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AMENABLE ACTIONS AND ALMOST INVARIANT SETS

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ABSTRACT. In this paper, we study the connections between properties of the action of a countable group Γ on a countable set X and the ergodic theoretic properties of the corresponding generalized Bernoulli shift, i.e., the corresponding shift action of Γ on M^X, where M is a measure space. In particular, we show that the action of Γ on X is amenable iff the shift $\Gamma \curvearrowright M^X$ has almost invariant sets.

1. INTRODUCTION

Let X be a countable set and Γ a countable, infinite group acting on X. Let M be a standard Borel space and ν an arbitrary Borel, probability measure on M which does not concentrate on a single point. Consider the measure space (M^X, ν^X) where ν^X stands for the product measure (which we will also denote by μ). The action of Γ on X gives rise to an action on M^X (called a *generalized Bernoulli shift*) by measure preserving transformations:

$$(\gamma \cdot c)(x) = c(\gamma^{-1} \cdot x), \quad \text{for } c \in M^X.$$

The classical Bernoulli shifts are obtained by letting $\mathbb{Z}$ act on itself by translation.

There are natural connections between many properties of the action of Γ on X and ergodic theoretic properties of the corresponding Bernoulli shift. We summarize some of those in Section 2. In studying generalized Bernoulli shifts, it is often useful to consider the unitary representations of Γ arising from the actions, namely the representation λ_X on $\ell^2(X)$ given by

$$(\lambda_X(\gamma) \cdot f)(x) = f(\gamma^{-1} \cdot x), \quad \text{for } f \in \ell^2(X),$$

and the *Koopman representation* κ on $L^2(M^X, \mu)$ given by

$$(\kappa(\gamma) \cdot f)(c) = f(\gamma^{-1} \cdot c), \quad \text{for } f \in L^2(M^X, \mu).$$

Since the representation κ trivially fixes the constants, we will often also consider its restriction κ_0 to $L^2_0(M^X, \mu) = \left\{ f \in L^2 : \int f = 0 \right\}$. We recall some basic definitions about unitary representations. Let π, σ be representations of a countable group Γ. If σ is isomorphic to a subrepresentation of π, we write $\sigma \leq \pi$. In particular, if $\sigma = 1_\Gamma$, the trivial (one-dimensional) representation of Γ, and $\sigma \leq \pi$, we say that

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π *has invariant vectors*. If $Q \subseteq \Gamma$ is finite, and $\epsilon > 0$, we say that a *unit* vector $v \in H$ is (Q, ϵ, π)*-invariant* if

$$\forall \gamma \in Q \quad \|\pi(\gamma) \cdot v - v\| < \epsilon.$$

If for all pairs (Q, ϵ), there exists a (Q, ϵ, π)-invariant vector, we say that π *has almost invariant vectors* and write $1_\Gamma \prec \pi$.

Recall that the action of Γ on X is called *amenable* if there exists a Γ-invariant mean on $\ell^\infty(X)$. The action is said to satisfy the *Følner condition* if for all finite $Q \subseteq \Gamma$ and all $\epsilon > 0$, there exists a finite $F \subseteq X$ such that

$$\forall \gamma \in Q \quad |F \mathbin{\triangle} \gamma \cdot F| < \epsilon |F|. \tag{1.1}$$

The following equivalences are well known and can be proved in exactly the same way as the corresponding ones for amenability of groups (see, for example, Bekka–de la Harpe–Valette [2]).

Theorem 1.1. *The following are equivalent for an action of* Γ *on* X*:*

(i) *the action is amenable;*
(ii) *the action satisfies the Følner condition;*
(iii) $1_\Gamma \prec \lambda_X$.

Clearly, all actions of amenable groups are amenable and if an action has a finite orbit, it is automatically amenable. There are also non-amenable groups which admit amenable actions with infinite orbits. Important examples are the non-amenable, inner amenable groups with infinite conjugacy classes (consider the action of Γ on $\Gamma \setminus \{1\}$ by conjugation; see Bédos–de la Harpe [1] for definitions and examples). Interestingly, free groups also admit transitive, faithful, amenable actions (van Douwen [3]). Y. Glasner and N. Monod in a recent paper [5] study the class of groups which admit transitive, faithful, amenable actions and give some history, references, and further examples. Grigorchuk–Nekrashevych [6] describe yet another example of faithful, transitive, amenable actions of free groups. On the other hand, every amenable action of a group with Kazhdan's property (T) has a finite orbit.

An action of a countable group Γ on a measure space (Y, μ) by measure preserving transformations *has almost invariant sets* if there is a sequence $\{A_n\}$ of measurable sets with measures bounded away from 0 and 1 such that for all $\gamma \in \Gamma$,

$$\mu(\gamma \cdot A_n \mathbin{\triangle} A_n) \to 0 \text{ as } n \to \infty.$$

It is easy to see that the existence of almost invariant sets implies the existence of almost invariant vectors for the Koopman representation κ_0 (look at the characteristic functions) but the converse may fail, as was first proved by Schmidt [11] (for another example, see Hjorth–Kechris [7, Theorem A3.2]). In fact, the existence of almost invariant sets depends only on the orbit equivalence relation which, in the ergodic case, is equivalent to non E_0-ergodicity (Jones–Schmidt [9]), while the existence of almost invariant vectors depends on the group action (see [7] again). Recall that E_0 is the equivalence relation on $2^{\mathbb{N}}$ defined by

$$(x_n) \, E_0 \, (y_n) \iff \exists m \forall n > m \; x_n = y_n.$$

An equivalence relation E on a measure space (Y, μ) is E_0*-ergodic* if for every Borel map $f \colon Y \to 2^{\mathbb{N}}$ which satisfies

$$x \, E \, y \implies f(x) \, E_0 \, f(y),$$

there is a single E_0 equivalence class whose preimage is μ-conull. For a discussion on E_0-ergodicity and the related concepts of almost invariant vectors and sets, see [7, Appendix A].

Now we can state the main theorem of this paper which connects the amenability of the action of Γ on X and the existence of almost invariant sets for the corresponding Bernoulli shift and almost invariant vectors for the Koopman representation.

Theorem 1.2. *Let an infinite, countable group Γ act on a countable set X. The following are equivalent:*

(i) *the action of Γ on X is amenable;*
(ii) *the action of Γ on M^X has almost invariant sets;*
(iii) *the Koopman representation κ_0 has almost invariant vectors.*

This result has an implication concerning orbit equivalence. Schmidt [11] showed that every non-amenable group Γ that does not have property (T) has at least two non-orbit equivalent, ergodic actions (this was extended later by Hjorth [8] to all non-amenable groups). The preceding result shows that if Γ is non-amenable but admits an action on X which is amenable and has infinite orbits (this class of groups is a subclass of non-property (T) groups), then one in fact has two ergodic, free a.e. generalized shifts which are not orbit equivalent: the generalized shift on 2^X and the usual shift on 2^Γ (ergodicity follows from Proposition 2.1 below and freeness can easily be achieved by adding an additional orbit to X, see Proposition 2.4). For example, for non-amenable, inner amenable groups Γ, the usual shift on 2^Γ and the conjugacy shift on $2^{\Gamma\setminus\{1\}}$ are not orbit equivalent. Also any non-abelian free group admits two non-orbit equivalent free, ergodic generalized shifts.

Since in most cases the existence of almost invariant vectors is easier to check than the existence of almost invariant sets, it will be interesting to know whether there are other cases in which the two concepts coincide. A relatively broad class of examples of measure preserving actions, studied by several authors (see the monograph Schmidt [12] for discussion and references and also Kechris [10]), consists of the actions by automorphisms on compact Polish groups (equipped with the Haar measure). The generalized Bernoulli shifts with a homogeneous base space M also fall into that class.

Question 1.3. Let Γ act on a compact Polish group G by automorphisms (which necessarily preserve the Haar measure). Is it true that the action has almost invariant sets iff the corresponding Koopman representation κ_0 has almost invariant vectors?

The rest of the paper is organized as follows: in Section 2, we recall some necessary and sufficient conditions for a Bernoulli shift to be ergodic, mixing, etc.; in Section 3, we carry out a detailed spectral analysis of the Koopman representation of generalized Bernoulli shifts and prove a few preliminary lemmas; and finally, in Section 4, we give a proof of Theorem 1.2.

Below Γ and G will always be countable, infinite groups and Q will denote a *finite* subset of the group.

2. Group actions and generalized shifts

In this section, we record several known facts which characterize when a generalized Bernoulli shift is ergodic, weakly mixing, mixing, or free a.e.

Proposition 2.1. *The following are equivalent:*

(i) *the action of* Γ *on* M^X *is ergodic;*
(ii) *the action of* Γ *on* M^X *is weakly mixing;*
(iii) *the action of* Γ *on* X *has infinite orbits;*
(iv) $1_\Gamma \not\leq \lambda_X$.

Proof. We shall need the following standard lemma from group theory (for a proof, see, e.g., [10, Lemma 4.4]):

Lemma 2.2 (Neumann). *Let* Γ *be a group acting on a set* X*. Then the following are equivalent:*

(a) *all orbits are infinite;*
(b) *for all finite* $F_1, F_2 \subseteq X$*, there exists* $\gamma \in \Gamma$ *such that* $\gamma \cdot F_1 \cap F_2 = \emptyset$.

(i) $\Rightarrow$ (iii) Suppose that there is a finite orbit $F \subseteq X$. Let $A \subseteq M$, $0 < \nu(A) < 1$. Then the set $\{c \in M^X : c(F) \subseteq A\}$ is non-trivial and invariant under the action.

(iii) $\Rightarrow$ (ii) It suffices to show that the diagonal action of Γ on $M^X \times M^X$ is ergodic. This action is the same as the Bernoulli shift corresponding to the disjoint sum of the action of Γ on X with itself. The latter action has infinite orbits by (iii). Suppose $A \subseteq M^{X \sqcup X}$ is invariant and $0 < \mu(A) < 1$. Then we can find $A' \subseteq M^{X \sqcup X}$ depending only on a finite set of coordinates $F \subseteq X \sqcup X$ such that $\mu(A' \triangle A) < \epsilon/3$ and $\mu(A') - \mu(A')^2 > \epsilon$ for some $\epsilon > 0$. By Lemma 2.2, there is $\gamma \in \Gamma$ such that $\gamma \cdot F \cap F = \emptyset$. By independence, $\mu(A' \cap \gamma \cdot A') = \mu(A')^2$. On the other hand,

$$\mu(A' \cap \gamma \cdot A') \geq \mu(A') - 3\mu(A \triangle A') > \mu(A') - \epsilon,$$

a contradiction.

(ii) $\Rightarrow$ (i) and (iii) $\Leftrightarrow$ (iv) are obvious. □

Recall that π is called a *c_0-representation* if for all $v \in H_\pi$, $\lim_{\gamma \to \infty} \langle \pi(\gamma) \cdot v, v \rangle = 0$.

Proposition 2.3. *The following are equivalent:*

(i) *the action of* Γ *on* M^X *is mixing;*
(ii) κ_0 *is a* c_0*-representation;*
(iii) λ_X *is a* c_0*-representation;*
(iv) *the stabilizers* $\Gamma_x = \{\gamma \in \Gamma : \gamma \cdot x = x\}$ *for* $x \in X$ *are finite.*

Proof. (ii) $\Rightarrow$ (iv) Let $A \subseteq M$, $0 < \nu(A) < 1$. Suppose Γ_x is infinite for some x and consider the set $B = \{c \in M^X : c(x) \in A\}$. Then $0 < \mu(B) < 1$ and $\gamma \cdot B = B$ for infinitely many γ so the shift is not mixing.

(iv) $\Rightarrow$ (ii) It suffices to show that the mixing condition is satisfied for sets $A, B \subseteq M^X$ depending only on finitely many coordinates. Let $F_1, F_2 \subseteq X$ be finite, A depend on F_1, and B depend on F_2. By (iv), there are only finitely $\gamma \in \Gamma$ for which $\gamma \cdot F_1 \cap F_2 \neq \emptyset$, hence

$$\lim_{\gamma \to \infty} \mu(\gamma \cdot A \cap B) = \mu(A)\mu(B)$$

and we are done.

Finally, the equivalences (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv) are easy to prove. □

Proposition 2.4. *If the measure* ν *has atoms, the following are equivalent:*

(i) *the action of* Γ *on* M^X *is free a.e.;*
(ii) *for each* $\gamma \in \Gamma \setminus \{1\}$*, the set* $\{x \in X : \gamma \cdot x \neq x\}$ *is infinite.*

If ν is non-atomic, (i) *is equivalent to*

(iii) *the action of Γ on X is faithful.*

Proof. Suppose first that ν has an atom $a \in M$. If for some $\gamma \neq 1$ the set $H_\gamma = \{x : \gamma \cdot x \neq x\}$ is finite, then

$$\mu(\gamma \cdot c = c) \geq \mu(\forall x \in H_\gamma \; c(x) = a) = \nu(\{a\})^{|H_\gamma|} > 0,$$

so the action of Γ on M^X is not free a.e.

Conversely, if H_γ is infinite for all $\gamma \neq 1$, find infinite sets $Y_\gamma \subseteq X$ such that $\gamma \cdot Y_\gamma \cap Y_\gamma = \emptyset$. Then

$$\begin{aligned}\mu(\gamma \cdot c = c) &\leq \mu\big(\forall x \in Y_\gamma \; c(x) = c(\gamma^{-1} \cdot x)\big) \\ &= \prod_{x \in Y_\gamma} \mu\big(c(x) = c(\gamma^{-1} \cdot x)\big) = 0.\end{aligned}$$

If the action of Γ on X is not faithful, then the action on M^X is not faithful either, so in particular it is not free. Conversely, if ν is non-atomic and $\gamma \cdot x \neq x$ for some $x \in X$,

$$\mu(\gamma \cdot c = c) \leq \mu\big(c(x) = c(\gamma^{-1} \cdot x)\big) = 0.$$

□

3. Spectral analysis of the Koopman representation

For each subgroup $\Delta \leq \Gamma$, we have the *quasi-regular representation* $\lambda_{\Gamma/\Delta}$ on $\ell^2(\Gamma/\Delta)$ given by

$$(\lambda_{\Gamma/\Delta}(\gamma) \cdot f)(\delta\Delta) = f(\gamma^{-1}\delta\Delta).$$

Notice that if S is a transversal for the action of Γ on X (i.e., $S \subseteq X$ and S intersects each orbit in exactly one point), then

$$\lambda_X \cong \textstyle\bigoplus_{x \in S} \lambda_{\Gamma/\Gamma_x}, \tag{3.1}$$

where Γ_x denotes the stabilizer of the point x. The first aim of this section is to verify that κ is also equivalent to a sum of quasi-regular representations. This is well-known but the authors were unable to find a specific reference.

Let $\{f_i : i \in I\}$ be a (finite or countably infinite) orthonormal basis for $L^2(M, \nu)$ such that $f_{i_0} \equiv 1$ for some $i_0 \in I$. Set $I_0 = I \setminus \{i_0\}$ and notice that since ν does not concentrate on a single point, $I_0 \neq \emptyset$. For a function $q \colon X \to I$, write

$$\operatorname{supp} q = q^{-1}(I_0)$$

and let $\mathcal{A} = \{q : |\operatorname{supp} q| < \infty\}$. For $q \in \mathcal{A}$, define $h_q \in L^2(M^X, \mu)$ by

$$h_q(c) = \prod_{x \in X} f_{q(x)}(c(x)).$$

Lemma 3.1. *The collection $\{h_q : q \in \mathcal{A}\}$ forms an orthonormal basis for $L^2(M^X)$.*

Proof. First we check that $\|h_q\| = 1$. Indeed,

$$\begin{aligned}\|h_q\|^2 = \langle h_q, h_q \rangle &= \int |h_q|^2 \, \mathrm{d}\mu \\ &= \prod_{x \in X} \int |f_{q(x)}(z)|^2 \, \mathrm{d}\nu(z) \\ &= \prod_{x \in X} \left\| f_{q(x)} \right\|^2 \\ &= 1.\end{aligned}$$

Now suppose $q_1 \neq q_2$. Then

$$\begin{aligned}\langle h_{q_1}, h_{q_2} \rangle &= \int h_{q_1} \overline{h_{q_2}} \, \mathrm{d}\mu \\ &= \prod_{x \in X} \int f_{q_1(x)}(z) \overline{f_{q_2(x)}(z)} \, \mathrm{d}\nu(z) \\ &= 0\end{aligned}$$

because $\int f_{q_1(x_0)}(z)\overline{f_{q_2(x_0)}(z)} \, \mathrm{d}\nu(z) = \left\langle f_{q_1(x_0)}, f_{q_2(x_0)} \right\rangle = 0$ for some x_0 for which $q_1(x_0) \neq q_2(x_0)$.

Finally, we verify that the h_q's are total in $L^2(M^X)$. Let $\mathcal{F}$ be the measure algebra of M^X. Fix an exhausting sequence $F_1 \subseteq F_2 \subseteq \cdots$ of finite subsets of X and denote by $\mathcal{F}_n$ the σ-subalgebra of $\mathcal{F}$ generated by the projections $\{p_x : x \in F_n\}$. Notice that $L^2(M^X, \mathcal{F}_n)$ is canonically isomorphic to $L^2(M^{F_n})$ which, in turn, is canonically isomorphic to $\bigotimes_{x \in F_n} L^2(M)$. Under this isomorphism, a function h_q with $\operatorname{supp} q \subseteq F_n$ corresponds to the tensor $\bigotimes_{x \in F_n} f_{q(x)}$. Hence $\{h_q : \operatorname{supp} q \subseteq F_n\}$ is total in $L^2(M^X, \mathcal{F}_n)$. But $\bigcup_n \mathcal{F}_n$ generates $\mathcal{F}$, so $\bigcup_n L^2(M^X, \mathcal{F}_n)$ is dense in $L^2(M^X)$ and we are done. □

Notice that Γ acts on $\mathcal{A}$ in a natural way:

$$(\gamma \cdot q)(x) = q(\gamma^{-1} \cdot x).$$

This action induces a representation on $L^2(M^X)$ (by permuting the basis $\{h_q : q \in \mathcal{A}\}$) and clearly this representation is equal to κ. Let now T be a transversal for the action of Γ on $\mathcal{A}$ (i.e., $T \subseteq \mathcal{A}$ and T intersects each orbit in exactly one point). Let for each $q \in \mathcal{A}$, Γ_q denote the stabilizer of q. The preceding discussion implies that

$$\kappa \cong \textstyle\bigoplus_{q \in T} \lambda_{\Gamma/\Gamma_q}.$$

Notice that the constant function $q_0 \equiv i_0$ is an orbit of the action of Γ on $\mathcal{A}$ consisting of a single element, so $q_0 \in T$. Let $T_0 = T \setminus \{q_0\}$. We have just proved

Proposition 3.2.

$$\kappa_0 \cong \textstyle\bigoplus_{q \in T_0} \lambda_{\Gamma/\Gamma_q}. \tag{3.2}$$

We also record a few facts about quasi-regular representations which will be used later.

Lemma 3.3. *Let G be a countable group and $K \leq H \leq G$ with $[H : K] < \infty$. Then $\lambda_{G/H} \leq \lambda_{G/K}$.*

Proof. Let $n = [H : K]$ and let $p \colon G/K \to G/H$ be the natural projection. Define the map $\Phi \colon \ell^2(G/H) \to \ell^2(G/K)$ by

$$\Phi(f) = \frac{1}{\sqrt{n}} f \circ p.$$

It is easy to check that Φ is an isometric embedding which intertwines $\lambda_{G/H}$ and $\lambda_{G/K}$. □

Lemma 3.4. *Let G be a countable group and $K \leq H \leq G$. Let $Q \subseteq G$, $\epsilon > 0$ and assume there is a $(Q, \epsilon, \lambda_{G/K})$-invariant vector. Then there exists a $(Q, \epsilon, \lambda_{G/H})$-invariant vector.*

Proof. Let $v \in \ell^2(G/K)$ be $(Q, \epsilon, \lambda_{G/K})$-invariant. By considering $|v|$ instead of v, we can assume that $v \geq 0$ ($|v|$ is $(Q, \epsilon, \lambda_{G/K})$-invariant by the triangle inequality). Define $w \in \ell^2(G/H)$ by

$$w(D) = \sqrt{\sum_{C \subseteq D} v^2(C)}, \quad D \in G/H$$

where C runs over elements of G/K. We have

$$\|w\|^2 = \sum_{D \in G/H} \sum_{C \subseteq D} v^2(C) = \sum_{C \in G/K} v^2(C) = \|v\|^2 = 1.$$

Furthermore, for each $\gamma \in Q$,

$$\begin{aligned}
\langle \gamma \cdot w, w \rangle &= \sum_{D \in G/H} w(\gamma^{-1} D) w(D) \\
&= \sum_{D \in G/H} \sqrt{\sum_{C \subseteq \gamma^{-1} D} v^2(C)} \sqrt{\sum_{C \subseteq D} v^2(C)} \\
&\geq \sum_{D \in G/H} \sum_{C \subseteq D} v(C) v(\gamma^{-1} C), \quad \text{by Cauchy-Schwartz}, \\
&= \sum_{C \in G/K} v(C) v(\gamma^{-1} C) \\
&= \langle \gamma \cdot v, v \rangle .
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\gamma \cdot w - w\|^2 &= 2\|w\|^2 - 2\langle \gamma \cdot w, w \rangle \\
&\leq 2\|v\|^2 - 2\langle \gamma \cdot v, v \rangle = \|\gamma \cdot v - v\|^2 < \epsilon^2
\end{aligned}$$

and w is $(Q, \epsilon, \lambda_{G/H})$-invariant. □

Lemma 3.5. *Let π_i, $i = 1, 2, \ldots$ be unitary representations of a countable group G on the Hilbert spaces H_i. Suppose that $1_G \prec \bigoplus_{i=1}^{\infty} \pi_i$. Then for each $Q \subseteq G$ and $\epsilon > 0$, there exists n and $v_n \in H_n$ which is (Q, ϵ, π_n)-invariant.*

Proof. Fix Q and ϵ. Let $H = \bigoplus_i H_i$. There exists $v \in H$, $v = \bigoplus_i v_i$, such that $\|\pi(\gamma) \cdot v - v\| < \epsilon/\sqrt{m}\,\|v\|$ where $m = |Q|$. We have

$$\begin{aligned}
&\|\pi(\gamma) \cdot \textstyle\bigoplus_i v_i - \bigoplus_i v_i\|^2 < \epsilon^2/m \,\|\bigoplus_i v_i\|^2 \quad \text{for all } \gamma \in Q,\\
&\sum_{\gamma \in Q} \sum_i \|\pi_i(\gamma) \cdot v_i - v_i\|^2 < \epsilon^2 \sum_i \|v_i\|^2\\
&\sum_i \sum_{\gamma \in Q} \|\pi_i(\gamma) \cdot v_i - v_i\|^2 < \sum_i \epsilon^2 \|v_i\|^2 .
\end{aligned}$$

Hence, for some i,

$$\sum_{\gamma \in Q} \|\pi_i(\gamma) \cdot v_i - v_i\|^2 < \epsilon^2 \|v_i\|^2 ,$$

and in particular, for each $\gamma \in Q$,

$$\|\pi_i(\gamma) \cdot v_i - v_i\|^2 < \epsilon^2 \|v_i\|^2 .$$

□

4. Proof of Theorem 1.2

We start with the implication (i) $\Rightarrow$ (ii). We shall need to use the Central Limit Theorem for random variables several times and we find it convenient to employ probabilistic notation. For all necessary background in probability theory, a good reference is Durrett [4]. In this section, we will use P instead of μ to denote the measure on M^X. Recall that a sequence ξ_k of random variables *converges in distribution* to ξ (written as $\xi_k \Rightarrow \xi$) if the distribution measures of ξ_k converge to the distribution measure of ξ in the weak* topology. For this, it is necessary and sufficient that $P(\xi_k \in A) \to P(\xi \in A)$ for every Borel set A for which $P(\xi \in \partial A) = 0$ (∂A denotes the topological boundary of A). The Central Limit Theorem states that if $\{\xi_k\}$ is a sequence of independent, identically distributed random variables with finite mean m and variance σ^2, then

$$\frac{\sum_{i=1}^k \xi_i - km}{\sigma\sqrt{k}}$$

converges in distribution to a standard normal random variable (see [4, Theorem 2.4.1]). Recall also that a distribution is *continuous* if the measure associated to it is non-atomic. Finally, a sequence ξ_k *converges in probability* to ξ if for all $\epsilon > 0$, $P(|\xi_k - \xi| > \epsilon) \to 0$ as $k \to \infty$. We need the following two lemmas.

Lemma 4.1. *Let $\xi_k, \eta_k, \zeta_k, k = 1, 2, \ldots$ be random variables such that $\xi_k \Rightarrow \xi$, where ξ is a random variable with continuous distribution, and η_k, ζ_k converge in probability to 0. Then $P(\eta_k \le \xi_k \le \zeta_k) \to 0$ as $k \to \infty$.*

Proof. Fix $\epsilon > 0$ and find δ such that $P(|\xi| \le \delta) < \epsilon$. Find N so big that for $k > N$, $\big|P(|\xi_k| \le \delta) - P(|\xi| \le \delta)\big| < \epsilon$, $P(\eta_k < -\delta) < \epsilon$, and $P(\zeta_k > \delta) < \epsilon$. Then, for all $k > N$,

$$P(\eta_k < \xi_k \le \zeta_k) \le P(|\xi_k| \le \delta) + P(\eta_k < -\delta) + P(\zeta_k > \delta) \le 4\epsilon.$$

□

Lemma 4.2. *Let $\xi_k \Rightarrow \xi$, $\alpha_k \in \mathbb{R}$, $\alpha_k \ge 0$, $\alpha_k \to 0$. Then $\alpha_k \xi_k \to 0$ in probability.*

Proof. It suffices to show that for all $\delta > 0$, $P(\alpha_k|\xi_k| > \delta) \to 0$. Fix $\epsilon > 0$. Find a such that $P(|\xi| > a) < \epsilon/2$ and $P(|\xi| = a) = 0$. For all large enough k, we will have $\big|P(|\xi_k| > a) - P(|\xi| > a)\big| < \epsilon/2$ and $\delta/\alpha_k > a$. For all those k (assuming also $\alpha_k > 0$),

$$P(|\xi_k| > \delta/\alpha_k) \leq P(|\xi_k| > a) < P(|\xi| > a) + \epsilon/2 < \epsilon.$$

□

Suppose now that the action of Γ on X is amenable. Without loss of generality, take $M = I = [-1, 1]$ and assume that the measure ν is centered at 0 (i.e., $\int_I x \,\mathrm{d}\nu(x) = 0$). We will find a sequence $\{A_k\}$ of subsets of I^X with measures bounded away from 0 and 1, satisfying for all $\gamma \in \Gamma$,

$$P(\gamma \cdot A_k \triangle A_k) \to 0 \text{ as } k \to \infty. \tag{4.1}$$

Enumerate $\Gamma = \{\gamma_n\}$. By (1.1), there exists a sequence $\{F_k\}$ of finite subsets of X satisfying

$$\forall i \leq k \quad \frac{|F_k \triangle \gamma_i \cdot F_k|}{|F_k|} < 1/k. \tag{4.2}$$

For each $x \in X$, let $p_x : I^X \to I$ be the corresponding projection function. We view the p_x's as independent, identically distributed, real random variables with distribution given by the measure ν. Note that all of their moments are finite because they are bounded. By our assumptions, the mean $\mathbf{E}\, p_x = 0$. Set $\sigma^2 = \operatorname{Var} p_x = \mathbf{E}\, p_x^2 > 0$. Let $r_k = |F_k|$ and set

$$A_k = \left\{ \sum_{x \in F_k} p_x > 0 \right\}.$$

First suppose that the sequence $\{r_k\}$ is bounded by a number K. Notice that $P(A_k)$ only depends on the number r_k and not on the actual set F_k. Therefore, in this case, we have only finitely many possibilities for $P(A_k)$, so $P(A_k)$ are bounded away from 0 and 1. Also, by (4.2), for $k > K$ and $i \leq k$, $\gamma_i \cdot F_k = F_k$, hence $\gamma_i \cdot A_k = A_k$ and the sequence $\{A_k\}$ is almost invariant.

Now consider the case when $\{r_k\}$ is unbounded. By taking a subsequence, we can assume that $r_k \to \infty$. We first show that the measures of A_k are bounded away from 0 and 1. Indeed, by the Central Limit Theorem,

$$P(A_k) = P\left(\frac{\sum_{x \in F_k} p_x}{\sqrt{r_k}\sigma} > 0 \right) \to P(\chi > 0) = 1/2,$$

where χ denotes a standard normal variable. Next we prove that (a subsequence of) A_k is almost invariant. By taking subsequences, we can assume that for each $\gamma \in \Gamma$, either $\{|\gamma \cdot F_k \triangle F_k|\}_k$ is bounded, or $|\gamma \cdot F_k \triangle F_k| \to \infty$. Fix $\gamma \in \Gamma$ and set $n_k = |\gamma \cdot F_k \setminus F_k| = |F_k \setminus \gamma \cdot F_k|$, $N_k = |\gamma \cdot F_k \cap F_k|$. Let

$$\xi_k = \sum_{x \in F_k \cap \gamma \cdot F_k} p_x,$$

$$\eta_k = \sum_{x \in F_k \setminus \gamma \cdot F_k} p_x,$$

$$\zeta_k = \sum_{x \in \gamma \cdot F_k \setminus F_k} p_x.$$

ξ_k, η_k, ζ_k are independent,

$$\mathbf{E}\,\xi_k = \mathbf{E}\,\eta_k = \mathbf{E}\,\zeta_k = 0, \quad \operatorname{Var}\xi_k = N_k\sigma^2, \quad \operatorname{Var}\eta_k = \operatorname{Var}\zeta_k = n_k\sigma^2,$$

and

$$A_k = \{\xi_k + \eta_k > 0\}\,, \quad \gamma \cdot A_k = \{\xi_k + \zeta_k > 0\}\,.$$

Suppose first that $\{n_k\}$ is bounded and let K be an upper bound for n_k. Notice that $|\eta_k|, |\zeta_k| \le K$. We have

$$\begin{aligned} P(A_k \setminus \gamma \cdot A_k) &= P(\xi_k + \eta_k > 0 \ \ \& \ \ \xi_k + \zeta_k \le 0) \\ &\le P(-K < \xi_k \le K) \\ &= P(-K/(\sigma\sqrt{N_k}) < \xi_k/(\sigma\sqrt{N_k}) \le K/(\sigma\sqrt{N_k})). \end{aligned} \tag{4.3}$$

By the Central Limit Theorem, $\xi_k/(\sigma\sqrt{N_k}) \Rightarrow \chi$ and clearly $K/(\sigma\sqrt{N_k}) \to 0$. By Lemma 4.1, the expression (4.3) converges to 0.

Now suppose $n_k \to \infty$. Let $\xi'_k = \xi_k/(\sigma\sqrt{N_k})$, $\eta'_k = \eta_k/(\sigma\sqrt{n_k})$, $\zeta'_k = \zeta_k/(\sigma\sqrt{n_k})$. By the Central Limit Theorem, $\xi'_k \Rightarrow \chi$, $\eta'_k \Rightarrow \chi$, $\zeta'_k \Rightarrow \chi$. We have

$$\begin{aligned} P(A_k \setminus \gamma \cdot A_k) &= P(\xi_k + \eta_k > 0 \ \ \& \ \ \xi_k + \zeta_k \le 0) \\ &= P(\zeta_k \le -\xi_k < \eta_k) \\ &= P(\sqrt{n_k}\zeta'_k \le -\sqrt{N_k}\xi'_k < \sqrt{n_k}\eta'_k) \\ &= P\left(\sqrt{\frac{n_k}{N_k}}\zeta'_k \le -\xi'_k < \sqrt{\frac{n_k}{N_k}}\eta'_k\right). \end{aligned} \tag{4.4}$$

By (4.2), $\sqrt{n_k/N_k} \to 0$. By Lemma 4.2, $\sqrt{n_k/N_k}\zeta'_k$, $\sqrt{n_k/N_k}\eta'_k \to 0$ in probability. Finally, by Lemma 4.1, (4.4) converges to 0.

The implication (ii) $\Rightarrow$ (iii) is clear so we proceed to show (iii) $\Rightarrow$ (i). By Theorem 1.1, it suffices to show that $1_\Gamma \prec \lambda_X$. Fix $Q \subseteq \Gamma$ and $\epsilon > 0$. We will find a (Q, ϵ, λ_X)-invariant vector in $\ell^2(X)$. By (iii), (3.2), and Lemma 3.5, there exists $q \in \mathcal{A}$ and $v_1 \in \ell^2(\Gamma/\Gamma_q)$ which is $(Q, \epsilon, \lambda_{\Gamma/\Gamma_q})$-invariant. Let $F = \operatorname{supp} q$ and notice that since $q \ne q_0$, $F \ne \emptyset$. Denote by Γ_F and $\Gamma_{(F)}$ the setwise and pointwise stabilizers of F, respectively. Since $\Gamma_q \le \Gamma_F \le \Gamma$, by Lemma 3.4, there exists $v_2 \in \ell^2(\Gamma/\Gamma_F)$ which is $(Q, \epsilon, \lambda_{\Gamma/\Gamma_F})$-invariant. Since $\Gamma_{(F)} \le \Gamma_F \le \Gamma$ and $[\Gamma_F : \Gamma_{(F)}] < \infty$, by Lemma 3.3, there exists $v_3 \in \ell^2(\Gamma/\Gamma_{(F)})$ which is $(Q, \epsilon, \lambda_{\Gamma/\Gamma_{(F)}})$-invariant. Fix $x \in F$. Since $\Gamma_{(F)} \le \Gamma_x \le \Gamma$, by Lemma 3.4, there exists $v_4 \in \ell^2(\Gamma/\Gamma_x)$ which is $(Q, \epsilon, \lambda_{\Gamma/\Gamma_x})$-invariant. Since by (3.1), $\lambda_{\Gamma/\Gamma_x} \le \lambda_X$, we are done.

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