result is proven. □

Remarks. 1. $\mathbb{U}(1,1)$ and $\mathbb{SU}(1,1)$ are thus connected Lie groups of dimension 4 and 3, respectively. Their Lie algebras (given a Lie group, G, of matrices, its Lie algebra, g, is given by $g = \{A \mid e^{tA} \in G \text{ for all } t \in \mathbb{R}\}$) are spanned by $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)$ for $\mathbb{SU}(1,1)$ and those three plus $\left(\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right)$ for $\mathbb{U}(1,1)$. Equivalently, $u(1,1) = \{A \mid A^*J + JA = 0\}$ and $su(1,1) = \{A \mid A^*J + JA = 0, \operatorname{Tr}(A) = 0\}$. This lets us see that the closed group, G, generated by $\{A(\alpha,z)\}_{z\in\partial\mathbb{D},\alpha\in\mathbb{D}}$ is $\mathbb{U}(1,1)$. For $\frac{d}{dt}A(te^{i\psi},z)\big|_{t=0} = \left(\begin{smallmatrix} 0 & -e^{-i\psi} \\ -e^{i\psi} & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)$ generate the Lie algebra for $\mathbb{SU}(1,1)$. Thus the group generated by $\{A(\alpha, z=0)\}_{\alpha\in\mathbb{D}}$ is $\mathbb{SU}(1,1)$. Since $A(\alpha=0, e^{2i\theta}) = e^{i\theta}\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right)$, G is $\mathbb{U}(1,1)$.

2. (10.4.16) implies $\mathbb{SU}(1,1)$ is a three-dimensional hyperboloid in $\mathbb{R}^4$.

Not only are the components of the polar decomposition in $\mathbb{U}(1,1)$, so are the components of standard similarities.

THEOREM 10.4.7. *Let* $A \in \mathbb{SU}(1,1)$. *Then*

(i) *If A is elliptic, there exists $V \in \mathbb{SU}(1,1)$ so that*

$$A = VUV^{-1} \tag{10.4.20}$$

with U diagonal and unitary.

(ii) *If A is hyperbolic, there exist $V \in \mathbb{SU}(1,1)$ so that*

$$A = VP(x,\varphi=0)V^{-1} \tag{10.4.21}$$

with $P(x,\varphi)$ given by (10.4.17) *and* $x > 0$.

(iii) *If A is parabolic, there exist $V \in \mathbb{SU}(1,1)$ so that*

$$A = VP_0V^{-1} \qquad \text{or} \qquad A = -VP_0V^{-1} \tag{10.4.22}$$

where

$$P_0 = \begin{pmatrix} 1+\frac{1}{2}\,i & -\frac{1}{2}\,i \\ \frac{1}{2}\,i & 1-\frac{1}{2}\,i \end{pmatrix} \tag{10.4.23}$$

Remarks. 1. There are natural canonical representations in the elliptic and hyperbolic cases, but not in the parabolic case, so we made an ad hoc choice in (iii) so that $P_0\binom{1}{1} = \binom{1}{1}$, $P_0\binom{i}{-i} = \binom{i}{-i} + \binom{1}{1}$.

2. There are similar results for $A \in \mathbb{U}(1,1)$ with an extra $(\det(A))^{1/2}$ factor in front.

PROOF. (i) By Theorem 10.4.3(a) and (g), A has an eigenvector u with $J(u,u) > 0$. Thus $u = \binom{1}{\alpha}$ with $\alpha \in \mathbb{D}$ and $Au = e^{i\theta}u$. The other eigenvector has $J(u,v) = 0$ so $v = \binom{\bar\alpha}{1}$ and $Av = e^{-i\theta}$. Let

$$V = \frac{1}{\rho}\begin{pmatrix} 1 & \alpha \\ \bar\alpha & 1 \end{pmatrix}$$

with $\rho = \sqrt{1-|\alpha|^2}$. Then $V \in \mathbb{SU}(1,1)$ and $V\binom{1}{0} = u$ and $V\binom{0}{1} = v$ so

$$AV = V\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

proving (10.4.20).

(ii) Since A is hyperbolic, there exist eigenvectors u, v with $J(u,u) = J(v,v) = 0$ and eigenvalues e^x, e^{-x} for $x > 0$. Thus $u = \binom{1}{\lambda}$, $v = \binom{1}{\mu}$ with $|\lambda| = |\mu| = 1$. Let $\lambda\mu = -e^{2i\theta}$, $\gamma = \lambda e^{-i\theta}$, so $\bar{\gamma} = -\mu e^{-i\theta}$. Define

$$\alpha = \frac{1-\gamma}{1+\gamma}$$

so $\bar{\alpha} = -\alpha$ and let

$$V = \frac{1}{\rho} \begin{pmatrix} e^{-i\theta/2} & \alpha e^{-i\theta/2} \\ \bar{\alpha} e^{i\theta/2} & e^{i\theta/2} \end{pmatrix}$$

with $\rho = (1 - |\alpha|^2)^{1/2}$ so $V \in \mathbb{SU}(1,1)$. Thus

$$V \begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_+ \begin{pmatrix} 1 \\ \gamma e^{i\theta} \end{pmatrix} = c_+ \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$$

where $c_+ = (1+\alpha)e^{-i\theta/2}\rho^{-1}$, and we used $\bar{\alpha} = -\alpha$ and $\frac{1-\alpha}{1+\alpha} = \gamma$. Similarly, since $\frac{1+\alpha}{1-\alpha} = \gamma^{-1} = \bar{\gamma}$, we get

$$V \begin{pmatrix} 1 \\ -1 \end{pmatrix} = c_- \begin{pmatrix} 1 \\ -\bar{\gamma} e^{i\theta} \end{pmatrix} = c_- \begin{pmatrix} 1 \\ \mu \end{pmatrix}$$

with $c_- = (1-\alpha)e^{i\theta/2}\rho^{-1}$. Since $P_+(1, \varphi = 0)\binom{1}{\pm 1} = e^{\pm x}\binom{1}{\pm 1}$, we have

$$AV \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = VP_+(1, \varphi = 0) \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

proving (10.4.21).

(iii) We consider the case $\operatorname{Tr}(A) = 2$ since $\operatorname{Tr}(A) = -2$ involves replacing P_0 by $-P_0$. (10.4.16) and $\operatorname{Tr}(A) = 2$ means

$$A = \begin{pmatrix} 1 + ia & -aie^{i\varphi} \\ aie^{i\varphi} & 1 - ia \end{pmatrix}$$

for some $a \in \mathbb{R}$ and $\varphi \in [0, 2\pi)$. Let $2a = e^x$ and

$$V = \begin{pmatrix} e^{i\varphi/2}\cosh(x) & e^{i\varphi/2}\sinh(x) \\ e^{-i\varphi/2}\sinh(x) & e^{i\varphi/2}\cosh(x) \end{pmatrix}$$

Then $A = VP_0V^{-1}$ by a direct calculation. $\square$

We define a complex conjugation, C, on $\mathbb{C}^2$ by

$$Cu = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bar{u} \tag{10.4.24}$$

that is, $C\binom{u_1}{u_2} = \binom{\bar{u}_2}{\bar{u}_1}$. Then

LEMMA 10.4.8. (a) *C is a complex conjugation, that is,*

$$C(au + bv) = \bar{a}Cu + \bar{b}Cv$$

for $u, v \in \mathbb{C}^2$, $a, b \in \mathbb{C}$, $C^2 = 1$, and $\|Cu\| = \|u\|$.

(b) *For all u, $J(Cu, u) = 0$.*

(c) *For all $A \in \mathbb{SU}(1,1)$, $CA = AC$.*

Remark. (c) is a unitary translation of the fact that $A \in \mathbb{SU}(1,1; J_r)$ is real.

PROOF. (a) Antilinearity is immediate, $C^2 = 1$ follows from $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\|Cu\| = \|u\|$ follows from the fact that $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is unitary.

(b) $J(Cu, u) = \overline{(Cu)_1}\, u_1 - \overline{Cu_2}\, u_2 = u_2 u_1 - u_1 u_2 = 0$.

(c) If A has the form (10.4.16), then

$$CAC = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar\alpha & \bar\gamma \\ \gamma & \alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A$$

by a direct calculation. □

This lemma sheds light on several aspects of Theorem 10.4.3. If $A \in \mathbb{SU}(1,1)$ and $Au = \lambda u$, then by (c) of the lemma, $A(Cu) = \bar\lambda Cu$. Thus, in the hyperbolic case, $Cu = e^{i\theta} u$ and so, by (b), $J(u,u) = 0$. In the elliptic case, if $Au_1 = \lambda u_1$, $Cu_1 = u_2$ and, by (b), $J(u_1, u_2) = 0$.

C is useful because it allows the following generalization of the fact that in the hyperbolic case, eigenvectors obey $J(u,u) = 0$.

THEOREM 10.4.9. *Suppose that $T_1, T_2, \dots$ is a sequence in $\mathbb{U}(1,1)$ so that for some nonzero $u, v \in \mathbb{C}^2$,*

$$\lim_{n\to\infty} \frac{\|T_n u\|}{\|T_n v\|} = 0 \tag{10.4.25}$$

Then $J(u,u) = 0$.

PROOF. Suppose Cu is not a multiple of u. Then $Cu = au + bv$ with $b \neq 0$ since (10.4.25) implies that u and v are independent. Since $\|T_n Cu\| \geq |b|\, \|T_n v\| - |a|\, \|T_n u\|$, (10.4.25) implies $\lim \frac{\|T_n u\|}{\|T_n Cu\|} = 0$. But $\|T_n Cu\| = \|CT_n u\| = \|T_n u\|$ so $\frac{\|T_n u\|}{\|T_n Cu\|} = 1$. It follows that Cu must be a multiple of u, that is, $Cu = e^{i\theta} u$. Thus $J(u,u) = J(e^{-i\theta} Cu, u) = e^{i\theta} J(Cu, u) = 0$ by Lemma 10.4.8(b). □

As a final subject in this section, we want to begin to discuss $\mathbb{U}(1,1)$ as the group of fractional linear transformations of $\bar{\mathbb{D}}$ onto $\bar{\mathbb{D}}$ and invariant measures on $\partial\mathbb{D}$, a subject central to Section 10.6. Given any 2×2 invertible matrix, we define $z \doteq Aw$ by (1.3.50)/(1.3.59) and L_A so (1.3.59) is $z = L_A(w)$. Notice that $L_{AB} = L_A L_B$ and that for a constant $c_{A,z}$,

$$A\begin{pmatrix} 1 \\ z \end{pmatrix} = c_{A,z} \begin{pmatrix} 1 \\ L_A(z) \end{pmatrix} \tag{10.4.26}$$

As usual, we view $\binom{1}{z}$ as the line $\{\binom{\lambda}{\lambda z} \mid \lambda \in \mathbb{C}\}$ with $\binom{1}{\infty}$ as the line $\{\binom{0}{\lambda} \mid \lambda \in \mathbb{C}\}$ so $\mathbb{C} \cup \{\infty\}$ is the Riemann sphere, $\mathbb{S}$, and L_A is a continuous map of $\mathbb{S}$ to itself. From this point of view, the map C of (10.4.24) takes $\binom{1}{\lambda}$ to $\binom{\bar\lambda}{1}$, which is in the line $\binom{1}{1/\bar\lambda}$, that is, $L_C(z) = \frac{1}{\bar z}$ is inversion in the unit circle.

PROPOSITION 10.4.10. (a) *A 2×2 matrix, A, with $|\det(A)| = 1$ has $A \in \mathbb{U}(1,1)$ if and only if L_A maps $\mathbb{D}$ onto $\mathbb{D}$ (and so also $\partial\mathbb{D}$ onto $\partial\mathbb{D}$).*

(b) *A with $\det(A) = 1$ is in $\mathbb{SL}(2,\mathbb{R})$ if and only if L_A maps $\mathbb{R}$ to $\mathbb{R}$ and $\mathbb{C}_+$ to $\mathbb{C}_+$.*

PROOF. (a) If $A \in \mathbb{U}(1,1)$, $J\left(A\binom{1}{z}, A\binom{1}{z}\right) = J\left(\binom{1}{z}, \binom{1}{z}\right) = 1 - |z|^2 = 0$ if $|z| = 1$. But $J\left(\binom{1}{w}, \binom{1}{w}\right) = 0$ if and only if $|w|^2 = 1$. Thus L_A maps $\partial\mathbb{D}$ to $\partial\mathbb{D}$. Moreover, $J\left(A\binom{1}{0}, A\binom{1}{0}\right) = J\left(\binom{1}{0}, \binom{1}{0}\right) = 1$ so $J\left(\binom{1}{L_A(0)}, \binom{1}{L_A(0)}\right) > 0$, so $L_A(0) \in \mathbb{D}$. Thus L_A maps $\mathbb{D}$ to $\mathbb{D}$.

Conversely, if f is any invertible linear map of $\mathbb{D}$ onto $\mathbb{D}$,

$$g(z) = \frac{f(z) - f(0)}{1 - \overline{f(0)}\, f(z)}$$

maps $\partial\mathbb{D}$ to $\partial\mathbb{D}$ and has $g(z) = 0$. By the Schwarz principle, $|\frac{g(z)}{z}| \leq 1$. By invertibility, g^{-1} also maps $\mathbb{D}$ to $\mathbb{D}$, $|\frac{g^{-1}(z)}{z}| \leq 1$ or $|\frac{z}{g(z)}| \leq 1$. Thus $|\frac{g(z)}{z}| = 1$, so $g(z) = e^{i\theta} z$ and $f(z)$ is given by

$$f(z) = \frac{f(0) + e^{i\theta} z}{1 + \overline{f(0)}\, e^{i\theta} z}$$

that is, $f = L_A$ with

$$A = (1 - |f(0)|^2)^{-1/2} \begin{pmatrix} e^{i\theta} & f(0) \\ e^{i\theta}\, \overline{f(0)} & 1 \end{pmatrix}$$

in $\mathbb{U}(1,1)$.

(b) Let J_r be given by Proposition 10.4.1. Then $WJ_rW^{-1} = J$ where $W = \left(\begin{smallmatrix} 1 & i \\ 1 & -i \end{smallmatrix}\right)$. Thus, by (10.4.4),

$$\mathbb{SL}(2,\mathbb{R}) = W^{-1}\mathbb{SU}(1,1)W \tag{10.4.27}$$

Notice that $L_W(z) = \frac{1+iz}{1-iz}$ maps $\mathbb{R}$ to $\partial\mathbb{D}$ and i to 0, and so $\mathbb{C}_+$ to $\mathbb{D}$. Thus part (a) and (10.4.27) imply that $SL(2,\mathbb{R})$ is precisely the set of A's with L_A maps $\mathbb{C}_+$ onto $\mathbb{C}_+$. □

Remark. The proof of (a) shows that $\{e^{i\psi}A(\alpha, z) \mid \alpha \in \mathbb{D},\, z \in \partial\mathbb{D},\, e^{i\psi} \in \partial\mathbb{D}\}$ is all of $\mathbb{U}(1,1)$.

Since $L_A \in \mathbb{U}(1,1)$ acts invertibly from $\partial\mathbb{D}$ to $\partial\mathbb{D}$, it defines a bijection on probability measures on $\partial\mathbb{D}$ by

$$\mu \mapsto L_A(\mu)$$

given by

$$\int f(e^{i\theta})\, d(L_A\mu)(\theta) = \int f(L_A(e^{i\theta}))\, d\mu(\theta) \tag{10.4.28}$$

With that definition, $L_{AB} = L_A L_B$, for if $(U_A f)(e^{i\theta}) = f(L_A(e^{i\theta}))$, then $U_A U_B = U_{BA}$. It is reasonable to use the same symbol, L_A, since if $x \in \partial\mathbb{D}$ and δ_x is a point mass at x_0, then $L_A(\delta_x) = \delta_{L_A(x)}$ since

$$\int f(L_A y)\, d\delta_x = f(L_A x) = \int f(y)\, d\delta_{L_A(x)}$$

Of course, $\partial\mathbb{D}$ has a distinguished measure, $\frac{d\theta}{2\pi}$, which we will denote as $dm(\theta)$ when we think of $\partial\mathbb{D}$ as a projective space for $\{u \mid J(u,u) = 0\}$, as we do here. Given $A \in \mathbb{U}(1,1)$, we define a positive function $N_A(e^{i\theta})$ on $\partial\mathbb{D}$ by letting

$$u_\theta = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\theta} \end{pmatrix} \tag{10.4.29}$$

and

$$N_A(e^{i\theta}) = \|Au_\theta\| \tag{10.4.30}$$

Here is what L_A does to dm:

THEOREM 10.4.11. *Let $A \in \mathbb{U}(1,1)$. Then $L_A m$ is a.c. w.r.t. m. Indeed,*

$$d(L_A m)(\theta) = N_{A^{-1}}(e^{i\theta})^{-2}\, dm(\theta) \tag{10.4.31}$$

PROOF. We will give two proofs of this important result. For both proofs, we note that multiplying A by $e^{i\eta}$ does not change either L_A or N_A, so we can, and will, suppose that $A \in \mathbb{SU}(1,1)$.

The first proof is a simple calculation. For A fixed, define $\psi(\theta)$ by

$$\begin{pmatrix} 1 \\ e^{i\psi(\theta)} \end{pmatrix} = C(\theta)A\begin{pmatrix} 1 \\ e^{i\theta} \end{pmatrix} \tag{10.4.32}$$

We claim that ψ is a C^∞ function of θ with $\frac{d\psi}{d\theta} = N_A(e^{i\theta})^{-2}$.

Since $A \in \mathbb{SU}(1,1)$, $A = \left(\begin{smallmatrix} \alpha & \gamma \\ \bar\gamma & \bar\alpha \end{smallmatrix}\right)$. Since $J(Au_\theta, Au_\theta) = 0$,

$$N_A(e^{i\theta})^2 = \|Au_\theta\|^2 = 2|(Au_\theta)_1|^2 = |\alpha + \gamma e^{i\theta}|^2 \tag{10.4.33}$$

On the other hand, by (10.4.32),

$$e^{i\psi(\theta)} = \frac{\bar\alpha e^{i\theta} + \bar\gamma}{\alpha + \gamma e^{i\theta}} \equiv G(\theta) \tag{10.4.34}$$

$G(\theta)$ is a C^∞ function of $e^{i\theta}$ which is never 0 since $|\gamma| < |\alpha|$, and so ψ is C^∞. Moreover,

$$\begin{aligned} \psi' &= -iG'(\theta)G(\theta)^{-1} \\ &= \left[\frac{e^{i\theta}(|\alpha|^2 - |\gamma|^2)}{(\alpha + \gamma e^{i\theta})^2}\right]\left[\frac{\alpha + \gamma e^{i\theta}}{\bar\alpha e^{i\theta} + \gamma}\right] \\ &= \frac{1}{|\alpha + \gamma e^{i\theta}|^2} \\ &= N_A(e^{i\theta})^{-2} \end{aligned}$$

by (10.4.33).

By a change of variables in Riemann integrals,

$$\begin{aligned} \int f(e^{i\theta})\, d(L_A m)(\theta) &= \int f(L_A(e^{i\theta}))\, dm(\theta) \\ &= \int f(e^{i\psi})\left(\frac{d\psi}{d\theta}\right)^{-1} dm(\psi) \end{aligned}$$

so we have to write $(\frac{d\psi}{d\theta})^{-1}$ as a function of ψ. We have

$$u_\psi = \frac{Au_\theta}{\|Au_\theta\|}$$

so $\|Au_\theta\|\|A^{-1}u_\psi\| = 1$. Thus

$$\begin{aligned} \left(\frac{d\psi}{d\theta}\right)^{-1} &= N_A(e^{i\theta})^2 \\ &= N_{A^{-1}}(e^{i\psi})^{-2} \end{aligned}$$

proving (10.4.31) and completing the first proof.

For the second proof, we let $\mathbb{V} = \{u \mid Cu = u\}$ where C is given by (10.4.24). $\mathbb{V}$ is a two-dimensional real vector space spanned by $\binom{1}{1}$ and $\binom{i}{-i}$. By Lemma 10.4.8(c), any $A \in \mathbb{SU}(1,1)$ maps $\mathbb{V}$ to $\mathbb{V}$, so in a real basis for $\mathbb{V}$, A has real matrix elements. Since $\det(A) = 1$ as a complex matrix on $\mathbb{C}^2$, $\det(A) = 1$ as a real matrix on $\mathbb{V}$. It follows that A^{-1} leaves Euclidean measure on $\mathbb{V}$ invariant.

Introduce polar coordinates on $\mathbb{V}$ by $\binom{re^{i\theta/2}}{re^{-i\theta/2}}$. We have just proven that $r\,dr\,d\theta$ is left invariant by A^{-1}. But the map in (r,θ) induced by A^{-1} is

$$A^{-1}:(r,\theta)=(rN_{A^{-1}}(e^{i\theta}),\tilde{\varphi}(\theta))$$

with $e^{i\tilde{\varphi}}=L_A(e^{i\theta})$. This has Jacobian

$$\mathcal{I}=\begin{pmatrix} N_{A^{-1}}(e^{i\theta}) & 0 \\ r\,\frac{\partial N_{A^{-1}}}{\partial\theta} & \frac{\partial\tilde{\varphi}}{\partial\theta}\end{pmatrix}$$

Invariance of $r\,dr\,d\theta$ implies $1=N_{A^{-1}}\det(\mathcal{I})=N_{A^{-1}}^2\frac{\partial\tilde{\varphi}}{\partial\theta}$ so $\frac{\partial\tilde{\varphi}}{\partial\theta}=N_{A^{-1}}^{-2}$. By (10.4.28),

$$d(L_Am)(\theta)=\frac{dL_{A^{-1}}\theta}{d\theta}\,d\theta=\frac{d\tilde{\varphi}}{d\theta}\,d\theta \qquad \square$$

Given $A\in\mathbb{U}(1,1)$ and a probability measure μ on $\partial\mathbb{D}$, we say that μ is A-*invariant* if and only if $L_A(\mu)=\mu$. One can easily describe all the invariant measures for a given $A\in\mathbb{U}(1,1)$. Let $\mathcal{I}(A)$ be the set of invariant measures for A.

Theorem 10.4.12. *Let $A\in\mathbb{U}(1,1)$ with $A_{\rm s}$ given by (10.4.3). Then*

(1) *If A is hyperbolic, the invariant measures are precisely the convex combinations of the point masses at the two eigenspaces of A.*
(2) *If A is parabolic, the unique invariant measure is the point mass at A's unique eigenspace.*
(3) *If A is elliptic and the eigenvalues of $A_{\rm s}$ are not roots of unity, then A has a unique invariant measure described as follows. If $\binom{1}{re^{i\varphi}}$ with $r<1$ is an eigenvector of A, the invariant measure is $P_r(\theta,-\varphi)\frac{d\theta}{2\pi}$ where P_r is the Poisson kernel* (1.3.14).
(4) *If A is elliptic and the eigenvalues of $A_{\rm s}$ are roots of unity, let m be the smallest integer so that $A^m=\mathbf{1}$ or $-\mathbf{1}$. Let $\theta_0=1,\theta_1,\dots,\theta_{m-1}$ be a reordering of $\{\varphi\mid\varphi=L_A^j1,\ j=0,1,\dots,m-1\}$ so that $0=\theta_0<\theta_1<\cdots<\theta_{m-1}<2\pi$. Let ν be an arbitrary probability measure on $[\theta_0,\theta_1)$. Then $\mu=\frac{1}{m}\sum_{j=0}^{m-1}L_{A^j}(\nu)$ is A-invariant and every A-invariant measure has this form.*

Proof. (1) Suppose first that $A=\begin{pmatrix}\cosh(x) & \sinh(x)\\ \sinh(x) & \cosh(x)\end{pmatrix}$ for $x>0$. Then

$$A^n\begin{pmatrix}1\\ e^{i\theta}\end{pmatrix}=c_n(\theta)\begin{pmatrix}1\\ \beta_n\end{pmatrix}$$

with

$$\beta_n=\frac{e^{nx}\cos(\frac{\theta}{2})+ie^{-nx}\sin(\frac{\theta}{2})}{e^{nx}\cos(\frac{\theta}{2})-ie^{-nx}\sin(\frac{\theta}{2})}$$

for

$$\begin{pmatrix}1\\ e^{i\theta}\end{pmatrix}=e^{i\theta/2}\left[\cos\left(\frac{\theta}{2}\right)\begin{pmatrix}1\\1\end{pmatrix}-i\sin\left(\frac{\theta}{2}\right)\begin{pmatrix}1\\-1\end{pmatrix}\right]$$

and $A(\binom{1}{\pm1})=e^{\pm x}\binom{1}{\pm1}$.

As $n\to\infty$, $\beta_n\to1$ exponentially fast so long as $\theta\neq\pi$ uniformly in each $[-\alpha,\alpha]$ with $\alpha<\pi$. Therefore, for any $f\in C(\partial\mathbb{D})$,

$$f(L_A^n(e^{i\theta}))\to\begin{cases}f(1) & \text{if }\theta\neq\pi\\ f(-1) & \text{if }\theta=\pi\end{cases} \tag{10.4.35}$$

so $L_A^n(\mu)\to\mu(\{-1\})\delta_{-1}+(1-\mu(\{-1\}))\delta_1$ weakly as $n\to\infty$.

It follows that any $\mu \in \mathcal{I}(A)$ is a convex combination of δ_1 and δ_{-1}. Since

$$\mathcal{I}(VAV^{-1}) = L_V[\mathcal{I}(A)] \tag{10.4.36}$$

and any hyperbolic element of $\mathbb{SU}(1,1)$ is similar to $\begin{pmatrix} \cosh(x) & \sinh(x) \\ \sinh(x) & \cosh(x) \end{pmatrix}$ (by Theorem 10.4.7), we obtain the result for arbitrary hyperbolic elements.

(2) The proof is similar to that for A hyperbolic using the fact that if A is parabolic, $A^n \binom{1}{e^{i\theta}} \to \binom{1}{e^{i\theta_0}}$ with an error of $O(\frac{1}{n})$ if $\binom{1}{e^{i\theta_0}}$ is the unique eigenvector.

(3) Let $A = \begin{pmatrix} e^{i\theta_0} & 0 \\ 0 & e^{-i\theta_0} \end{pmatrix}$ with θ_0 an irrational multiple of 2π. By (10.4.28), if $L_A\mu = \mu$ and $\hat{\mu}_k = \int e^{-ik\theta}\, d\mu$, then $\hat{\mu}_k = e^{2ik\theta_0}\hat{\mu}_k$, so for $k \neq 0$, $\hat{\mu}_k = 0$ and $\mu = \frac{d\theta}{2\pi}$ by uniqueness of solutions of the moment problem. This proves the result for A of the special form. For any elliptic A, if $\binom{1}{re^{i\varphi}}$ is an eigenvector of A, then

$$A = V \begin{pmatrix} e^{i\theta_0} & 0 \\ 0 & e^{-i\theta_0} \end{pmatrix} V^{-1}$$

with

$$V = \frac{1}{\rho} \begin{pmatrix} 1 & re^{i\varphi} \\ re^{-i\varphi} & 1 \end{pmatrix}$$

where $\rho = (1-|r|^2)^{1/2}$. By (10.4.28), if θ_0 is irrational,

$$\mathcal{I}(A) = \{L_V(dm)\}$$

By (10.4.31),

$$(L_V(dm))(e^{i\theta}) = N_{V^{-1}}(e^{i\theta})^{-2}\, dm$$

Since

$$V^{-1} = \frac{1}{\rho} \begin{pmatrix} 1 & -re^{i\varphi} \\ -re^{-i\varphi} & 1 \end{pmatrix}$$

$N_{V^{-1}}(e^{i\theta})^2 = 2\|V^{-1}\mu_\theta\|^2 = \rho^{-2}|1 - re^{i(\varphi+\theta)}|^2$, so

$$N_{V^{-1}}(e^{i\theta})^{-2} = \frac{(1-|r|^2)}{(1+r^2-2r\cos(\theta+\varphi))} = P_r(\theta,-\varphi)$$

(4) By (10.4.28), it suffices to do the case where $A = \begin{pmatrix} e^{i\theta_0} & 0 \\ 0 & e^{-i\theta_0} \end{pmatrix}$ with $\theta_0 = \frac{\pi n}{m}$ with m and n relatively prime. In this case, L_A on $\partial\mathbb{D}$ permutes the intervals $\{[\frac{2\pi j}{m}, \frac{2\pi(j+1)}{m})\}_{j=0}^{m-1}$, from which the result is immediate. $\square$

Remark. We will provide a quantitative extension of (10.4.35) in Lemma 10.6.10.

The next topic in this section involves determining when $\mathcal{I}(A) \cap \mathcal{I}(B) \neq \emptyset$ for general $A, B \in \mathbb{U}(1,1)$. We will need this later (see Theorems 10.4.18 and 12.6.3) for $\mathcal{I}(A(\alpha,z)) \cap \mathcal{I}(A(\beta,z))$. A key role will be played by

Definition. $A \in \mathbb{SU}(1,1)$ is called a *reflection* if $A^2 = -1$ and $\operatorname{Tr}(A) = 0$.

This is a slight misnomer since A has eigenvalues $\pm i$ and it is iA which has eigenvalues $+1$ and -1 and is the "true" reflection. But since $L_A = L_{iA}$, we will use this terminology.

THEOREM 10.4.13. *Let $A, B \in \mathbb{SU}(1,1)$ be distinct and different from $\pm\mathbf{1}$. Suppose also $A \neq -B$. Then $\mathcal{I}(A) \cap \mathcal{I}(B) \neq \emptyset$ if and only if*

(a) *when A and B are both nonelliptic (i.e., each is hyperbolic or parabolic), if and only if A and B have a common eigenvector. $\mathcal{I}(A)\cap\mathcal{I}(B)$ is either $\{\delta_{e^{i\theta_0}}\}$ for a single θ_0 (if they have a single common eigenvector) or $\{\lambda\delta_{\theta_0}+(1-\lambda)\delta_{\theta_1}\mid\lambda\in[0,1]\}$ if they have a pair of common eigenvectors (i.e., if both are hyperbolic and they commute).*

(b) *when A is nonelliptic and B is elliptic, if and only if A is hyperbolic, B is a reflection, and L_B permutes the two eigenspaces of A. $\mathcal{I}(A)\cap\mathcal{I}(B)$ is then $\{\frac{1}{2}\delta_{\theta_0}+\frac{1}{2}\delta_{\theta_1}\}$ where θ_0,θ_1 are the two eigenspaces for A.*

(c) *when A and B are reflections, $\mathcal{I}(A)\cap\mathcal{I}(B)$ is always nonempty and has a single element $\frac{1}{2}\delta_{\theta_0}+\frac{1}{2}\delta_{\theta_1}$ where θ_0,θ_1 are the eigenspaces for $C=AB$, which is always hyperbolic.*

(d) *when A and B are both elliptic and at least one is not a reflection, if and only if A and B commute. In that case, when A and B both have eigenvalues which are roots of unity, there is a C with $A=\pm C^p$, $B=\pm C^q$, and p,q relatively prime and suitable choice of $\pm$, and $\mathcal{I}(A)\cap\mathcal{I}(B)=\mathcal{I}(C)$. If either A or B has eigenvalues which are not roots of unity, $\mathcal{I}(A)\cap\mathcal{I}(B)$ is a single element of the form $P_r(\theta,-\varphi)\frac{d\theta}{2\pi}$ for suitable r,φ.*

PROOF. (a) In this case, $\mathcal{I}(A)$ and $\mathcal{I}(B)$ consist of measures with point masses at the eigenspaces, so the result is immediate.

(b) $\mathcal{I}(A)$ consists of measures with only one or two pure points. By Theorem 10.4.12, $\mathcal{I}(B)$ never has a measure with one pure point and only has a measure with two pure points if B is a reflection. Thus, if B is not a reflection, $\mathcal{I}(A)\cap\mathcal{I}(B)=\emptyset$. Since B has no eigenspaces in $\partial\mathbb{D}$, L_B has no fixed points. But since $B^2=-1$, $L_B^2=\mathbf{1}$, and thus L_B has two point orbits. If the two eigenspaces of A are in such an orbit, $\mathcal{I}(A)\cap\mathcal{I}(B)=\{\frac{1}{2}\delta_{\theta_0}+\frac{1}{2}\delta_{\theta_0}\}$. If $L_B(e^{i\theta_0})\neq e^{i\theta_1}$, $\mathcal{I}(A)\cap\mathcal{I}(B)=0$.

(c) Suppose $A=\left(\begin{smallmatrix} i & 0\\ 0 & -i\end{smallmatrix}\right)$ and $B=\left(\begin{smallmatrix} ia & b\\ \bar b & -ia\end{smallmatrix}\right)$ with a real, $|a|^2-|b|^2=1$ and $b\neq 0$. Then $\operatorname{Tr}(AB)=-2a$. Since $|a|>1$, $C=AB$ is hyperbolic. Moreover, $BCB^{-1}=BA=C^{-1}$ (since $B^2=A^2=-1$) and $ACA^{-1}=-BA^{-1}=BA=C^{-1}$. Thus C is hyperbolic and A,B interchange its eigenspaces so, by (b), $\mathcal{I}(A)\cap\mathcal{I}(C)=\mathcal{I}(B)\cap\mathcal{I}(C)$ is a single element. Given arbitrary reflections A,B, there is a $V\in\mathbb{SU}(1,1)$ with $VAV^{-1}=\left(\begin{smallmatrix} i & 0\\ 0 & -i\end{smallmatrix}\right)$ and $VBV^{-1}=\left(\begin{smallmatrix} \alpha & \beta\\ \bar\beta & \bar\alpha\end{smallmatrix}\right)$. Since $B\neq\pm A$, $\beta\neq 0$, and since $\operatorname{Tr}(B)=0$, $\operatorname{Re}\alpha=0$, so $VBV^{-1}=\left(\begin{smallmatrix} ia & b\\ \bar b & -ia\end{smallmatrix}\right)$ with a real. The special case thus applies.

(d) We will need the lemma below that if A and B are noncommuting elliptic elements, the group, $\mathcal{G}(A,B)$, generated by A and B (i.e., the closure of all finite strings of A, B, A^{-1}, and B^{-1}) contains a hyperbolic element. Given this lemma, we note that if ν is invariant for A and B, it is also invariant for A^{-1}, B^{-1}, and so for all of $\mathcal{G}(A,B)$. If $C\in\mathcal{G}(A,B)$ is hyperbolic and if A is not a reflection, C and A have no common invariant measure by (b), and so $\mathcal{I}(A)\cap\mathcal{I}(B)=\emptyset$. □

LEMMA 10.4.14. *Let A and B in $\mathbb{U}(1,1)$ be noncommuting elliptic elements. Let $\mathcal{G}(A,B)$ be the smallest closed subgroup of $\mathbb{U}(1,1)$ containing A and B. Then $\mathcal{G}(A,B)$ contains a hyperbolic element.*

PROOF. Without loss, we can take $A,B\in\mathbb{SU}(1,1)$. By using the covariance $\mathcal{G}(VAV^{-1},VBV^{-1})=V\mathcal{G}(A,B)V^{-1}$, we can suppose A has $\binom{1}{0}$ and $\binom{0}{1}$ as eigenvectors, that is, a rotation about 0 in its action on $\mathbb{D}$. We will say A has rotation angle 2θ if $A\binom{1}{0} = e^{i\theta}\binom{1}{0}$ (since then $L_A(e^{i\varphi}) = e^{i(2\theta+\varphi)}$). If 2θ is irrational or $2\theta = \frac{2\pi\ell}{k}$ with k an even integer, $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$ is in the closed group generated by A, so we can suppose A is a reflection in that case. BAB^{-1} is also a reflection, and since $B\binom{1}{0} \neq \binom{1}{0}$ (for if it were, B would commute with A), it is distinct from A. Thus $C = (BAB^{-1})A$ is hyperbolic by the calculation in the proof of Theorem 10.4.13(c).

Thus we are reduced to the case where $2\theta = \frac{2\pi\ell}{k}$ with k odd at least 3. Then some power of A (the one with $\frac{2\pi\ell}{k}$ closest to π) has $2\theta_0 \in (\frac{2\pi}{3}, \frac{4\pi}{3})$ so, in particular,

$$\sin^2(\theta_0) > \cos^2(\theta_0) \tag{10.4.37}$$

Define B_n inductively by

$$B_1 = BAB^{-1} \qquad B_{n+1} = B_n A B_n^{-1} \tag{10.4.38}$$

Thus B_n are also rotations by angle θ_0, and no B_n has $\binom{1}{0}$ as eigenvector. We will show that for n large,

$$|\mathrm{Tr}(AB_n)| > 2 \tag{10.4.39}$$

so AB_n is hyperbolic, completing the proof. The intuition is that the "center" $\binom{1}{z_n}$ of the rotation B_n gets further and further from 0 and closer to $\partial\mathbb{D}$, since the angle of rotation $2\theta_0$ is close to π and $\binom{1}{z_{n+1}} = cB_n\binom{1}{0}$ rotating about z_n. As $|z_n| \to 1$, its matrix elements go to ∞, forcing a large trace for AB_n. Here are the calculations that verify this intuition.

Given $z \in \mathbb{D}$ and $\theta \in [0, 2\pi)$, let $B(z,\theta)$ be the unique element of $\mathbb{SU}(1,1)$ with

$$B(z,\theta)\begin{pmatrix} 1 \\ z \end{pmatrix} = e^{i\theta}\begin{pmatrix} 1 \\ z \end{pmatrix} \tag{10.4.40}$$

We know, by (10.4.21),

$$B(z,\theta) = \begin{pmatrix} a(z,\theta) & c(z,\theta) \\ \overline{c(z,\theta)} & \overline{a(z,\theta)} \end{pmatrix} \tag{10.4.41}$$

and we claim that if $z = re^{i\varphi}$, then

$$a(z,\theta) = \cos(\theta) + i\,\frac{(1+r^2)}{(1-r^2)}\sin(\theta) \tag{10.4.42}$$

$$c(z,\theta) = -e^{-i\varphi}(i\sin(\theta))\,\frac{2r}{1-r^2} \tag{10.4.43}$$

One can check with this choice that (10.4.40) holds, or conversely, solve (10.4.40) for a and c. The reader should confirm that $|a|^2 - |c|^2 = 1$.

Notice that

$$B(z,\theta)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a(z,\theta)\begin{pmatrix} 1 \\ \frac{\overline{c(z,\theta)}}{a(z,\theta)} \end{pmatrix} \tag{10.4.44}$$

which implies that

$$B(z,\theta)B(0,\theta_0)B(z,\theta)^{-1} = B\left(\frac{\overline{c(z,\theta)}}{a(z,\theta)}, \theta_0\right)$$

and use, by (10.4.38),

$$B_n = B(z_n, \theta_0)$$

with

$$z_{n+1} = \frac{\overline{c(z_n,\theta_0)}}{a(z_n,\theta_0)}$$

Thus, if $|z_n| = r_n$,

$$r_{n+1}^2 = \frac{4r_n \sin^2(\theta_0)}{[(1-r_n^2)^2\cos^2(\theta_0) + (1+r_n^2)^2\sin^2(\theta_0)]}$$

or

$$1 - r_{n+1}^2 = \frac{(1-r_n^2)^2}{[1 + r_n^4 + 2r_n^2(\sin^2(\theta_0) - \cos^2(\theta_0))]}$$

By (10.4.37), the denominator is larger than 1 and thus

$$1 - r_{n+1}^2 \le (1-r_n^2)^2$$

which implies that

$$1 - r_n^2 \le (1-r_1^2)^{2^{n-1}} \tag{10.4.45}$$

so, in particular, $r_n \to 1$ as $n \to \infty$ (and much faster than exponentially!).

To finish the argument, note that, by (10.4.41),

$$\begin{aligned}\operatorname{Tr}(B(z_n,\theta_0)B(0,\theta_0)) &= 2\operatorname{Re}(e^{i\theta_0}a(z_n,\theta_0)) \\ &= 2\operatorname{Re}\biggl[e^{i\theta_0}\biggl[\cos(\theta_0) + i\sin(\theta_0)\biggl(\frac{1+r_n^2}{1-r_n^2}\biggr)\biggr]\biggr] \\ &= 2\cos^2(\theta_0) + \sin^2(\theta_0)\,\frac{1+r_n^2}{1-r_n^2}\end{aligned}$$

goes to ∞ as $n \to \infty$, since $r_n \to 1$. Thus for n large, $B(z_n,\theta_0)B(0,\theta_0)$ is hyperbolic. □

This lemma not only implies the hardest part of Theorem 10.4.13, but also the following striking result:

THEOREM 10.4.15. *Every compact subgroup of* $\mathbb{U}(1,1)$ *is abelian.*

PROOF. Let $\mathbb{K}$ be a compact subgroup of $\mathbb{U}(1,1)$. If A is hyperbolic or parabolic, $\|A^n\| \to \infty$, so A^n has no limit and thus A cannot lie in any compact subgroup. Thus $\mathbb{K}$ can only have elliptic elements and $\pm\mathbf{1}$. If $\mathbb{K}$ is nonabelian, it must have noncommuting elliptic elements A and B. But then $\mathcal{G}(A,B) \subset \mathbb{K}$ has hyperbolic elements, violating the compactness of $\mathbb{K}$. Thus $\mathbb{K}$ must be abelian. □

Our final topic in this section is a return to the $A(\alpha,z)$ of (10.4.2) and asks about when $A(\alpha_1,z)$ and $A(\alpha_2,z)$ have a common invariant measure. As a warmup, we consider the analogous matrices for discrete Schrödinger operators (Jacobi matrices with $a_n \equiv 1$), namely,

$$J(a) = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \tag{10.4.46}$$

For if $a_n \equiv 1$, then $p_n(x)$ obey $p_{n+1}(x) = (x - b_{n+1})p_n(x) - p_{n-1}(x)$, so

$$\begin{pmatrix} p_{n+1}(x) \\ p_n(x) \end{pmatrix} = \begin{pmatrix} x - b_{n+1} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_n(x) \\ p_{n-1}(x) \end{pmatrix}$$

In (10.4.46), a is real, so $J(a) \in \mathbb{SL}(2,\mathbb{R})$, which is isomorphic to $\mathbb{SU}(1,1)$, so we can talk of elliptic, parabolic, and hyperbolic elements and invariant measures ($\partial\mathbb{D}$ is mapped by the isomorphism to $\mathbb{R} \cup \{\infty\} \cong \mathbb{P}(1)$, the real projective line, which can be realized as pairs of vectors $\pm\binom{\cos(\theta)}{\sin(\theta)}$, $\theta \in [0,\pi)$).

THEOREM 10.4.16. *Let $a, b \in \mathbb{R}$ be distinct. Then $J(a)$ and $J(b)$ have no common invariant measures.*

PROOF. $J(a)^{-1} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & a \end{smallmatrix}\right)$ so

$$J(a)J(b)^{-1} = \begin{pmatrix} 1 & a-b \\ 0 & 1 \end{pmatrix} \qquad J(b)^{-1}J(a) = \begin{pmatrix} 1 & 0 \\ b-a & 1 \end{pmatrix}$$

are both parabolic with invariant measures $\delta_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}$ and $\delta_{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)}$. Thus $J(a)J(b)^{-1}$ and $J(b)^{-1}J(a)$ have no common invariant measure, so the group generated by $J(a)$ and $J(b)$ has no common invariant measure. □

EXAMPLE 10.4.17. The nice result for the Jacobi case that there are never invariant measures for any pair of one-step transfers does not extend to the general OPUC case. If $z = -1$, $\operatorname{Tr}(A(\alpha, z)) = 0$, so $A(\alpha, z = -1)$ is a reflection. Thus for *any* pair $\alpha, \beta \in \mathbb{D}$, $A(\alpha, -1)$ and $A(\beta, -1)$ have an invariant measure. If $z = 1$ and $\alpha = -e^{i\psi}\tanh(x)$, then

$$A = \begin{pmatrix} \cosh(x) & e^{-i\psi}\sinh(x) \\ e^{i\psi}\sinh(x) & \cosh(x) \end{pmatrix}$$

is hyperbolic and

$$A\begin{pmatrix} 1 \\ \pm e^{i\psi} \end{pmatrix} = e^{\pm x}\begin{pmatrix} 1 \\ \pm e^{i\psi} \end{pmatrix}$$

It follows that $A(\alpha, z = 1)$ and $A(\beta, z = 1)$ have a common invariant measure if and only if $\arg\alpha = \arg\beta$ or $\arg\alpha = \pi + \arg\beta$, that is, if and only if $\operatorname{Im}(\bar\alpha\beta) = 0$. □

Here is the general result:

THEOREM 10.4.18. *Let $\alpha, \beta \in \mathbb{D}$ be distinct. Then*

(a) *If $\operatorname{Im}(\alpha\bar\beta) = 0$ (i.e., $\beta = 0$ or $\alpha = \lambda\beta$ with $\lambda \in \mathbb{R}$), then $\mathcal{I}(A(\alpha, z)) \cap \mathcal{I}(A(\beta, z)) \neq \emptyset$ if and only if $z = 1$ or $z = -1$.*

(b) *If $\operatorname{Im}(\alpha\bar\beta) \neq 0$, and if $\theta_0 \in (0, \pi)$ is given by*

$$\sin\left(\frac{\theta_0}{2}\right) = \frac{1}{2}\,\frac{|\operatorname{Im}(\bar\alpha\beta)|}{|\alpha - \beta|} \tag{10.4.47}$$

then $\mathcal{I}(A(\alpha, z)) \cap \mathcal{I}(A(\beta, z)) \neq \emptyset$ if and only if $z = -1$, $e^{i\theta_0}$, or $e^{-i\theta_0}$.

Remarks. 1. Thus, there are either two or three exceptional points.

2. $|\operatorname{Im}(\bar\alpha\beta)| = |\operatorname{Im}(\bar\alpha(\beta - \alpha))| \leq |\alpha||\beta - \alpha|$ so the right side of (10.4.47) is at most $\frac{1}{2}$ and so $\theta_0 \in (0, \frac{\pi}{3})$.

PROOF. By the discussion in Example 10.4.17, $\mathcal{I}(A(\alpha, z = -1)) \cap \mathcal{I}(A(\beta, z = -1)) \neq \emptyset$. So we consider $z \neq -1$.

The 12 matrix element of $A(\alpha, z)A(\beta, z)$ is $-z\bar\beta - \bar\alpha$, so the 12 matrix element of $[A(\alpha, z), A(\beta, z)]$ is $(1 - z)(\bar\beta - \bar\alpha) \neq 0$ if $\alpha \neq \beta$ and $z \neq 1$. Since $A(\alpha, z = 1)$ is always hyperbolic, $A(\alpha, z)$ and $A(\beta, z)$ are never commuting elliptic matrices.

If $z \neq -1$, $A(\alpha, z)$ is not a reflection, so if $A(\alpha, z)$ is elliptic, it and $A(\beta, z)$ cannot have a common invariant measure by Theorem 10.4.13 and the argument above that $A(\alpha, z)$ and $A(\beta, z)$ are never commuting elliptic matrices. Thus we need only consider the possibility that $A(\alpha, z)$ and $A(\beta, z)$ have a common eigenvector $\left(\begin{smallmatrix} 1 \\ w \end{smallmatrix}\right)$ with $w \in \partial\mathbb{D}$.

If $A\binom{1}{w} = \binom{a_1}{a_2}$, then $\binom{1}{w}$ is an eigenvector of A if and only if $a_2 = wa_1$. Thus $\binom{1}{w}$ is an eigenvector of A if and only if

$$(z - \bar{\alpha}w)w = -\alpha z + w \tag{10.4.48}$$

This holds for α and β if and only if it holds for α and

$$(\bar{\beta} - \bar{\alpha})w^2 = (\beta - \alpha)z \tag{10.4.49}$$

Given α, β distinct, define

$$x = \frac{\beta - \alpha}{\bar{\beta} - \bar{\alpha}} \in \partial\mathbb{D} \tag{10.4.50}$$

so (10.4.49) becomes

$$w^2 = xz \tag{10.4.51}$$

Under (10.4.51), (10.4.48) is equivalent to

$$w(1 - z) = (\alpha - \bar{\alpha}x)z \tag{10.4.52}$$

We claim this is equivalent to

$$w^2(1 - z)^2 = (\alpha - \bar{\alpha}x)^2 z^2 \tag{10.4.53}$$

For clearly (10.4.52) implies (10.4.53) and, given a solution of (10.4.51) and (10.4.53), we can choose the sign of w so that (10.4.52) holds.

Clearly, (10.4.51) and (10.4.53) hold if and only if

$$\begin{aligned}(1 - z)^2 &= x^{-1}(\alpha - \bar{\alpha}x)^2 z \\ &= -|\alpha - \bar{\alpha}x|^2 z\end{aligned} \tag{10.4.54}$$

since $x^{-1} = \bar{x}$. By a simple calculation and (10.4.47),

$$\begin{aligned}|\alpha - \bar{\alpha}x|^2 &= \frac{|\mathrm{Im}(\bar{\alpha}\beta)|^2}{|\alpha - \beta|^2} = 4\sin^2\left(\frac{\theta_0}{2}\right) \\ &= 2 - 2\cos(\theta_0)\end{aligned}$$

so (10.4.54) is

$$z^2 - 2\cos(\theta_0)z + 1 = 0 \tag{10.4.55}$$

where solutions are $z = e^{\pm i\theta_0}$.

Thus, if $\mathcal{I}(A(\alpha, z)) \cap \mathcal{I}(A(\beta, z)) \neq \emptyset$, as we have shown, z must solve (10.4.55). Conversely, if $z \neq +1$ solves (10.4.55), and we define w by (10.4.52), then w and z obey (10.4.51) and (10.4.52), and so (10.4.49) for α and β, and so $\mathcal{I}(A(\alpha, z)) \cap \mathcal{I}(A(\beta, z)) \neq \emptyset$. If $z = 1$ solves (10.4.55), then $\mathrm{Im}(\bar{\alpha}\beta) = 0$, in which case we saw in Example 10.4.17 that $\mathcal{I}(A(\alpha, z = 1)) \cap \mathcal{I}(A(\beta, z = 1)) \neq \emptyset$. □

One can go on and ask about common elements of more than two $\mathcal{I}(A(\alpha, z))$. Here is a typical result:

THEOREM 10.4.19. *Let $\{\alpha_j\}_{j=1}^k$ be a collection of points of $\mathbb{D}$. Then $\cap_{j=1}^k \mathcal{I}(A(\alpha_j, z = -1))$ is nonempty if and only if the collection of points lies on a circle that intersects $\partial\mathbb{D}$ orthogonally.*

Remarks. 1. As usual, with fractional linear transformations, "circle" means circle or line.

2. If $k = 2$, there is always such a circle — the one through α_1, α_2, and $-\frac{1}{\bar{\alpha}_1}$. If $k = 3$, the points determine a unique circle which may or may not be orthogonal to $\partial\mathbb{D}$. If $k \geq 4$, the first three points determine a unique circle which may or may not contain the other points.

PROOF. As in the proof of Theorem 10.4.13(c), using the fact that fractional linear transformations are conformal (angle preserving), we can suppose $\alpha_1 = 0$, in which case the "circles" through α_1 orthogonal to $\partial\mathbb{D}$ are straight lines and the condition of lying on such a line is $\mathrm{Im}(\bar{\alpha}_i \alpha_j) = 0$ for all i, j.

Let $C_j \equiv A(\alpha_j, z = -1)$. By a direct calculation using $C_1 C_j = (1 - |\alpha_j|^2)^{-1/2} \left(\begin{smallmatrix} 1 & \bar{\alpha}_j \\ \alpha_j & 1 \end{smallmatrix}\right)$, one sees that $[C_1 C_i, C_1 C_j] = 0$ if and only if $\mathrm{Im}(\bar{\alpha}_i \alpha_j) = 0$. If the commutator is 0, C_1, C_i, and C_j all leave invariant $\frac{1}{2}\delta_{w_0} + \frac{1}{2}\delta_{w_1}$ where w_0, w_1 are the common eigenspaces for $C_1 C_i$ and $C_1 C_j$.

Conversely, if there is a common invariant measure, it must be $\frac{1}{2}\delta_{w_0} + \frac{1}{2}\delta_{w_1}$ with w_0, w_1 the eigenspaces of $C_0 C_i$ and $C_0 C_j$. If they have this pair of eigenspaces, they must commute. $\square$

Remarks and Historical Notes. The literature on $\mathbb{U}(1,1)$ and $\mathbb{SL}(2,\mathbb{R})$ is enormous in part because they arise in so many contexts. Obviously, as groups of fractional linear transformations, they play a critical role in complex analysis. As we will see, they are also groups of isometries of the hyperbolic plane, and so they are important in hyperbolic geometry — a subject central to the modern theory of three-dimensional manifolds. Finally, the group $\mathbb{SL}(2,\mathbb{Z}) \subset \mathbb{SL}(2,\mathbb{R})$ of elements of $\mathbb{SL}(2,\mathbb{R})$ with integral coefficients is a core part of modern algebraic number theory. For this reason, the basics, through Theorem 10.4.12, have been discussed and rediscovered many times and are part of the folklore. So I will make no attempt to trace their history except to note that the use of the group $\mathbb{U}(1,1)$ to describe conformal maps of $\mathbb{D}$ to itself and its connection to hyperbolic geometry were born full blown from Poincaré's great 1880 work which established his reputation; see Yandell [**1111**].

If the disk $\mathbb{D}$ is given the real Riemannian metric $(1 - |z|^2)^{-2}(dx^2 + dy^2)$, it is a manifold with constant negative curvature. This is the Poincaré metric and the Poincaré model of the hyperbolic plane. The group $\mathbb{SU}(1,1)/\{\mathbf{1}, -\mathbf{1}\}$, acting via fractional linear transformations, is precisely the group of all orientation-preserving isometries of $\mathbb{D}$ with this metric. The geometry on $\mathbb{D}$ induced by this metric is called *hyperbolic geometry.* One often writes H^2 instead of $\mathbb{D}$.

It is useful to think of two other alternative geometries. $\mathbb{R}^2$ with the standard Euclidean metric has as its orientation-preserving isometry group, the group of translations and rotations (about any axis), $\mathbb{E}(2) \cong \mathbb{R}^2 \circledS \mathrm{SO}(2)$. This is Euclidean geometry. The curvature here is constant and zero. The third geometry is on the two-sphere $\mathbb{S}^2$ with the usual rotation invariant metric. The group of orientation-preserving isometries here is $\mathbb{SO}(3)$. This model has constant positive curvature and is called spherical geometry.

Each isometry group is three-dimensional. The Lie algebra [**554**] has three generators, R, T_1, T_2 where R generates rotations about some fixed point (conventionally 0 in the $\mathbb{D}$ case) and T_1, T_2 infinitesimal "translations." The Lie relations are

$$[R, T_1] = T_2$$
$$[R, T_2] = -T_1$$
$$[T_1, T_2] = \sigma R$$

with $\sigma = +1$ for $\mathbb{S}^2$, $\sigma = 0$ for $\mathbb{R}^2$, and $\sigma = -1$ for H^2. For $\mathbb{SU}(1,1)$, an explicit realization is

$$R = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad T_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad T_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

so, for example, $e^{aT_1} = (1 - \frac{a^2}{4})^{-1/2}\begin{pmatrix} 1 & a/2 \\ a/2 & 1 \end{pmatrix}$ generating the fractional linear transformation $z \mapsto \frac{z+a/2}{1+az/2}$ for real a.

Lemma 10.4.14 is essentially a result in hyperbolic geometry. It is interesting to consider the analogs of Theorem 10.4.15 in the other geometries. Like $\mathbb{U}(1,1)$, $\mathbb{E}(2)$ has no nonabelian compact groups. For $\mathbb{SO}(3)$, one should ask about compact, proper subgroups. It is easy to see the only proper subgroup of positive dimension are the rotations about a single axis (abelian) and the infinite dihedral group. The finite subgroups are all known and classified: There are, in fact, finite nonabelian subgroups associated with the Platonic solids and the dihedral groups; see [**972**, p. 15] for the beautiful Klein-Weyl analysis of the set of discrete subgroups. It is interesting to note why Lemma 10.4.14 fails to extend to the spherical geometry case because rotation returns on back to the initial axis.

The special role of $z = \pm 1$ in Theorem 10.4.18 seems to violate the notion that a circle has no end. One can understand this in the context of the applications of Theorem 10.4.18 to measures with random Verblunsky coefficients chosen from α_1 and α_2. To get "equivalence" of z and 1, we need to rotate the circle from 1 to z which maps the Verblunsky coefficients $\alpha_j(d\mu) \to z^{j+1}\alpha_j(d\mu)$. If one considers random coefficients where $\alpha_j(d\mu)$ is chosen from $\alpha_1 z^{j+1}$ and $\alpha_2 z^{j+1}$, then $\pm z$ have special roles instead of ± 1.

Theorem 10.4.16 is due to Ishii-Matsuda [**545**], although their proof is different.

A theorem like Theorem 10.4.18 appears in Katsnelson [**616**], but his result is, unfortunately, false. In our notation, his theorem asserts that when $\mathrm{Im}(\bar\alpha_1\alpha_2) \neq 0$, $\mathcal{I}(A(\alpha, z)) \cap \mathcal{I}(A(\alpha_2, z))$ is always empty and if $\mathrm{Im}(\bar\alpha_1\alpha_2) = 0$, it is empty if and only if $z \neq \pm 1$. The error in his proof is that he asserts $A(\alpha_1, z)A(\alpha_2, z)^{-1}$ is Hermitian, which is true if and only if $\mathrm{Im}(\bar\alpha_1\alpha_2) = 0$. In response to our findings, he has a correction; see [**617**].

10.5. Lyapunov Exponents and the Growth of Norms in $\mathbb{U}(1,1)$

Spectral analysis will depend on the large n behavior of $\varphi_n(z)$ when $z \in \partial\mathbb{D}$. It will be useful to consider the transfer matrix

$$T_n(z) = \tfrac{1}{2}\begin{pmatrix} \varphi_n(z) + \psi_n(z) & \varphi_n(z) - \psi_n(z) \\ \varphi_n^*(z) - \psi_n^*(z) & \varphi_n^*(z) + \psi_n^*(z) \end{pmatrix} \tag{10.5.1}$$

discussed already in Section 3.2 (see (3.2.27)), which is the matrix product

$$T_n(z) = A(\alpha_{n-1}, z) \dots A(\alpha_0, z) \tag{10.5.2}$$

with $A(\alpha, z)$ given by (10.4.2). We sometimes use $T_n(z, \{\alpha_j\}_{j=1}^\infty)$ when we want to make the α_j's explicit.