STRICTLY ERGODIC SUBSHIFTS AND ASSOCIATED OPERATORS

DAVID DAMANIK

Dedicated to Barry Simon on the occasion of his 60th birthday.

Abstract. We consider ergodic families of Schrödinger operators over base dynamics given by strictly ergodic subshifts on finite alphabets. It is expected that the majority of these operators have purely singular continuous spectrum supported on a Cantor set of zero Lebesgue measure. These properties have indeed been established for large classes of operators of this type over the course of the last twenty years. We review the mechanisms leading to these results and briefly discuss analogues for CMV matrices.

1. Introduction

When I was a student in the mid-1990's at the Johann Wolfgang Goethe Universität in Frankfurt, my advisor Joachim Weidmann and his students and postdocs would meet in his office for coffee every day and discuss mathematics and life. One day we walked in and found a stack of preprints on the coffee table. What now must seem like an ancient practice was not entirely uncommon in those days: In addition to posting preprints on the archives, people would actually send out hardcopies of them to their peers around the world.

In this particular instance, Barry Simon had sent a series of preprints, all dealing with singular continuous spectrum. At the time I did not know Barry personally but was well aware of his reputation and immense research output. I was intrigued by these preprints. After all, we had learned from various sources (including the Reed-Simon books!) that singular continuous spectrum is sort of a nuisance and something whose absence should be proven in as many cases as possible. Now we were told that singular continuous spectrum is generic?

Soon after reading through the preprint series it became clear to me that my thesis topic should have something to do with this beast: singular continuous spectrum. Coincidentally, only a short while later I came across a beautifully written paper by Sütő [130] that raised my interest in the Fibonacci operator. I had studied papers on the almost Mathieu operator earlier. For that operator, singular continuous spectrum does occur, but only in very special cases, that is, for special choices of the coupling constant, the frequency, or the phase. In the Fibonacci case, however, singular continuous spectrum seemed to be the rule. At least there was no sensitive dependence on the coupling constant or the frequency as I learned from [15, 130, 131].

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Another feature, which occurs in the almost Mathieu case, but only at special coupling, seemed to be the rule for the Fibonacci operator: zero-measure spectrum.

So I set out to understand what about the Fibonacci operator was responsible for this persistent occurrence of zero-measure singular continuous spectrum. Now, some ten years later, I still do not really understand it. In fact, as is always the case, the more you understand (or think you understand), the more you realize how much else is out there, still waiting to be understood.[1]

Thus, this survey is meant as a snapshot of the current level of understanding of things related to the Fibonacci operator and also as a thank-you to Barry for having had the time and interest to devote a section or two of his OPUC book to subshifts and the Fibonacci CMV matrix. Happy Birthday, Barry, and thank you for being an inspiration to so many generations of mathematical physicists!

2. Strictly Ergodic Subshifts

In this section we define strictly ergodic subshifts over a finite alphabet and discuss several examples that have been studied from many different perspectives in a great number of papers.

2.1. Basic Definitions.

We begin with the definitions of the the basic objects:

Definition 2.1 (full shift)**.** *Let $\mathcal{A}$ be a finite set, called the alphabet. The two-sided infinite sequences with values in $\mathcal{A}$ form the full shift $\mathcal{A}^{\mathbb{Z}}$. We endow $\mathcal{A}$ with the discrete topology and the full shift with the product topology.*

Definition 2.2 (shift transformation)**.** *The shift transformation T acts on the full shift by $[T\omega]_n = \omega_{n+1}$.*

Definition 2.3 (subshift)**.** *A subset Ω of the full shift is called a subshift if it is closed and T-invariant.*

Thus, our base dynamical systems will be given by (Ω, T), where Ω is a subshift and T is the shift transformation. This is a special class of topological dynamical systems that is interesting in its own right. Basic questions regarding them concern the structure of orbits and invariant (probability) measures. The situation is particularly simple when orbit closures and invariant measures are unique:

Definition 2.4 (minimality)**.** *Let Ω be a subshift and $\omega \in \Omega$. The orbit of ω is given by $\mathcal{O}_\omega = \{T^n\omega : n \in \mathbb{Z}\}$. If $\mathcal{O}_\omega$ is dense in Ω for every $\omega \in \Omega$, then Ω is called minimal.*

Definition 2.5 (unique ergodicity)**.** *Let Ω be a subshift. A Borel measure μ on Ω is called T-invariant if $\mu(T(A)) = \mu(A)$ for every Borel set $A \subseteq \Omega$. Ω is called uniquely ergodic if there is a unique T-invariant Borel probability measure on Ω.*

By compactness of Ω, the set of T-invariant Borel probability measure on Ω is non-empty. It is also convex and the extreme points are exactly the ergodic measures, that is, probability measures for which $T(A) = A$ implies that either $\mu(A) = 0$ or $\mu(A) = 1$. Thus, a subshift is uniquely ergodic precisely when there is a unique ergodic measure on it.

We will focus our main attention on subshifts having both of these properties. For convenience, one often combines these two notions into one:

[1] Most recently, I have come to realize that I do not understand why the Lyapunov exponent vanishes on the spectrum, even at large coupling. Who knows what will be next...

Definition 2.6 (strict ergodicity)**.** *A subshift Ω is called strictly ergodic if it is both minimal and uniquely ergodic.*

2.2. **Examples of Strictly Ergodic Subshifts.** Let us list some classes of strictly ergodic subshifts that have been studied by a variety of authors and from many different perspectives (e.g., symbolic dynamics, number theory, spectral theory, operator algebras, etc.) in the past.

2.2.1. *Subshifts Generated by Sequences.* Here we discuss a convenient way of defining a subshift starting from a sequence $s \in \mathcal{A}^{\mathbb{Z}}$. Since $\mathcal{A}^{\mathbb{Z}}$ is compact, $\mathcal{O}_s$ has a non-empty set of accumulation points, denoted by Ω_s. It is readily seen that Ω_s is closed and T-invariant. Thus, we call Ω_s the *subshift generated by* s. Naturally, we seek conditions on s that imply that Ω_s is minimal or uniquely ergodic.

Every word w (also called block or string) of the form $w = s_m \dots s_{m+n-1}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ is called a *subword of* s (of *length* n, denoted by $|w|$). We denote the set of all subwords of s of length n by $\mathcal{W}_s(n)$ and let

$$\mathcal{W}_s = \bigcup_{n \geq 1} \mathcal{W}_s(n).$$

If $w \in \mathcal{W}_s(n)$, let $\cdots < m_{-1} < 0 \leq m_0 < m_1 < \cdots$ be the integers m for which $s_m \dots s_{m+n-1} = w$. The sequence s is called *recurrent* if $m_n \to \pm\infty$ as $n \to \pm\infty$ for every $w \in \mathcal{W}_s$. A recurrent sequence s is called *uniformly recurrent* if $(m_{j+1} - m_j)_{j \in \mathbb{Z}}$ is bounded for every $w \in \mathcal{W}_s$. Finally, a uniformly recurrent sequence s is called *linearly recurrent* if there is a constant $C < \infty$ such that for every $w \in \mathcal{W}_s$, the gaps $m_{j+1} - m_j$ are bounded by $C|w|$.

We say that $w \in \mathcal{W}_s$ *occurs in* s *with a uniform frequency* if there is $\mathrm{d}_s(w) \geq 0$ such that, for every $k \in \mathbb{Z}$,

$$\mathrm{d}_s(w) = \lim_{n \to \infty} \frac{1}{n} \left| \{m_j\}_{j \in \mathbb{Z}} \cap [k, k+n) \right|,$$

and the convergence is uniform in k.

For results concerning the minimality and unique ergodicity of the subshift Ω_s generated by a sequence s, we recommend the book by Queffélec [120]; see in particular Section IV.2. Let us recall the main findings.

Proposition 2.7. *If s is uniformly recurrent, then Ω_s is minimal. Conversely, if Ω is minimal, then every $\omega \in \Omega$ is uniformly recurrent. Moreover, $\mathcal{W}_{\omega_1} = \mathcal{W}_{\omega_2}$ for every $\omega_1, \omega_2 \in \Omega$.*

The last statement permits us to define a set $\mathcal{W}_\Omega$ for any minimal subshift Ω so that $\mathcal{W}_\Omega = \mathcal{W}_\omega$ for every $\omega \in \Omega$.

Proposition 2.8. *Let s be recurrent. Then, Ω_s is uniquely ergodic if and only if each subword of s occurs with a uniform frequency.*

As a consequence, Ω_s is strictly ergodic if and only if each subword w of s occurs with a uniform frequency $\mathrm{d}_s(w) > 0$.

An interesting class of strictly ergodic subshifts is given by those subshifts that are generated by linearly recurrent sequences [65, 104]:

Proposition 2.9. *If s is linearly recurrent, then Ω_s is strictly ergodic.*

2.2.2. *Sturmian Sequences.* Suppose s is a uniformly recurrent sequence. We saw above that Ω_s is a minimal subshift and all elements of Ω_s have the same set of subwords, $\mathcal{W}_{\Omega_s} = \mathcal{W}_s$. Let us denote the cardinality of $\mathcal{W}_s(n)$ by $p_s(n)$. The map $\mathbb{Z}^+ \to \mathbb{Z}^+$, $n \mapsto p_s(n)$ is called the *complexity function of* s (also called *factor, block* or *subword complexity function*).

It is clear that a periodic sequence gives rise to a bounded complexity function. It is less straightforward that every non-periodic sequence gives rise to a complexity function that grows at least linearly. This fact is a consequence of the following celebrated theorem due to Hedlund and Morse [116]:

Theorem 2.10. *If s is recurrent, then the following statements are equivalent:*

(i) *s is periodic, that is, there exists k such that $s_m = s_{m+k}$ for every $m \in \mathbb{Z}$.*
(ii) *p_s is bounded, that is, there exists p such that $p_s(n) \le p$ for every $n \in \mathbb{Z}^+$.*
(iii) *There exists $n_0 \in \mathbb{Z}^+$ such that $p_s(n_0) \le n_0$.*

Proof. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious, so we only need to show (iii) $\Rightarrow$ (i).

Let $R_s(n)$ be the directed graph with $p_s(n)$ vertices and $p_s(n+1)$ edges which is defined as follows. Every subword $w \in \mathcal{W}_s(n)$ corresponds to a vertex of $R_s(n)$. Every $\tilde{w} \in \mathcal{W}_s(n+1)$ generates an edge of $R_s(n)$ as follows. Write $\tilde{w} = axb$, where $a, b \in \mathcal{A}$ and x is a (possibly empty) string. Then draw an edge from the vertex ax to the vertex xb.

We may assume that $\mathcal{A}$ has cardinality at least two since otherwise the theorem is trivial. Thus, $p_s(1) \ge 2 > 1$. Obviously, p_s is non-decreasing. Thus, by assumption (iii), there must be $1 \le n_1 < n_0$ such that $p_s(n_1) = p_s(n_1 + 1)$. Consider the graph $R_s(n_1)$. Since s is recurrent, there must be a directed path from w_1 to w_2 for every pair $w_1, w_2 \in \mathcal{W}_s(n_1)$. On the other hand, $R_s(n_1)$ has the same number of vertices and edges. It follows that $R_s(n_1)$ is a simple cycle and hence s is periodic of period $p_s(n_1)$. $\square$

The graph $R_s(n)$ introduced in the proof above is called the *Rauzy graph* associated with s and n. It is an important tool for studying (so-called) combinatorics on words. This short proof of the Hedlund-Morse Theorem is just one of its many applications.

Corollary 2.11. *If s is recurrent and not periodic, then $p_s(n) \ge n+1$ for every $n \in \mathbb{Z}^+$.*

This raises the question whether aperiodic sequences of minimal complexity exist.

Definition 2.12. *A sequence s is called Sturmian if it is recurrent and satisfies $p_s(n) = n + 1$ for every $n \in \mathbb{Z}^+$.*

Remarks. (a) There are non-recurrent sequences s with complexity $p_s(n) = n+1$. For example, $s_n = \delta_{n,0}$. The subshifts generated by such sequences are trivial and we therefore restrict our attention to recurrent sequences.
(b) We have seen that growth strictly between bounded and linear is impossible for a complexity function. It is an interesting open problem to characterize the increasing functions from $\mathbb{Z}^+$ to $\mathbb{Z}^+$ that arise as complexity functions.

Note that a Sturmian sequence is necessarily defined on a two-symbol alphabet $\mathcal{A}$. Without loss of generality, we restrict our attention to $\mathcal{A} = \{0, 1\}$. The following result gives an explicit characterization of all Sturmian sequences with respect to this normalization; compare [113, Theorem 2.1.13].

Theorem 2.13. *A sequence $s \in \{0,1\}^{\mathbb{Z}}$ is Sturmian if and only if there are $\theta \in (0,1)$ irrational and $\phi \in [0,1)$ such that either*

$$s_n = \chi_{[1-\theta,1)}(n\theta + \phi) \quad or \quad s_n = \chi_{(1-\theta,1]}(n\theta + \phi) \tag{1}$$

for all $n \in \mathbb{Z}$.

Remark. In (1), we consider the 1-periodic extension of the function $\chi_{[1-\theta,1)}(\cdot)$ (resp., $\chi_{(1-\theta,1]}(\cdot)$) on $[0,1)$.

Each sequence of the form (1) generates a subshift. The following theorem shows that the resulting subshift only depends on θ. A proof of this result may be found, for example, in the appendix of [44].

Theorem 2.14. *Assume $\theta \in (0,1)$ is irrational, $\phi \in [0,1)$, and $s_n = \chi_{[1-\theta,1)}(n\theta + \phi)$. Then the subshift generated by s is given by*

$$\Omega_s = \left\{ n \mapsto \chi_{[1-\theta,1)}(n\theta + \tilde{\phi}) : \tilde{\phi} \in [0,1) \right\} \cup \left\{ n \mapsto \chi_{(1-\theta,1]}(n\theta + \tilde{\phi}) : \tilde{\phi} \in [0,1) \right\}.$$

Moreover, Ω_s is strictly ergodic.

Let us call a subshift *Sturmian* if it is generated by a Sturmian sequence. We see from the previous theorem that there is a one-to-one correspondence between irrational numbers θ and Sturmian subshifts. We call θ the *slope* of the subshift.

Example (Fibonacci case). The Sturmian subshift corresponding to the inverse of the golden mean,

$$\theta = \frac{\sqrt{5}-1}{2},$$

is called the *Fibonacci subshift* and its elements are called *Fibonacci sequences*.

An important property of Sturmian sequences is their hierarchical, or *S-adic*, structure. That is, there is a natural level of hierarchies such that on each level, there is a unique decomposition of the sequence into blocks of two types. The starting level is just the decomposition into individual symbols. Then, one may pass from one level to the next by a set of rules that is determined by the coefficients in the continued fraction expansion of the slope θ.

Let

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \tag{2}$$

be the continued fraction expansion of θ with uniquely determined $a_k \in \mathbb{Z}^+$. Truncation of this expansion after k steps yields rational numbers p_k/q_k that obey

$$p_0 = 0, \quad p_1 = 1, \quad p_k = a_k p_{k-1} + p_{k-2}, \tag{3}$$

$$q_0 = 1, \quad q_1 = a_1, \quad q_k = a_k q_{k-1} + q_{k-2}. \tag{4}$$

These rational numbers are known to be best approximants to θ. See Khinchin [94] for background on continued fraction expansions.

We define words $(w_k)_{k\in\mathbb{Z}_0^+}$ over the alphabet $\{0,1\}$ as follows:

$$w_0 = 0, \quad w_1 = 0^{a_1-1}1, \quad w_{k+1} = w_k^{a_{k+1}} w_{k-1} \text{ for } k \geq 1. \tag{5}$$

Theorem 2.15. *Let Ω be a Sturmian subshift with slope θ and let the words w_k be defined by (2) and (5). Then, for every $k \in \mathbb{Z}^+$, each $\omega \in \Omega$ has a unique partition, called the k-partition of ω, into blocks of the form w_k or w_{k-1}. In this partition, blocks of type w_k occur with multiplicity a_{k+1} or $a_{k+1}+1$ and blocks of type w_{k-1} occur with multiplicity one.*

Sketch of proof. The first step is to use the fact that p_k/q_k are best approximants to show that the restriction of $\chi_{[1-\theta,1)}(n\theta)$ to the interval $[1, q_k]$ is given by w_k, $k \in \mathbb{Z}^+$; compare [15]. The recursion (5) therefore yields a k-partition of $\chi_{[1-\theta,1)}(n\theta)$ on $[1,\infty)$. Since every $\omega \in \Omega$ may be obtained as an accumulation point of shifts of this sequence, it can then be shown that a unique partition of ω is induced; see [38]. The remaining claims follow quickly from the recursion (5) □

Example (Fibonacci case, continued). In the Fibonacci case, $a_k = 1$ for every k. Thus, both (p_k) and (q_k) are sequences of Fibonacci numbers (i.e., $p_{k+1} = q_k = F_k$, where $F_0 = F_1 = 1$ and $F_{k+1} = F_k + F_{k-1}$ for $k \geq 1$) and the words w_k are obtained by the simple rule

$$w_0 = 0, \quad w_1 = 1, \quad w_k = w_{k-1}w_{k-2} \text{ for } k \geq 2. \tag{6}$$

Thus, the sequence $(w_k)_{k\in\mathbb{Z}^+}$ is given by $1,\ 10,\ 101,\ 10110,\ 10110101, \ldots$, which may also be obtained by iterating the rule

$$1 \mapsto 10, \quad 0 \mapsto 1, \tag{7}$$

starting with the symbol 1.

For the proofs omitted in this subsection and much more information on Sturmian sequences and subshifts, we refer the reader to [16, 44, 113, 117].

2.2.3. *Codings of Rotations.* Theorem 2.13 shows that Sturmian sequences are obtained by coding an irrational rotation of the torus according to a partition of the circle into two half-open intervals. It is natural to generalize this and consider codings of rotations with respect to a more general partition of the circle. Thus, let $[0,1) = I_1 \cup \ldots \cup I_l$ be a partition into l half-open intervals. Choosing numbers $\lambda_1, \ldots, \lambda_l$, we consider the sequences

$$s_n = \sum_{j=1}^{l} \lambda_j \chi_{I_j}(n\theta + \phi). \tag{8}$$

Subshifts generated by sequences of this form will be said to be associated with codings of rotations.

Theorem 2.16. *Let $\theta \in (0,1)$ be irrational and $\phi \in [0,1)$. If s is of the form (8), then Ω_s is strictly ergodic. Moreover, the complexity function satisfies $p_s(n) = an+b$ for every $n \geq n_0$ and suitable integers a, b, n_0.*

See [77] for a proof of strict ergodicity and [1] for a proof of the complexity statement. In fact, the integers a, b, n_0 can be described explicitly; see [1, Theorem 10].

2.2.4. *Arnoux-Rauzy and Episturmian Subshifts.* Let us consider a minimal subshift Ω over the alphabet $\mathcal{A}_m = \{0, 1, 2, \dots, m-1\}$, where $m \geq 2$. A word $w \in \mathcal{W}_\Omega$ is called *right-special* (resp., *left-special*) if there are distinct symbols $a, b \in \mathcal{A}_m$ such that $wa, wb \in \mathcal{W}_\Omega$ (resp., $aw, bw \in \mathcal{W}_\Omega$). A word that is both right-special and left-special is called *bispecial.* Thus, a word is right-special (resp., left-special) if and only if the corresponding vertex in the Rauzy graph has out-degree (resp., in-degree) ≥ 2.

Note that the complexity function of a Sturmian subshift obeys $p(n+1)-p(n) = 1$ for every n, and hence for every length, there is a unique right-special factor and a unique left-special factor, each having exactly two extensions.

Arnoux-Rauzy subshifts and episturmian subshifts relax this restriction on the possible extensions somewhat, and they are defined as follows: A minimal subshift Ω is called an *Arnoux-Rauzy subshift* if $p_\Omega(1) = m$ and for every $n \in \mathbb{Z}^+$, there is a unique right-special word in $\mathcal{W}_\Omega(n)$ and a unique left-special word in $\mathcal{W}_\Omega(n)$, both having exactly m extensions. This implies in particular that $p_\Omega(n) = (m-1)n+1$. Arnoux-Rauzy subshifts over $\mathcal{A}_2$ are exactly the Sturmian subshifts.

A minimal subshift Ω is called *episturmian* if $\mathcal{W}_\Omega$ is closed under reversal (i.e., for every $w = w_1 \dots w_n \in \mathcal{W}_\Omega$, we have $w^R = w_n \dots w_1 \in \mathcal{W}_\Omega$) and for every $n \in \mathbb{Z}^+$, there is exactly one right-special word in $\mathcal{W}_\Omega(n)$.

Proposition 2.17. *Every Arnoux-Rauzy subshift is episturmian and every episturmian subshift is strictly ergodic.*

See [63, 88, 122, 137] for these results and more information on Arnoux-Rauzy and episturmian subshifts.

2.2.5. *Codings of Interval Exchange Transformations.* Interval exchange transformations are defined as follows. Given a probability vector $\lambda = (\lambda_1, \dots, \lambda_m)$ with $\lambda_i > 0$ for $1 \leq i \leq m$, let $\mu_0 = 0$, $\mu_i = \sum_{j=1}^i \lambda_j$, and $I_i = [\mu_{i-1}, \mu_i)$. Let $\tau \in S_m$, the symmetric group. Then $\lambda^\tau = (\lambda_{\tau^{-1}(1)}, \dots, \lambda_{\tau^{-1}(m)})$ is also a probability vector and we can form the corresponding μ_i^τ and I_i^τ. Denote the unit interval $[0, 1)$ by I. The (λ, τ) interval exchange transformation is then defined by

$$T : I \to I, \;\; T(x) = x - \mu_{i-1} + \mu^\tau_{\tau(i)-1} \text{ for } x \in I_i, \; 1 \leq i \leq m.$$

It exchanges the intervals I_i according to the permutation τ.

The transformation T is invertible and its inverse is given by the $(\lambda^\tau, \tau^{-1})$ interval exchange transformation.

The symbolic coding of $x \in I$ is $\omega_n(x) = i$ if $T^n(x) \in I_i$. This induces a subshift over the alphabet $\mathcal{A} = \{1, \dots, m\}$: $\Omega_{\lambda,\tau} = \overline{\{\omega(x) : x \in I\}}$. Every Sturmian subshift can be described by the exchange of two intervals.

Keane [90] proved that if the orbits of the discontinuities μ_i of T are all infinite and pairwise distinct, then T is minimal. In this case, the coding is one-to-one and the subshift is minimal and aperiodic. This holds in particular if τ is irreducible and λ is irrational. Here, τ is called irreducible if $\tau(\{1, \dots, k\}) \neq (\{1, \dots, k\})$ for every $k < m$ and λ is called irrational if the λ_i are rationally independent.

Keane also conjectured that all minimal interval exchange transformations give rise to a uniquely ergodic system. This was disproved by Keynes and Newton [92] using five intervals, and then by Keane [91] using four intervals (the smallest possible number). The conjecture was therefore modified in [91] and then ultimately proven by Masur [114], Veech [135], and Boshernitzan [19]: For every irreducible $\tau \in S_m$ and for Lebesgue almost every λ, the subshift $\Omega_{\lambda,\tau}$ is uniquely ergodic.

2.2.6. *Substitution Sequences.* All the previous examples were generalizations of Sturmian sequences. We now discuss a class of examples that generalize a certain aspect of the Fibonacci sequence

$$s_n = \chi_{[1-\theta,0)}(n\theta), \quad \theta = \frac{\sqrt{5}-1}{2}.$$

We saw above (see (6) and its discussion) that this sequence, restricted to the right half line, is obtained by iterating the map (7). That is,

$$1 \mapsto 10 \mapsto 101 \mapsto 10110 \mapsto 10110101 \mapsto \cdots$$

has $(s_n)_{n\geq 1}$ as its limit. In other words, $(s_n)_{n\geq 1}$ is invariant under the substitution rule (7).

Definition 2.18 (substitution)**.** *Denote the set of words over the alphabet $\mathcal{A}$ by $\mathcal{A}^*$. A map $S : \mathcal{A} \to \mathcal{A}^*$ is called a substitution. The naturally induced maps on $\mathcal{A}^*$ and $\mathcal{A}^{\mathbb{Z}^+}$ are denoted by S as well.*

Examples. (a) Fibonacci: $1 \mapsto 10,\ 0 \mapsto 1$
(b) Thue-Morse: $1 \mapsto 10,\ 0 \mapsto 01$
(c) Period doubling: $1 \mapsto 10,\ 0 \mapsto 11$
(d) Rudin-Shapiro: $1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 42,\ 4 \mapsto 43$

Definition 2.19 (substitution sequence)**.** *Let S be a substitution. A sequence $s \in \mathcal{A}^{\mathbb{Z}^+}$ is called a substitution sequence if it is a fixed point of S.*

If $S(a)$ begins with the symbol a and has length at least two, it follows that $|S^n(a)| \to \infty$ as $n \to \infty$ and $S^n(a)$ has $S^{n-1}(a)$ as a prefix. Thus, the limit of $S^n(a)$ as $n \to \infty$ defines a substitution sequence s. In the examples above, we obtain the following substitution sequences.
(a) Fibonacci: $s_{\mathrm{F}} = 1011010110110\ldots$
(b) Thue-Morse: $s_{\mathrm{TM}}^{(1)} = 100101100110\ldots$ and $s_{\mathrm{TM}}^{(0)} = 0110100110010110\ldots$
(c) Period doubling: $s_{\mathrm{PD}} = 1011101010111011 10\ldots$
(d) Rudin-Shapiro: $s_{\mathrm{RS}}^{(1)} = 1213124212134313\ldots$ and $s_{\mathrm{RS}}^{(4)} = 4342431343421242\ldots$

We want to associate a subshift Ω_s with a substitution sequence s. Since the iteration of S on a suitable symbol a naturally defines a one-sided sequence s, we have to alter the definition of Ω_s used above slightly. One possible way is to extend s to a two-sided sequence $\tilde{s}$ arbitrarily and then define

$$\Omega_s = \{\omega \in \mathcal{A}^{\mathbb{Z}} : \omega = T^{n_j}\tilde{s} \text{ for some sequence } n_j \to \infty\}.$$

A different way is to define Ω_s to be the set of all ω's with $\mathcal{W}_\omega \subseteq \mathcal{W}_s$. Below we will restrict our attention to so-called primitive substitutions and for them, these two definitions are equivalent.

To ensure that Ω_s is strictly ergodic, we need to impose some conditions on S. A very popular sufficient condition is primitivity.

Definition 2.20. *A substitution S is called primitive if there is $k \in \mathbb{Z}^+$ such that for every pair $a, b \in \mathcal{A}$, $S^k(a)$ contains the symbol b.*

It is easy to check that our four main examples are primitive. Moreover, if S is primitive, then every power of S is primitive. Thus, even if $S(a)$ does not begin with a for any symbol $a \in \mathcal{A}$, we may replace S by a suitable S^m and then find such an a, which in turn yields a substitution sequence associated with S^m by iteration.

Theorem 2.21. *Suppose S is primitive and s is an associated substitution sequence. Then, s is linearly recurrent. Consequently, Ω_s is strictly ergodic.*

See [55, 66]. Linear recurrence clearly also implies that $p_s(n) = O(n)$. Fixed points of non-primitive substitutions may have quadratic complexity. However, there are non-primitive substitutions that have fixed points which are linearly recurrent and hence define strictly ergodic subshifts; see [45] for a characterization of linearly recurrent substitution generated subshifts.

2.2.7. *Subshifts with Positive Topological Entropy.* All the examples discussed so far have linearly bounded complexity. One may wonder if strict ergodicity places an upper bound on the growth of the complexity function. Here we want to mention the existence of strictly ergodic subshifts that have a very fast growing complexity function. In fact, it is possible to have growth that is arbitrarily close to the maximum possible one on a logarithmic scale.

Given a sequence s over an alphabet $\mathcal{A}$, $|\mathcal{A}| \ge 2$, its (*topological*) *entropy* is given by

$$h_s = \lim_{n \to \infty} \frac{1}{n} \log p_s(n).$$

The existence of the limit follows from the fact that $n \mapsto \log p_s(n)$ is subadditive. Moreover,

$$0 \le h_s \le \log |\mathcal{A}|,$$

where $|\cdot|$ denotes cardinality.

The following was shown by Hahn and Katznelson [75]:

Theorem 2.22. (a) *If s is a uniformly recurrent sequence over the alphabet $\mathcal{A}$, then $h_s < \log |\mathcal{A}|$.*
(b) *For every $\delta \in (0,1)$, there are alphabets $\mathcal{A}^{(j)}$ and sequences $s^{(j)}$ over $\mathcal{A}^{(j)}$, $j \in \mathbb{Z}^+$, such that $|\mathcal{A}^{(j)}| \to \infty$ as $j \to \infty$, $h_{s^{(j)}} \ge \log \left[|\mathcal{A}^{(j)}|(1-\delta) \right]$, and every $\Omega_{s^{(j)}}$ is strictly ergodic.*

3. Associated Schrödinger Operators and Basic Results

In this section we associate Schrödinger operators with a subshift Ω and a sampling function f mapping Ω to the real numbers. In subsequent sections we will study spectral and dynamical properties of these operators.

Let Ω be a strictly ergodic subshift with invariant measure μ and let $f : \Omega \to \mathbb{R}$ be continuous. Then, for every $\omega \in \Omega$, we define a potential $V_\omega : \mathbb{Z} \to \mathbb{R}$ by $V_\omega(n) = f(T^n \omega)$ and a bounded operator H_ω acting on $\ell^2(\mathbb{Z})$ by

$$[H_\omega \psi](n) = \psi(n+1) + \psi(n-1) + V_\omega(n)\psi(n).$$

Example. The most common choice for f is $f(\omega) = g(\omega_0)$ with some $g : \mathcal{A} \to \mathbb{R}$. This is a special case of a locally constant function that is completely determined by the values of ω_n for n's from a finite window around the origin, that is, f is called *locally constant* if it is of the form $f(\omega) = h(\omega_{-M} \ldots \omega_N)$ for suitable integers $M, N \ge 0$ and $h : \mathcal{A}^{N+M+1} \to \mathbb{R}$. Clearly, every locally constant f is continuous.

The family $\{H_\omega\}_{\omega\in\Omega}$ is an ergodic family of discrete one-dimensional Schrödinger operators in the sense of Carmona and Lacroix [24]. By the general theory it follows that the spectrum and the spectral type of H_ω are μ-almost surely ω-independent [24, Sect. V.2]:

Theorem 3.1. *There exist sets* $\Omega_0 \subseteq \Omega$, $\Sigma, \Sigma_{\rm pp}, \Sigma_{\rm sc}, \Sigma_{\rm ac} \subseteq \mathbb{R}$ *such that* $\mu(\Omega_0) = 1$ *and*

$$\sigma(H_\omega) = \Sigma \tag{9}$$
$$\sigma_{\rm pp}(H_\omega) = \Sigma_{\rm pp} \tag{10}$$
$$\sigma_{\rm sc}(H_\omega) = \Sigma_{\rm sc} \tag{11}$$
$$\sigma_{\rm ac}(H_\omega) = \Sigma_{\rm ac} \tag{12}$$

for every $\omega \in \Omega_0$.

Here, $\sigma(H), \sigma_{\rm pp}(H), \sigma_{\rm sc}(H), \sigma_{\rm ac}(H)$ denote the spectrum, the closure of the set of eigenvalues, the singular continuous spectrum and the absolutely continuous spectrum of the operator H, respectively.

Since Ω is minimal and f is continuous, a simple argument involving strong approximation shows that (9) even holds everywhere, rather than almost everywhere:

Theorem 3.2. *For every* $\omega \in \Omega$, $\sigma(H_\omega) = \Sigma$.

Proof. By symmetry it suffices to show that for every pair $\omega_1, \omega_2 \in \Omega$, $\sigma(H_{\omega_1}) \subseteq \sigma(H_{\omega_2})$. Due to minimality, there exists a sequence $(n_j)_{j\ge1}$ such that $T^{n_j}\omega_2 \to \omega_1$ as $j \to \infty$. By continuity of f, $H_{T^{n_j}\omega_2}$ converges strongly to H_{ω_1} as $j \to \infty$. Thus,

$$\sigma(H_{\omega_1}) \subseteq \overline{\bigcup_{j\ge1} \sigma(H_{T^{n_j}\omega_2})} = \sigma(H_{\omega_2}).$$

Here, the first step follows by strong convergence and the second step is a consequence of the fact that each of the operators $H_{T^{n_j}\omega_2}$ is unitarily equivalent to H_{ω_2} and hence has the same spectrum. □

Far more subtle is the result that (12) also holds everywhere:

Theorem 3.3. *For every* $\omega \in \Omega$, $\sigma_{\rm ac}(H_\omega) = \Sigma_{\rm ac}$.

For strictly ergodic models, such as the ones considered here, there are two proofs of Theorem 3.3 in the literature. It was shown, based on unique ergodicity, by Kotani in [102]. A proof based on minimality was given by Last and Simon in [106].

A map $A \in C(\Omega, \mathrm{SL}(2,\mathbb{R}))$ induces an $\mathrm{SL}(2,\mathbb{R})$-cocycle over T as follows:

$$\tilde{A} : \Omega \times \mathbb{R}^2 \to \Omega \times \mathbb{R}^2, \; (\omega, v) \mapsto (T\omega, A(\omega)v).$$

Note that when we iterate this map n times, we get

$$\tilde{A}^n(\omega, v) = (T^n\omega, A_n(\omega)v),$$

where $A_n(\omega) = A(T^{n-1}\omega)\cdots A(\omega)$. We are interested in the asymptotic behavior of the norm of $A_n(\omega)$ as $n \to \infty$. The multiplicative ergodic theorem ensures the existence of $\gamma_A \geq 0$, called the *Lyapunov exponent*, such that

$$\begin{aligned}\gamma_A &= \lim_{n\to\infty} \frac{1}{n} \int \log \|A_n(\omega)\| \, d\mu(\omega) \\ &= \inf_{n\geq 1} \frac{1}{n} \int \log \|A_n(\omega)\| \, d\mu(\omega) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \|A_n(\omega)\| \quad \text{ for } \mu\text{-a.e. } \omega.\end{aligned}$$

In the study of the operators H_ω the following cocycles are relevant:

$$A^{f,E}(\omega) = \begin{pmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{pmatrix},$$

where f is as above and E is a real number, called the *energy*. We regard f as fixed and write $\gamma(E)$ instead of $\gamma_{A^{f,E}}$ to indicate that our main interest is in the mapping $E \mapsto \gamma(E)$. Let

$$\mathcal{Z} = \{E \in \mathbb{R} : \gamma(E) = 0\}.$$

Note that we leave the dependence on Ω and f implicit.

These cocycles are important in the study of H_ω because $A_n^{f,E}$ is the transfer matrix for the associated difference equation. That is, a sequence u solves

$$u(n+1) + u(n-1) + V_\omega(n)u(n) = Eu(n) \tag{13}$$

if and only if it solves

$$\begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix} = A_n^{f,E} \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix},$$

as is readily verified.

4. Absence of Absolutely Continuous Spectrum

Let Ω be a strictly ergodic subshift and $f : \Omega \to \mathbb{R}$ locally constant. It follows that the resulting potentials V_ω take on only finitely many values. In this section we study the absolutely continuous spectrum of H_ω, equal to Σ_{ac} for every $\omega \in \Omega$ by Theorem 3.3. In 1982, Kotani made one of the deepest and most celebrated contributions to the theory of ergodic Schrödinger operators by showing that Σ_{ac} is completely determined by the Lyapunov exponent, or rather the set $\mathcal{Z}$. Namely, his results, together with earlier ones by Ishii and Patur, show that Σ_{ac} is given by the essential closure of $\mathcal{Z}$. In 1989, Kotani found surprisingly general consequences of his theory in the case of potentials taking on finitely many values. We will review these results below.

By assumption, the potentials V_ω take values in a fixed finite subset $\mathcal{B}$ of $\mathbb{R}$. Thus, they can be regarded as elements of $\mathcal{B}^{\mathbb{Z}}$, equipped with product topology. Let ν be the measure on $\mathcal{B}^{\mathbb{Z}}$ which is the push-forward of μ under the mapping

$$\Omega \to \mathcal{B}^{\mathbb{Z}}, \quad \omega \to V_\omega.$$

Recall that the support of ν, denoted by $\operatorname{supp}\nu$, is the complement of the largest open set U with $\nu(U) = 0$. Let

$$\begin{aligned}(\operatorname{supp}\nu)_+ &= \big\{V|_{\mathbb{Z}_0^+} : V \in \operatorname{supp}\nu\big\} \\ (\operatorname{supp}\nu)_- &= \big\{V|_{\mathbb{Z}^-} : V \in \operatorname{supp}\nu\big\},\end{aligned}$$

where $\mathbb{Z}_0^+ = \{0, 1, 2, \dots\}$ and $\mathbb{Z}^- = \{\dots, -3, -2, -1\}$.

Definition 4.1. *The measure ν is called deterministic if every $V_+ \in (\operatorname{supp}\nu)_+$ comes from a unique $V \in \operatorname{supp}\nu$ and every $V_- \in (\operatorname{supp}\nu)_-$ comes from a unique $V \in \operatorname{supp}\nu$.*

Consequently, if ν is deterministic, there is a bijection $C : (\operatorname{supp}\nu)_- \to (\operatorname{supp}\nu)_+$ such that for every $V \in \operatorname{supp}\nu$, $V|_{\mathbb{Z}_0^+} = C(V|_{\mathbb{Z}^-})$ and $V|_{\mathbb{Z}^-} = C^{-1}(V|_{\mathbb{Z}_0^+})$.

Definition 4.2. *The measure ν is called topologically deterministic if it is deterministic and the map C is a homeomorphism.*

Thus, when ν is topologically deterministic, we can continuously recover one half line from the other for elements of $\operatorname{supp}\nu$.

Let us only state the part of Kotani theory that is of immediate interest to us here:

Theorem 4.3. (a) *If $\mathcal{Z}$ has zero Lebesgue measure, then Σ_{ac} is empty.*
(b) *If $\mathcal{Z}$ has positive Lebesgue measure, then ν is topologically deterministic.*

This theorem holds in greater generality; see [100, 101, 102, 125]. The underlying dynamical system (Ω, T, μ) is only required to be measurable and ergodic and the set $\mathcal{B}$ can be any compact subset of $\mathbb{R}$. Part (a) is a particular consequence of the Ishii-Kotani-Pastur identity

$$\Sigma_{\mathrm{ac}} = \overline{\mathcal{Z}}^{\mathrm{ess}},$$

where the essential closure of a set $S \subseteq \mathbb{R}$ is given by

$$\overline{S}^{\mathrm{ess}} = \{E \in \mathbb{R} : \operatorname{Leb}((E - \varepsilon, E + \varepsilon) \cap S) > 0 \text{ for every } \varepsilon > 0\}.$$

The following result was proven by Kotani in 1989 [101]. Here it is crucial that the set $\mathcal{B}$ is finite.

Theorem 4.4. *If ν is topologically deterministic, then $\operatorname{supp}\nu$ is finite. Consequently, all potentials in $\operatorname{supp}\nu$ are periodic.*

Combining Theorems 4.3 and 4.4 we arrive at the following corollary.

Corollary 4.5. *If Ω and f are such that $\operatorname{supp}\nu$ contains an aperiodic element, then $\mathcal{Z}$ has zero Lebesgue measure and Σ_{ac} is empty.*

Note that by minimality, the existence of one aperiodic element is equivalent to all elements being aperiodic. This completely settles the issue of existence/purity of absolutely continuous spectrum. In the periodic case, the spectrum of H_ω is purely absolutely continuous for every $\omega \in \Omega$, and in the aperiodic case, the spectrum of H_ω is purely singular for every $\omega \in \Omega$.

5. Zero-Measure Spectrum

Suppose throughout this section that Ω is strictly ergodic, $f : \Omega \to \mathbb{R}$ is locally constant, and the resulting potentials V_ω are aperiodic.[2] This section deals with the Lebesgue measure of the set Σ, which is the common spectrum of the operators H_ω, $\omega \in \Omega$. It is widely expected that Σ always has zero Lebesgue measure. This is supported by positive results for large classes of subshifts and functions. We

[2]Even when we make explicit assumptions on Ω and f, aperiodicity of the potentials will always be assumed implicitly; for example, in Theorem 5.5 and Corollary 5.8.

present two approaches to zero-measure spectrum, one based on trace map dynamics and sub-exponential upper bounds for $\|A_n^{f,E}(\omega)\|$ for energies in the spectrum, and another one based on uniform convergence of $\frac{1}{n}\log\|A_n^{f,E}(\omega)\|$ to $\gamma(E)$ for all energies. Both approaches have in common that they establish the identity

$$\Sigma = \mathcal{Z}. \tag{14}$$

Zero-measure spectrum then follows immediately from Corollary 4.5.

5.1. **Trace Map Dynamics.** Zero-measure spectrum follows once one proves

$$\Sigma \subseteq \mathcal{Z}. \tag{15}$$

Note, however, that the Lyapunov exponent is always positive away from the spectrum. Thus, $\mathcal{Z} \subseteq \Sigma$, and (15) is in fact equivalent to (14).

A trace map is a dynamical system that may be associated with a family $\{H_\omega\}$ under suitable circumstances. It is given by the iteration of a map $T : \mathbb{R}^k \to \mathbb{R}^k$. Iteration of this map on some energy-dependent initial vector, v_E, will then describe the evolution of a certain sequence of transfer matrix traces. Typically, these iterates will diverge rather quickly. The stable set, B_∞, is defined to be the set of energies for which $T^n v_E$ does not diverge quickly. The inclusion (15) is then established in a two-step procedure:

$$\Sigma \subseteq B_\infty \subseteq \mathcal{Z}. \tag{16}$$

Again, by the remark above, this establishes equality and hence

$$\Sigma = B_\infty = \mathcal{Z}.$$

For the sake of clarity of the main ideas, we first discuss the trace-map approach for the Fibonacci subshift Ω_{F} and $f : \Omega_{\mathrm{F}} \to \mathbb{R}$ given by $f(\omega) = g(\omega_0)$, $g(0) = 0$, $g(1) = \lambda > 0$. See [25, 98, 118, 130, 131] for the original literature concerning this special case.

Given the partition result, Theorem 2.15, and the recursion (6), it is natural to decompose transfer matrix products into factors of the form $M_k(E)$, where

$$M_{-1}(E) = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}, \quad M_0(E) = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix} \tag{17}$$

and

$$M_{k+1}(E) = M_{k-1}(E)M_k(E), \quad \text{for } k \geq 0. \tag{18}$$

Proposition 5.1. *Let* $x_k = x_k(E) = \frac{1}{2}\mathrm{Tr} M_k(E)$. *Then,*

$$x_{k+2} = 2x_{k+1}x_k - x_{k-1} \quad \textit{for } k \in \mathbb{Z}_0^+ \tag{19}$$

and

$$x_{k+1}^2 + x_k^2 + x_{k-1}^2 - 2x_{k+1}x_kx_{k-1} = 1 + \frac{\lambda^2}{4} \quad \textit{for } k \in \mathbb{Z}_0^+. \tag{20}$$

Proof. The recursion (19) follows readily from (18). Using (19), one checks that the left-hand side of (20) is independent of k. Evaluation for $k = 0$ then yields the right-hand side. See [98, 118, 130] for more details. $\square$

The recursion (19) is called the *Fibonacci trace map*. The x_k's may be obtained by the iteration of the map $T : \mathbb{R}^3 \to \mathbb{R}^3$, $(x, y, z) \mapsto (xy - z, x, y)$ on the initial vector $((E - \lambda)/2, E/2, 1)$.

Proposition 5.2. *The sequence $(x_k)_{k\geq -1}$ is unbounded if and only if*

$$|x_{k_0-1}| \leq 1, \quad |x_{k_0}| > 1, \quad |x_{k_0+1}| > 1 \tag{21}$$

for some $k_0 \geq 0$. In this case, the k_0 is unique, and we have

$$|x_{k+2}| > |x_{k+1}x_k| > 1 \quad \text{for } k \geq k_0 \tag{22}$$

and

$$|x_k| > C^{F_{k-k_0}} \quad \text{for } k \geq k_0 \tag{23}$$

and some $C > 1$. If $(x_k)_{k\geq -1}$ is bounded, then

$$|x_k| \leq 1 + \frac{\lambda}{2} \quad \text{for every } k. \tag{24}$$

Proof. Suppose first that (21) holds for some $k_0 \geq 0$. Then, by (19),

$$|x_{k_0+2}| \geq |x_{k_0+1}x_{k_0}| + (|x_{k_0+1}x_{k_0}| - |x_{k_0-1}|) > |x_{k_0+1}x_{k_0}| > 1.$$

By induction, we get (22), and also that the k_0 is unique. Taking log's, we see that $\log |x_k|$ grows faster than a Fibonacci sequence for $k \geq k_0$, which gives (23).

Conversely, suppose that (21) fails for every $k_0 \geq 0$. Consider a value of k for which $|x_k| > 1$. Since $x_{-1} = 1$, it follows that $|x_{k-1}| \leq 1$ and $|x_{k+1}| \leq 1$. Thus, the invariant (20) shows that

$$\begin{aligned} |x_k| &\leq |x_{k+1}x_{k-1}| + \left(|x_{k+1}x_{k-1}|^2 - x_{k+1}^2 - x_{k-1}^2 + 1 + \frac{\lambda^2}{4} \right)^{1/2} \\ &= |x_{k+1}x_{k-1}| + \left((1 - x_{k+1}^2)(1 - x_{k-1}^2) + \frac{\lambda^2}{4} \right)^{1/2}, \end{aligned}$$

which implies that the sequence $(x_k)_{k\geq -1}$ is bounded and obeys (24). □

The dichotomy described in Proposition 5.2 motivates the following definition:

$$B_\infty = \left\{ E \in \mathbb{R} : |x_k| \leq 1 + \frac{\lambda}{2} \text{ for every } k \right\}. \tag{25}$$

This set provides the link between the spectrum and the set of energies for which the Lyapunov exponent vanishes.

Theorem 5.3. *Let $\Omega = \Omega_{\mathrm{F}}$ be the Fibonacci subshift and let $f : \Omega \to \mathbb{R}$ be given by $f(\omega) = g(\omega_0)$, $g(0) = 0$, $g(1) = \lambda > 0$. Then, $\Sigma = B_\infty = \mathcal{Z}$ and Σ has zero Lebesgue measure.*

Proof. We show the two inclusions in (16). Let $\sigma_k = \{E : |x_k| \leq 1\}$. On the one hand, σ_k is the spectrum of an F_k-periodic Schrödinger operator H_k. It is not hard to see that $H_k \to H$ strongly, where H is the Schrödinger operator with potential $V(n) = \lambda \chi_{[1-\theta,1)}(n\theta)$ and hence Σ is contained in the closure of $\bigcup_{\tilde{k}\geq k} \sigma_{\tilde{k}}$ for every k. On the other hand, Proposition 5.2 shows that $\sigma_{k+1} \cup \sigma_{k+2} \subseteq \sigma_k \cup \sigma_{k+1}$ and $B_\infty = \bigcap_k \sigma_k \cup \sigma_{k+1}$. Thus,

$$\Sigma \subseteq \bigcap_k \overline{\bigcup_{\tilde{k}\geq k} \sigma_{\tilde{k}}} = \bigcap_k \sigma_k \cup \sigma_{k+1} = B_\infty.$$

This is the first inclusion in (16). The second inclusion follows once we can show that for every $E \in B_\infty$, we have that $\log \|M_n\| \lesssim n$, where the implicit constant depends only on λ. From the matrix recursion (18) and the Cayley-Hamilton Theorem, we obtain

$$M_{k+1} = M_{k-1}M_k^2M_k^{-1} = M_{k-1}(2x_kM_k - \mathrm{Id})M_k^{-1} = 2x_kM_{k-1} - M_{k-2}^{-1}.$$

If $E \in B_\infty$, then $2|x_k| \leq 2+\lambda$, and hence we obtain by induction that $\|M_k\| \leq C^k$. Combined with the partition result, Theorem 2.15, this yields the claim since the F_k grow exponentially. □

The same strategy works in the Sturmian case, as shown by Bellissard et al. [15], although the analysis is technically more involved. Because of (5), we now consider instead of (18) the matrices defined by the recursion

$$M_{k+1}(E) = M_{k-1}(E)M_k(E)^{a_{k+1}},$$

where the a_k's are the coefficients in the continued fraction expansion (2) of θ. This recursion again gives rise to a trace map for $x_k = \frac{1}{2}\mathrm{Tr}M_k(E)$ which involves Chebyshev polynomials. These traces obey the invariant (20) and the exact analogue of Proposition 5.2 holds. After these properties are established, the proof may be completed as above. Namely, B_∞ is again defined by (25) and the same line of reasoning yields the two inclusions in (16). We can therefore state the following result:

Theorem 5.4. *Let Ω be a Sturmian subshift with irrational slope $\theta \in (0,1)$ and let $f : \Omega \to \mathbb{R}$ be given by $f(\omega) = g(\omega_0)$, $g(0) = 0$, $g(1) = \lambda > 0$. Then, $\Sigma = B_\infty = \mathcal{Z}$ and Σ has zero Lebesgue measure.*

We see that every operator family associated with a Sturmian subshift admits a trace map and an analysis of this dynamical system allows one to prove the zero-measure property.

Another class of operators for which a trace map always exists and may be used to prove zero-measure spectrum is given by those that are generated by a primitive substitution. The existence of a trace map is even more natural in this case and not hard to verify; see, for example, [3, 4, 5, 99, 119] for general results and [8, 10, 124] for trace maps with an invariant. However, its analysis is more involved and has been completed only in 2002 by Liu et al. [111], leading to Theorem 5.5 below, after a number of earlier works had established partial results [13, 14, 22]. Bellissard et al., on the other hand, had proved their Sturmian result already in 1989 – shortly after Kotani made his crucial observation leading to Corollary 4.5.

Theorem 5.5. *Let Ω be a subshift generated by a primitive substitution $S : \mathcal{A} \to \mathcal{A}^*$ and let $f : \Omega \to \mathbb{R}$ be given by $f(\omega) = g(\omega_0)$ for some function $g : \mathcal{A} \to \mathbb{R}$. Then, the associated trace map admits a stable set, B_∞, for which we have $\Sigma = B_\infty = \mathcal{Z}$. Consequently, Σ has zero Lebesgue measure.*

5.2. **Uniform Hyperbolicity.** Recall that the Lyapunov exponent associated with the Schrödinger cocycle $A^{f,E}$ obeys

$$\gamma(E) = \lim_{n\to\infty} \frac{1}{n} \log \|A^{f,E}(T^{n-1}\omega)\cdots A^{f,E}(\omega)\| \tag{26}$$

for μ-almost every $\omega \in \Omega$.

Definition 5.6 (uniformity)**.** *The cocycle $A^{f,E}$ is called uniform if the convergence in* (26) *holds for every $\omega \in \Omega$ and is uniform in ω. It is called uniformly hyperbolic* *if it is uniform and* $\gamma(E) > 0$. *Define*

$$\mathcal{U} = \{E \in \mathbb{R} : A^{f,E} \textit{ is uniformly hyperbolic } \}.$$

Uniform hyperbolicity of $A^{f,E}$ is equivalent to E belonging to the resolvent set as shown by Lenz [108] (see also Johnson [86]):

Theorem 5.7. $\mathbb{R} \setminus \Sigma = \mathcal{U}$.

Recall that we assumed at the beginning of this section that the potentials V_ω are aperiodic. Thus, combining Corollary 4.5 and Theorem 5.7, we arrive at the following corollary.

Corollary 5.8. *If* $A^{f,E}$ *is uniform for every* $E \in \mathbb{R}$, *then* $\Sigma = \mathcal{Z}$ *and* Σ *has zero Lebesgue measure.*

We thus seek a sufficient condition on Ω and f such that $A^{f,E}$ is uniform for every $E \in \mathbb{R}$, which holds for as many cases of interest as possible. Such a condition was recently found by Damanik and Lenz in [46]. In [48] it was then shown by the same authors that this condition holds for the majority of the models discussed in Section 2.

Definition 5.9 (condition (B))**.** *Let* Ω *be a strictly ergodic subshift with unique* T*-invariant measure* μ. *It satisfies the Boshernitzan condition* (B) *if*

$$\limsup_{n\to\infty} \left(\min_{w \in \mathcal{W}_\Omega(n)} n \cdot \mu\left([w]\right) \right) > 0. \tag{27}$$

Remarks. (a) $[w]$ denotes the cylinder set

$$[w] = \{\omega \in \Omega : \omega_1 \dots \omega_{|w|} = w\}.$$

(b) It suffices to assume that Ω is minimal and there exists some T-invariant measure μ with (27). Then, Ω is necessarily uniquely ergodic.
(c) The condition (27) was introduced by Boshernitzan in [20]. His main purpose was to exhibit a useful sufficient condition for unique ergodicity. The criterion proved to be particularly useful in the context of interval exchange transformations [19, 136], where unique ergodicity holds almost always, but not always [91, 114, 135].

It was shown by Damanik and Lenz that condition (B) implies uniformity for all energies and hence zero-measure spectrum [46].

Theorem 5.10. *If* Ω *satisfies* (B), *then* $A^{f,E}$ *is uniform for every* $E \in \mathbb{R}$.

Remarks. (a) The Boshernitzan condition holds for almost all of the subshifts discussed in Section 2. For example, it holds for every Sturmian subshift, almost every subshift generated by a coding of a rotation with respect to a two-interval partition, a dense set of subshifts associated with general codings of rotations, almost every subshift associated with an interval exchange transformation, almost every episturmian subshift, and every linearly recurrent subshift; see [48].[3]
(b) There was earlier work by Lenz who proved uniformity for all energies assuming a stronger condition, called (PW) for positive weights [107]. Essentially, (PW) requires (27) with lim sup replaced by lim inf. The condition (PW) holds for all

[3]Here, notions like "dense" or "almost all" are with respect to the natural parameters of the class of models in question. We refer the reader to [48] for detailed statements of these applications of Theorem 5.10.

linearly recurrent subshifts but it fails, for example, for almost every Sturmian subshift.
(c) Lenz in turn was preceded and inspired by Hof [77] who proved uniform existence of the Lyapunov exponents for Schrödinger operators associated with primitive substitutions. Extensions of [77] to linearly recurrent systems, including higher-dimensional ones, were found by Damanik and Lenz [40].

In particular, all zero-measure spectrum results obtained by the trace map approach also follow from Theorem 5.10. Moreover, the applications of Theorem 5.10 cover operator families that are unlikely to be amenable to the trace map approach. However, as we will see later, the trace map approach yields additional information that is crucial in a study of detailed spectral and dynamical properties. Thus, it is worthwhile to carry out an analysis of the trace map whenever possible.

5.3. **The Hausdorff Dimension of the Spectrum.** Once one knows that the spectrum has zero Lebesgue measure it is a natural question if anything can be said about its Hausdorff dimension. There is very important unpublished work by Raymond [121] who proved in the Fibonacci setting of Theorem 5.3 that the Hausdorff dimension is strictly smaller than one for λ large enough ($\lambda \geq 5$ is sufficient) and it converges to zero as $\lambda \to \infty$. Strictly positive lower bounds for the Hausdorff dimension of the spectrum at all couplings λ follow from the Hausdorff continuity results of [29, 83], to be discussed in Section 7. Several aspects of Raymond's work were used and extended in a number of papers [35, 51, 53, 95, 112]. In particular, Liu and Wen carried out a detailed analysis of the Hausdorff dimension of the spectrum in the general Sturmian case in the spirit of Raymond's approach; see [112].

6. Absence of Point Spectrum

We have seen that two of the three properties that are expected to hold in great generality for the operators discussed in this paper hold either always (absence of absolutely continuous spectrum) or almost always (zero-measure spectrum). In this section we turn to the third property that is expected to be the rule—the absence of point spectrum. As with zero-measure spectrum, no counterexamples are known and there are many positive results that have been obtained by essentially two different methods. Both methods rely on local symmetries of the potential. The existence of square-summable eigenfunctions is excluded by showing that these local symmetries are reflected in the solutions of the difference equation (13). Since there are exactly two types of symmetries in one dimension, the effective criteria for absence of eigenvalues that implement this general idea therefore rely on translation and reflection symmetries, respectively. In the following, we explain these two methods and their range of applicability.

6.1. **Local Repetitions.** In 1976, Gordon showed how to use the Cayley-Hamilton theorem to derive quantitative solution estimates from local repetitions in the potential [71]. The first major application of this observation was in the context of the almost Mathieu operator: For super-critical coupling and Liouville frequencies, there is purely singular continuous spectrum for all phases, as shown by Avron and Simon in 1982 [6].[4] The first application of direct relevance to this survey was

[4] As a consequence, positive Lyapunov exponents do not in general imply spectral localization.

obtained by Delyon and Petritis [58] in 1986 who proved absence of eigenvalues for certain codings of rotations, including most Sturmian models. Further applications will be mentioned below.

Gordon's Lemma is a deterministic criterion and may be applied to a fixed potential $V : \mathbb{Z} \to \mathbb{R}$. Analogous to the discussion in Section 3, we define transfer matrices $A_n^E = T_{n-1} \cdots T_0$, where

$$T_j = \begin{pmatrix} E - V(j) & -1 \\ 1 & 0 \end{pmatrix}.$$

Then, a sequence u solves

$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n) \tag{28}$$

if and only if it solves $U(n) = A_n^E U(0)$, where

$$U(j) = \begin{pmatrix} u(j) \\ u(j-1) \end{pmatrix}.$$

Lemma 6.1. *Suppose the potential V obeys $V(m+p) = V(m)$, $0 \le m \le p-1$. Then,*

$$\max \{\|U(2p)\|, \|U(p)\|\} \ge \frac{1}{2\max\{|\mathrm{Tr} A_p^E|, 1\}} \|U(0)\|.$$

Proof. This is immediate from the Cayley-Hamilton Theorem, applied to the matrix A_p^E and the vector $(u(0), u(-1))^T$. $\square$

For obvious reasons, we call this criterion the *two-block* Gordon Lemma. A slight variation of the argument gives the following (*three-block*) version of Gordon's Lemma.

Lemma 6.2. *Suppose the potential V obeys $V(m+p) = V(m)$, $-p \le m \le p-1$. Then,*

$$\max \{\|U(2p)\|, \|U(p)\|, \|U(-p)\|\} \ge \frac{1}{2}.$$

Remark. The original criterion from [71] uses four blocks. For the application to the almost Mathieu operator, this is sufficient; but for Sturmian models, for example, the improvements above are indeed needed, as we will see below. The two-block version can be found in Sütő's paper [130] and the three-block version was proved in [58] by Delyon and Petritis.

The two-block version is especially useful in situations where a trace map exists and we have bounds on trace map orbits for energies in the spectrum. Note that the two-block version gives a stronger conclusion. This will be crucial in the next section when we discuss continuity properties of spectral measures with respect to Hausdorff measures in the context of quantum dynamics.

6.2. **Palindromes.** Gordon-type criteria give quantitative estimates for solutions of (28) in the sense that repetitions in the potential are reflected in solutions, albeit in a weak sense. One would hope that local reflection symmetries in the potential give similar information. Unfortunately, such a result has not been found yet. However, it is possible to exclude square-summable solutions in this way by an indirect argument. Put slightly simplified, *if* a solution is square-summable, *then* local reflection symmetries are mirrored by solutions and these solution symmetries in turn prevent the solution from being square-summable.

The original criterion for absence of eigenvalues in this context is due to Jitomirskaya and Simon [85] and it was developed in the context of the almost Mathieu operator to prove, just as the result by Avron and Simon did, an unexpected occurrence of singular continuous spectrum. An adaptation of the Jitomirskaya-Simon method to the subshift context can be found in a paper by Hof et al. [78]. Let us state their result:[5]

Lemma 6.3. *Let $V : \mathbb{Z} \to \mathbb{R}$ be given. There is a constant B, depending only on $\|V\|_\infty$, with the following property: if there are $n_j \to \infty$ and l_j with $B^{n_j}/l_j \to 0$ as $j \to \infty$ such that V is symmetric about n_j on an interval of length l_j centered at n_j for every j, then the Schrödinger operator H with potential V has empty point spectrum.*

Sketch of proof. Suppose that V satisfies the assumptions of the lemma. Assume that u is a square-summable solution of (28), normalized so that $\|u\|_2 = 1$. Fix some j and reflect u about n_j. Call the reflected sequence $u^{(j)}$. Since the potential is reflection-symmetric on an interval of length l_j about n_j, the Wronskian of u and $u^{(j)}$ is constant on this interval. By $\|u\|_2 = 1$, it is pointwise bounded in this interval by $2/l_j$. From this, it follows that u and $u^{(j)}$ are close (up to a sign) near n_j. Now apply transfer matrices and compare u and $u^{(j)}$ near zero. The assumption $B^{n_j}/l_j \to 0$ then implies that, for j large, u and $u^{(j)}$ are very close near zero. In other words, u is bounded away from zero near $2n_j$ for all large j. This contradicts $u \in \ell^2(\mathbb{Z})$. □

Thus, eigenvalues can be excluded if the potential contains infinitely many suitably located palindromes. Here, a *palindrome* is a word that is the same when read backwards. Sequences obeying the assumption of Lemma 6.3 are called *strongly palindromic* in [78].

Hof et al. also prove the following general result for subshifts:

Proposition 6.4. *Suppose Ω is an aperiodic minimal subshift. If $\mathcal{W}_\Omega$ contains infinitely many palindromes, then the set of strongly palindromic ω's in Ω is uncountably infinite.*

In any event, since the set $\mathcal{C}_\Omega = \{\omega \in \Omega : \sigma_{\mathrm{pp}}(H_\omega) = \emptyset\}$ is a G_δ set as shown by Simon [126] (see also Choksi and Nadkarni [27] and Lenz and Stollmann [109]), it is a dense G_δ set as soon as it is non-empty by minimality of Ω and unitary equivalence of H_ω and $H_{T\omega}$.

Thus, when excluding eigenvalues we are interested in three kinds of results. We say that eigenvalues are *generically absent* if $\mathcal{C}_\Omega$ is a dense G_δ set. To prove generic absence of eigenvalues it suffices to treat one $\omega \in \Omega$, as explained in the previous paragraph. Absence of eigenvalues holds *almost surely* if $\mu(\mathcal{C}_\Omega) = 1$. To prove almost sure absence of eigenvalues one only has to show $\mu(\mathcal{C}_\Omega) > 0$ by ergodicity and T-invariance of $\mathcal{C}_\Omega$. Finally, absence of eigenvalues is said to hold *uniformly* if $\mathcal{C}_\Omega = \Omega$.

6.3. **Applications.** Let us now turn to applications of the two methods just described. We emphasize that absence of eigenvalues is expected to hold in great generality and no counterexamples are known.

[5] There is also a half-line version, which is the palindrome analogue of Lemma 6.1; see [36].

As in Section 5, things are completely understood in the Sturmian case and absence of eigenvalues holds uniformly.

Theorem 6.5. *Let Ω be a Sturmian subshift with irrational slope $\theta \in (0, 1)$ and let $f : \Omega \to \mathbb{R}$ be given by $f(\omega) = g(\omega_0)$, $g(0) = 0$, $g(1) = \lambda > 0$. Then, H_ω has empty point spectrum for every $\omega \in \Omega$.*

Sketch of proof. Given any $\lambda > 0$ and $\omega \in \Omega$, absence of point spectrum follows if Lemma 6.1 can be applied to V_ω for infinitely many values of p. Considering only p's of the form q_k, where the q_k's are associated with θ via (4), the trace bounds established in Theorem 5.4 show that we can focus our attention on the existence of infinitely many two-block structures aligned at the origin. Using Theorem 2.15, a case-by-case analysis through the various levels of the hierarchy detects these structures and completes the proof. □

Remarks. (a) For details, see [37, 38]. In fact, the argument above has to be extended slightly for θ's with $\limsup a_k = 2$. To deal with these exceptional cases, one also has to consider p's of the form $q_k + q_{k-1}$.
(b) Here is a list of earlier partial results for Sturmian models: Delyon and Petritis proved absence of eigenvalues almost surely for every $\lambda > 0$ and Lebesgue almost every θ [58]. Their proof employs Lemma 6.2. Using Lemma 6.1, Sütő proved absence of eigenvalues for $\lambda > 0$, $\theta = (\sqrt{5} - 1)/2$, and $\phi = 0$ [130], and hence generic absence of eigenvalues in the Fibonacci case. His proof and result were extended to all irrational θ's by Bellissard et al. [15].[6] Hof et al. proved generic absence of eigenvalues for every $\lambda > 0$ and every θ using Lemma 6.3 [78]. Kaminaga then showed an almost sure result for every $\lambda > 0$ and every θ [89]. His proof is based on Lemma 6.2 and refines the arguments of Delyon and Petritis.
(c) If most of the continued fraction coefficients are small, eigenvalues cannot be excluded using a four-block Gordon Lemma. This applies in particular in the Fibonacci case where $a_k \equiv 1$. The reason for this is that there simply are no four-block structures in the potential. See [41, 42, 87, 134] for papers dealing with local repetitions in Sturmian sequences.
(d) The palindrome method is very useful to prove generic results. However, it cannot be used to prove almost sure or uniform results for linearly recurrent subshifts (e.g., subshifts generated by primitive substitutions). Namely, for these subshifts, the strongly palindromic elements form a set of zero μ-measure as shown by Damanik and Zare [55].

Let us now turn to subshifts generated by codings of rotations. The key papers were mentioned above [58, 78, 89].

Theorem 6.6. *Suppose Ω is the subshift generated by a sequence of the form (8) with irrational $\theta \in (0, 1)$, some $\phi \in [0, 1)$, and a partition on the circle into l half-open intervals. Let $f : \Omega \to \mathbb{R}$ be given by $f(\omega) = g(\omega_0)$ with some non-constant function g. Suppose that the continued fraction coefficients of θ satisfy*

$$\limsup_{k\to\infty} a_k \geq 2l. \tag{29}$$

Then, H_ω has empty point spectrum for μ-almost every $\omega \in \Omega$.

[6] They do not state the result explicitly in [15], but given their analysis of the trace map and the structure of the potential, it follows as in [130].

Remarks. (a) For every $l \in \mathbb{Z}^+$, the condition (29) holds for Lebesgue almost every θ. In fact, almost every θ has unbounded continued fraction coefficients; see [94].
(b) The proof of Theorem 6.6, given in [58, 89], is based on Lemma 6.2.
(c) Hof et al. prove a generic result using Lemma 6.3 for every θ provided that the partition of the circle has a certain symmetry property, which is always satisfied in the case $l = 2$ [78].
(d) It is possible to prove a result similar to Theorem 6.6 for a locally constant f. In this case, the number $2l$ in (29) has to be replaced by a larger integer, determined by the size of the window $f(\omega)$ depends upon. Still, this gives almost sure absence of eigenvalues for almost every θ.

A large number of papers deal with the eigenvalue problem for Schrödinger operators generated by primitive substitutions; for example, [7, 14, 22, 30, 31, 32, 34, 59, 78].[7] We first describe general results that can be obtained using the two general methods we discussed and then turn to some specific examples, where more can be said.

We start with an application of Lemma 6.2. Fix some strictly ergodic subshift Ω and define, for $w \in \mathcal{W}_\Omega$, the *index of* w to be

$$\mathrm{ind}(w) = \sup\{r \in \mathbb{Q} : w^r \in \mathcal{W}_\Omega\}.$$

Here, w^r denotes the word $(xy)^m x$, where $m \in \mathbb{Z}^+$, $w = xy$, and $r = m + |x|/|w|$. The *index of* Ω is given by

$$\mathrm{ind}(\Omega) = \sup\{\mathrm{ind}(w) : w \in \mathcal{W}_\Omega\} \in [1, \infty].$$

Then, the following result was shown in [32] using three-block Gordon.

Theorem 6.7. *Suppose Ω is generated by a primitive substitution and* $\mathrm{ind}(\Omega) > 3$. *Let $f : \Omega \to \mathbb{R}$ be given by $f(\omega) = g(\omega_0)$ with some non-constant function $g : \mathcal{A} \to \mathbb{R}$. Then, H_ω has empty point spectrum for μ-almost every $\omega \in \Omega$.*

Remarks. (a) See [31] for a weaker result, assuming $\mathrm{ind}(\Omega) \geq 4$.
(b) The result extends to the case of a locally constant f.
(c) Consider the case of the period doubling substitution. Since $s_{\mathrm{PD}} = 101110101011101110\ldots$, we see that $\mathrm{ind}(\Omega) \geq \mathrm{ind}(10) \geq 3.5 > 3$. Thus, Theorem 6.7 implies almost sure absence of eigenvalues, recovering the main result of [30].

A substitution belongs to *class P* if there is a palindrome p and, for every $a \in \mathcal{A}$, a palindrome q_a such that $S(a) = pq_a$. Here, p is allowed to be the empty word and, if p is not empty, q_a may be the empty word. Clearly, if a subshift is generated by a class P substitution, it contains arbitrarily long palindromes. Thus, by Proposition 6.4, it contains uncountably many strongly palindromic elements. The following result from [78] is therefore an immediate consequence.

Theorem 6.8. *Suppose Ω is generated by a primitive substitution S that belongs to class P. Let $f : \Omega \to \mathbb{R}$ be given by $f(\omega) = g(\omega_0)$ with some non-constant function $g : \mathcal{A} \to \mathbb{R}$. Then, eigenvalues are generically absent.*

[7] There are also papers dealing with Schrödinger operators associated with non-primitive substitutions [45, 61, 62, 110]. The subshifts considered in these papers are, however, linearly recurrent and hence strictly ergodic, so that the theory is quite similar.

Notice that the Fibonacci, period doubling, and Thue-Morse subshifts are generated by class P substitutions. See [78] for more examples. The Rudin-Shapiro subshift, on the other hand, is not generated by a class P substitution. In fact, it does not contain arbitrarily long palindromes [2, 7].

We mentioned earlier that the proof of Theorem 6.8 cannot give a stronger result since the set of strongly palindromic sequences is always of zero measure for substitution subshifts [55]. Moreover, it was shown in [32] that the three-block Gordon argument cannot prove more than an almost everywhere statement in the sense that for every minimal aperiodic subshift Ω, there exists an element $\omega \in \Omega$ such that ω does not have the infinitely many three block structures needed for an application of Lemma 6.2. Thus, proofs of uniform results should use Lemma 6.1 in a crucial way. Theorem 6.5 shows that a uniform result is known in the Fibonacci case, for example, and Lemma 6.1 along with trace map bounds was indeed the key to the proof of this theorem.

Another example for which a uniform result is known is given by the period doubling substitution [34]. The trace map bounds are weaker than in the Fibonacci case, but a combination of two-block and three-block arguments was shown to work. Further applications of this idea can be found in [110].

The other two examples from Section 2, the Thue-Morse and Rudin-Shapiro substitutions, are not as well understood as Fibonacci and period doubling. Almost sure or uniform absence of eigenvalues for these cases are open, though expected. Generic results can be found in [59, 103].

The eigenvalue problem in the context of the other examples mentioned in Section 2 has been studied only in a small number of papers. For Arnoux-Rauzy subshifts, see [54]; and for interval exchange transformations, see [60].

7. Quantum Dynamics

In this section we focus on the time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t}\psi = H\psi, \quad \psi(0) = \psi_0, \tag{30}$$

where H is a Schrödinger operator in $\ell^2(\mathbb{Z})$ with a potential $V : \mathbb{Z} \to \mathbb{R}$, typically from a strictly ergodic subshift, and $\psi_0 \in \ell^2(\mathbb{Z})$. By the spectral theorem, (30) is solved by $\psi(t) = e^{-itH}\psi_0$. Thus, the question we want to study is the following: Given some potential V and some initial state $\psi_0 \in \ell^2(\mathbb{Z})$, what can we say about $e^{-itH}\psi_0$ for large times t?

7.1. Spreading of Wavepackets.

Since ψ_0 is square-summable, it is in some sense localized near the origin. For simplicity, one often considers the special case $\psi_0 = \delta_0$—the delta-function at the origin. With time, $\psi(t)$ will in general spread out in space. Our goal is to measure this spreading of the wavepacket and relate spreading rates to properties of the potential. As a general rule of thumb, spreading rates decrease with increased randomness of the potential. We will make this more explicit below.

A popular way of measuring the spreading of wavepackets is the following. For $p > 0$, define

$$\langle |X|^p_{\psi_0} \rangle (T) = \sum_n |n|^p a(n,T), \tag{31}$$

where

$$a(n,T) = \frac{2}{T} \int_0^\infty e^{-2t/T} |\langle e^{-itH} \psi_0, \delta_n \rangle|^2 \, dt. \tag{32}$$

Clearly, the faster $\langle |X|^p_{\psi_0} \rangle (T)$ grows, the faster $e^{-itH}\psi_0$ spreads out, at least averaged in time.[8] One typically wants to obtain power-law bounds on $\langle |X|^p_{\delta_0} \rangle (T)$ and hence it is natural to define the following quantities: For $p > 0$, define the *upper* (resp., *lower*) *transport exponent* $\beta^\pm_{\delta_0}(p)$ by

$$\beta^\pm_{\psi_0}(p) = \lim_{T\to\infty} {}^{\sup}_{\inf} \frac{\log \langle |X|^p_{\psi_0} \rangle (T)}{p \log T}$$

Both functions $p \mapsto \beta^\pm_{\psi_0}(p)$ are nondecreasing and obey $0 \le \beta^-_{\psi_0}(p) \le \beta^+_{\psi_0}(p) \le 1$.

For periodic V, $\beta^\pm_{\psi_0}(p) \equiv 1$ (*ballistic transport*); while for random V, $\beta^\pm_{\psi_0}(p) \equiv 0$ (a weak version of *dynamical localization*—stronger results are known). For V's that are intermediate between periodic and random, and in particular Sturmian V's, it is expected that the transport exponents take values between 0 and 1.

7.2. **Spectral Measures and Subordinacy Theory.** By the spectral theorem, $\langle e^{-itH}\psi_0, \psi_0 \rangle = \int e^{-itE} \, d\mu_{\psi_0}(E)$, where μ_{ψ_0} is the spectral measure associated with H and ψ_0. Thus, it is natural to investigate quantum dynamical questions by relating them to properties of the spectral measure corresponding to the initial state. This approach is classical and the Riemann-Lebesgue Lemma and Wiener's Theorem may be interpreted as statements in quantum dynamics. The RAGE theorem establishes basic dynamical results in terms of the standard decomposition of the Hilbert space into pure point, singular continuous, and absolutely continuous subspaces. We refer the reader to Last's well-written article [105] for a review of these early results.

The results just mentioned are very satisfactory for initial states whose spectral measure has an absolutely continuous component. This is, to some extent, also true for pure point measures. However, if the measure is purely singular continuous, it is desirable to obtain results that go beyond Wiener's Theorem and the RAGE theorem.

Last also addressed this issue in [105] and proposed a decomposition of spectral measures with respect to Hausdorff measures. This was motivated by earlier results of Guarneri [72] and Combes [28] who proved dynamical lower bounds for initial states with uniformly Hölder continuous spectral measures. By approximation with uniformly Hölder continuous measures, Last proved in [105] that these bounds extend to measures that are absolutely continuous with respect to a suitable Hausdorff measure:

Theorem 7.1. *If μ_{ψ_0} has a non-trivial component that is absolutely continuous with respect to the α-dimensional Hausdorff measure h^α on $\mathbb{R}$, then*

$$\beta^-_{\psi_0}(p) \ge \alpha \quad \textit{for every } p > 0. \tag{33}$$

[8]Taking time averages is natural since the operators of interest in this paper have purely singular continuous spectrum; compare Wiener's Theorem. While Wiener's Theorem would suggest taking a Cesàro time average, the Abelian time average we choose is more convenient for technical purposes. The transport exponents are the same for both ways of time averaging.

Remarks. (a) Here, h^α is defined by

$$h^\alpha(S) = \lim_{\delta \to 0} \inf_{\delta\text{-covers}} \sum |I_m|^\alpha,$$

where $S \subseteq \mathbb{R}$ is a Borel set and a δ-cover is a countable collection of intervals I_m of length bounded by δ such that the union of these intervals contains the set in question. Note that h^1 coincides with Lebesgue measure and h^0 is the counting measure.
(b) For further developments of quantum dynamical lower bounds in terms of continuity or dimensionality properties of spectral measures, see [11, 12, 73, 74].
(c) The result and its proof have natural analogues in higher dimensions; see [105].

While a bound like (33) is nice, it needs to be complemented by effective methods for verifying the input to Theorem 7.1. In the context of one-dimensional Schrödinger operators, it is always extremely useful to connect a problem at hand to properties of solutions to the difference equation (28). The classical decomposition of spectral measures can be studied via subordinacy theory as shown by Gilbert and Pearson [69]; see also [68, 93]. Subordinacy theory has proved to be one of the major tools in one-dimensional spectral theory and many important results have been obtained with its help. Jitomirskaya and Last were able to refine subordinacy theory to the extent that Hausdorff-dimensional spectral issues can be investigated in terms of solution behavior [81, 82, 83]. The key result is the Jitomirskaya-Last inequality, which explicitly relates the Borel transform of the spectral measure to solutions in the half-line setting [82, Theorem 1.1].

Using the maximum modulus principle together with the Jitomirskaya-Last inequality, Damanik et al. then proved the following result for operators on the line [37]:

Theorem 7.2. *Suppose $\Sigma \subseteq \mathbb{R}$ is a bounded set and there are constants γ_1, γ_2 such that for each $E \in \Sigma$, every solution u of* (28) *with $|u(-1)|^2 + |u(0)|^2 = 1$ obeys the estimate*

$$C_1(E) L^{\gamma_1} \le \left(\sum_{n=1}^{L} |u(n)|^2 \right)^{1/2} \le C_2(E) L^{\gamma_2} \tag{34}$$

for $L > 0$ sufficiently large and suitable constants $C_1(E), C_2(E)$. Let $\alpha = 2\gamma_1/(\gamma_1+\gamma_2)$. Then, for any $\psi_0 \in \ell^2(\mathbb{Z})$, the spectral measure for the pair (H, ψ_0) is absolutely continuous with respect to h^α on Σ. In particular, the bound (33) *holds for every non-trivial initial state whose spectral measure is supported in Σ.*

This shows that suitable bounds for solutions of (28) imply statements on Hausdorff-dimensional spectral properties, which in turn yield quantum dynamical lower bounds. There is an extension to multi-dimensional Schrödinger operators by Kiselev and Last [96].

Two remarks are in order. First, while there are some important applications of the method just presented, proving the required solution estimates is often quite involved. The number of known applications is therefore still relatively small. Second, dynamical bounds in terms of Hausdorff-dimensional properties are strictly one-sided. It is not possible to prove dynamical upper bounds purely in terms of dimensional properties. There are a number of examples that demonstrate this phenomenon. For example, modifications of the super-critical almost Mathieu operator lead to spectrally localized operators with almost ballistic transport [57, 67].

Another important example that is spectrally, but not dynamically, localized is given by the random dimer model [56, 84].

7.3. **Plancherel Theorem.** There is another approach to dynamical bounds that is also based on solution (or rather, transfer matrix) estimates, which relates dynamics to integrals over Lebesgue measure, as opposed to integrals over the spectral measure. Compared with the approach discussed above, it has two main advantages: One can prove both upper and lower bounds in this way, and the proof of a lower bound is so soft that it applies to a greater number of models.

The key to this approach is a formula due to Kato, which follows quickly from the Plancherel Theorem:

Lemma 7.3.

$$2\pi \int_0^\infty e^{-2t/T} |\langle e^{-itH}\psi_0, \delta_n\rangle|^2 \, dt = \int_{-\infty}^\infty \left|\langle (H-E-\tfrac{i}{T})^{-1}\psi_0, \delta_n\rangle\right|^2 \, dE. \tag{35}$$

Proof. Consider the function

$$F(t) = \begin{cases} e^{-t/T}\langle e^{-itH}\psi_0, \delta_n\rangle & t \ge 0, \\ 0 & t < 0. \end{cases}$$

Using the spectral theorem, it is readily verified that the Fourier transform of F obeys $\widehat{F}(-E) = i\langle (H-E-\frac{i}{T})^{-1}\psi_0, \delta_n\rangle$. Thus, (35) follows if we apply the Plancherel theorem to F. $\square$

For simplicity, let us consider the case $\psi_0 = \delta_0$. Note that

$$u(n) = \langle (H-E-i/T)^{-1}\delta_0, \delta_n\rangle$$

solves the difference equation (28) (with E replaced by $E + i/T$) away from the origin and can therefore be studied by means of transfer matrices! In particular, we may infer a bound from below in terms of $\|A_n^{E+i/T}\|^{-1}$. Thus, upper bounds on transfer matrix norms are of interest.

Theorem 7.4. *Suppose that the transfer matrices obey the bound $\|A_n^E\| \le C|n|^\alpha$ for every $n \ne 0$, some fixed energy $E \in \mathbb{R}$ and suitable constants C, α. Then,*

$$\beta^-_{\delta_0}(p) \ge \frac{1}{1+2\alpha} - \frac{1+8\alpha}{p+2\alpha p}$$

for every $p > 0$.

Remarks. (a) This is the one-energy version of a more general result due to Damanik and Tcheremchantsev [51]. See [50] for extensions of [51] and supplementary material and [67] for related work.
(b) An interesting application of Theorem 7.4 (and its proof) to the random dimer model may be found in the paper [84] by Jitomirskaya et al., which confirms a prediction of Dunlap et al. [64].
(c) There is also a version of Theorem 7.4 for more general initial states ψ_0 [49].
(d) The idea of the proof of Theorem 7.4 is simple. A Gronwall-type perturbation argument derives upper bounds on $\|A_n^{\tilde{E}}\|$ for $\tilde{E}$ close to E and n not too large. The right-hand side of (35) may then be estimated from below by integrating only over a small neighborhood of E, where u is controlled by the upper bound on the transfer matrix. The bound for $\beta^-_{\delta_0}(p)$ then follows by rather straightforward arguments.
(e) The paper [52] (using some ideas from [133]) shows that a combination of the

two approaches may sometimes (e.g., in the Fibonacci case) give better bounds.
(f) Killip et al. used (35) to prove dynamical upper bounds for the slow part of the wavepacket [95]. Their work inspired the use of (35) in [51].

Since (35) is an identity, rather than an inequality, it can be used to bound $a(n,T)$ from both below and above. Clearly, proving an upper bound is more involved and will require assumptions that are global in the energy. It was shown by Damanik and Tcheremchantsev that the following assumption on transfer matrix growth is sufficient to allow one to infer an upper bound for the transport exponents [53]:

Theorem 7.5. *Let $K \ge 4$ be such that $\sigma(H) \subseteq [-K+1, K-1]$. Suppose that, for some $C \in (0,\infty)$ and $\alpha \in (0,1)$, we have*

$$\int_{-K}^{K} \left(\max_{1 \le n \le CT^\alpha} \left\| A_n^{E+\frac{i}{T}} \right\|^2 \right)^{-1} dE = O(T^{-m}) \tag{36}$$

and

$$\int_{-K}^{K} \left(\max_{1 \le -n \le CT^\alpha} \left\| A_n^{E+\frac{i}{T}} \right\|^2 \right)^{-1} dE = O(T^{-m}) \tag{37}$$

for every $m \ge 1$. Then, $\beta^+_{\delta_0}(p) \le \alpha$ for every $p > 0$.

7.4. **Applications.** Let us discuss the applications of these general methods to Schrödinger operators with potentials from strictly ergodic subshifts.

We begin with the Fibonacci case. In fact, every approach to quantum dynamical bounds has been tested on this example and there are many papers proving dynamical results for it; for example, [29, 35, 37, 51, 52, 53, 83, 95].

Upper bounds for transfer matrices were established by Iochum and Testard [80] who proved, for zero phase, that the norms of the transfer matrices grow no faster than a power law for every energy in the spectrum. The power can be chosen uniformly on the spectrum and depends only on the sampling function f. Notice that this improves on the statement that the Lyapunov exponent vanishes on the spectrum. An extension to Sturmian subshifts whose slope has (essentially) bounded continued fraction coefficients was obtained by Iochum et al. [79]. Note that upper bounds for transfer matrix norms yield the input to Theorem 7.4 and one half of the input to Theorem 7.2. The other half of the input to Theorem 7.2, lower bounds for solutions, was obtained in [29, 83]. The proof of these bounds uses the bound for the trace map for energies from the spectrum, Gordon's two-block lemma, and a mass-reproduction technique based on cyclic permutations of repeated blocks.[9]

Theorem 7.6. *Let $V(n) = \lambda \chi_{[1-\theta,1)}(n\theta)$, where $\theta = (\sqrt{5}-1)/2$ and $\lambda > 0$. Then,*

$$\beta^-_{\delta_0}(p) \ge \begin{cases} \frac{p+2\kappa}{(p+1)(\alpha+\kappa+1/2)} & p \le 2\alpha+1, \\ \frac{1}{\alpha+1} & p > 2\alpha+1. \end{cases}, \tag{38}$$

where κ is an absolute constant ($\kappa \approx 0.0126$) and $\alpha \asymp \log \lambda$.

[9]Using partitions (cf. Theorem 2.15), these solution estimates described in this paragraph can be shown for all elements of the subshift [37, 39].

Remarks. (a) In the form stated, the result is from [52]. The bound (38) is the best known dynamical lower bound for the Fibonacci operator and is a culmination of the sequence of works [29, 37, 51, 83, 95] leading up to [52].
(b) When we write $\alpha \asymp \log \lambda$, we mean that α is a positive λ-dependent quantity that satisfies $C_1 \log \lambda \le \alpha \le C_2 \log \lambda$ for positive constants C_1, C_2 and all large λ. See [52] for the explicit dependence of α on λ.

To apply Theorem 7.5 to the Fibonacci operator, one has to prove the estimates (36) and (37). This was done in [53]. Let us describe the main idea. Clearly, to prove the desired lower bounds for transfer matrix norms, it suffices to prove lower bounds for transfer matrix traces. We know a way to establish such lower bounds: Lemma 5.2. Since all relevant energies in (36) and (37) are non-real, we know that the trace map will eventually enter the escape region described in Lemma 5.2. The point is to control the number of iterates it takes for this to occur. To this end, define the complex analogue of the set σ_k from Section 5 by

$$\sigma_k^{\mathbb{C}} = \{z \in \mathbb{C} : |x_k(z)| \le 1\}.$$

Notice that the x_k's are polynomials and hence defined for all complex z. As before, being in the complement of two consecutive $\sigma_k^{\mathbb{C}}$'s is a sufficient condition for escape at an explicit rate; compare Lemma 5.2, whose proof extends to complex energies. It is therefore useful to bound the imaginary width of these sets from above. This will give an upper bound on the number of iterates it takes at a given energy to enter the escape region. For λ sufficiently large, the connected components of $\sigma_k^{\mathbb{C}}$ can be studied with the help of Koebe's Distortion Theorem; see [53] for details. The resulting dynamical upper bound has the same asymptotics for large λ as the lower bound above:

Theorem 7.7. *Let $V(n) = \lambda \chi_{[1-\theta,1)}(n\theta)$, where $\theta = (\sqrt{5}-1)/2$ and $\lambda \ge 8$. Then,*

$$\beta_{\delta_0}^{+}(p) \le \tilde{\alpha} \quad \text{for every } p > 0,$$

where $\tilde{\alpha} \in (0,1)$ and $\tilde{\alpha} \asymp (\log \lambda)^{-1}$.

In particular, for the Fibonacci operator with $\lambda \ge 8$, all transport exponents $\{\beta_{\delta_0}^{\pm}(p)\}_{p>0}$ are strictly between zero and one.

The dynamical lower bounds have been established for more general models; see [29, 37, 43, 49, 50, 51]. On the other hand, Theorem 7.7 is the only explicit result of this kind, but as mentioned in [53], the ideas of [35] should permit one to extend this theorem to more general slopes and all elements of the subshift.

8. CMV Matrices Associated with Subshifts

Given a strictly ergodic subshift Ω and a continuous/locally constant function $f : \Omega \to \mathbb{D}$, we can define $\alpha_n(\omega) = f(T^n \omega)$ for $n \in \mathbb{Z}$ and $\omega \in \Omega$. Let $\mathcal{C}_\omega$ be the CMV matrix associated with Verblunsky coefficients $\{\alpha_n(\omega)\}_{n \ge 0}$ and $\mathcal{E}_\omega$ the extended CMV matrix associated with Verblunsky coefficients $\{\alpha_n(\omega)\}_{n \in \mathbb{Z}}$. That is, with $\rho_n(\omega) = (1 - |\alpha_n(\omega)|)^{-1/2}$, $\mathcal{C}_\omega$ is given by

$$\begin{pmatrix} \bar{\alpha}_0(\omega) & \bar{\alpha}_1(\omega)\rho_0(\omega) & \rho_1(\omega)\rho_0(\omega) & 0 & 0 & \dots \\ \rho_0(\omega) & -\bar{\alpha}_1(\omega)\alpha_0(\omega) & -\rho_1(\omega)\alpha_0(\omega) & 0 & 0 & \dots \\ 0 & \bar{\alpha}_2(\omega)\rho_1(\omega) & -\bar{\alpha}_2(\omega)\alpha_1(\omega) & \bar{\alpha}_3(\omega)\rho_2(\omega) & \rho_3(\omega)\rho_2(\omega) & \dots \\ 0 & \rho_2(\omega)\rho_1(\omega) & -\rho_2(\omega)\alpha_1(\omega) & -\bar{\alpha}_3(\omega)\alpha_2(\omega) & -\rho_3(\omega)\alpha_2(\omega) & \dots \\ 0 & 0 & 0 & \bar{\alpha}_4(\omega)\rho_3(\omega) & -\bar{\alpha}_4(\omega)\alpha_3(\omega) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

and $\mathcal{E}_\omega$ is the analogous two-sided infinite matrix. See [127, 128] for more information on CMV and extended CMV matrices.

For these unitary operators in ℓ^2, we can ask questions similar to the ones considered above in the context of Schrödinger operators. That is, is the spectrum of zero Lebesgue measure, are spectral measures purely singular continuous, etc. Since we are dealing with ergodic models, it is more natural to consider the whole-line situation. On the other hand, from the point of view of orthogonal polynomials on the unit circle, the half-line situation is more relevant. The zero-measure property is independent of the setting, whereas the spectral type for half-line models is almost always (i.e., when the "boundary condition" is varied) pure point as soon as zero-measure spectrum is established. The latter statement follows quickly from spectral averaging; compare [128, Theorem 10.2.2]. Thus, the key problem for CMV matrices associated with subshifts is proving zero-measure spectrum. In fact, Simon conjectured the following; see [128, Conjecture 12.8.2].

Simon's Subshift Conjecture. Suppose $\mathcal{A}$ is a subset of $\mathbb{D}$, the subshift Ω is minimal and aperiodic and let $f : \Omega \to \mathbb{D}$, $f(\omega) = \omega(0)$. Then, Σ has zero Lebesgue measure.

Here, Σ is the common spectrum of the operators $\mathcal{E}_\omega$, $\omega \in \Omega$. Equivalently, it is the common essential spectrum of $\mathcal{C}_\omega$, $\omega \in \Omega$.

Simon proved the zero-measure property for the Fibonacci case by means of the trace map approach [128, Section 12.8]. Since the approach based on the Boshernitzan condition has a wider scope in the Schrödinger case, it is natural to try and extend it to the CMV case. This was done by Damanik and Lenz in [47] where the following result was shown.

Theorem 8.1. *Suppose the subshift Ω is aperiodic and satisfies the Boshernitzan condition. Let $f : \Omega \to \mathbb{D}$ be locally constant. Then, Σ has zero Lebesgue measure.*

This proves Simon's Subshift Conjecture for a large number of models since we saw above that many of the prominent aperiodic subshifts satisfy the Boshernitzan condition.

Regarding the spectral type, it should not be hard to extend the material from Section 6 to the CMV case. This will imply purely singular continuous spectrum for $\mathcal{E}_\omega$ for many subshifts Ω and many (generic, almost all, all) $\omega \in \Omega$. However, as was noted above, the Aleksandrov measures associated with $\mathcal{C}_\omega$ will almost surely be pure point whenever Theorem 8.1 applies.

Quantum dynamics, on the other hand, is less natural in the CMV case than in the Schrödinger case, and has not really been studied.[10] Most of the ideas leading to the results presented in Section 7 should have CMV counterparts. In particular,

[10]Simon did extend the Jitomirskaya-Last theory to OPUC in [128]. This theory has its roots in quantum dynamics; compare [57, 72, 82, 83, 105].

it should be possible to prove absolute continuity of spectral measures with respect to suitable Hausdorff measures for extended CMV matrices over Fibonacci-like subshifts.

9. Concluding Remarks

The material presented in this survey is motivated by and closely related to the theory of quasicrystals; compare, for example, [9, 115]. More specifically, the surveys [33, 132] deal with the Fibonacci operator and its generalizations and the interested reader may find references to the original physics literature in those papers.

Regarding future research in this field, it would be interesting to see how far one can take the philosophy that potentials taking finitely many values preclude localization phenomena. Since the Bernoulli Anderson model is localized [23], this cannot hold in full generality. On the other hand, Gordon potentials are much more prevalent in the subshift case than in the uniformly almost periodic case. Moreover, for smooth quasi-periodic potentials, it is expected that the Lyapunov exponent is positive at all energies if the coupling is large enough. This is known for trigonometric potentials [76], analytic potentials [21, 70, 129], and Gevrey potentials [97]. See also [17, 26] for recent results in the C^r category. These potentials should be contrasted with those coming from quasi-periodic subshifts satisfying the Boshernitzan condition. The Boshernitzan condition is independent of the coupling constant and yields vanishing Lyapunov exponent throughout the spectrum. Since it is satisfied on a dense set of sampling (step-)functions, upper-semicontinuity arguments allow one to derive surprising phenomena that hold generically in the C^0 category [18].

To shed some light on this, it should be helpful to analyze more examples. That is, take one of the popular base transformations of the torus (e.g., shifts, skew-shifts, or expanding maps) and define an ergodic family of potentials by choosing a sampling function on the torus that takes finitely many values. These models, with the exception of rotations of the circle, are not well understood! There is a serious issue about the competition between the flat pieces of the sampling function and the randomness properties of the base transformation (expressed, e.g., in terms of mixing properties). For example, take a 1-periodic step function f and consider $V_\omega(n) = \lambda f(2^n\omega)$, $\lambda > 0$, $\omega \in [0,1)$. Is it true that the Lyapunov exponent is positive? For all λ's or all large λ's? For all energies or all but finitely many?

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Mathematics 253–37, California Institute of Technology, Pasadena, CA 91125, USA
E-mail address: damanik@caltech.edu
URL: http://www.math.caltech.edu/people/damanik.html