OPUC and Me

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Bolyai Prize Talk – January 6, 2016

The Bolyai Prize

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OPUC

Orthogonal Polynomials on the Real Line (OPRL) have roots in the work of French mathematicians in the late 18^{th} century with key developments by Jacobi, Hermite and the Russian school in the 19^{th}. But Orthogonal Polynomials on the Unit Circle (OPUC) are the invention of Szegő, especially in a two part paper of 1920-21.

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Schrödinger Operators

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In some ways, the study of NRQM is understanding the Schrödinger operator, $-\Delta + V$. In the 1970's I looked at multiparticle systems and sometimes one particle systems on $\mathbb{R}^3$ or $\mathbb{R}^\nu$.

Schrödinger Operators

One issue that often came up was proving that these operators had only absolutely continuous spectrum (which could be viewed as scattering states) and discrete point spectrum (bound states). The issue was to show that there was no singular continuous spectrum – what my advisor, Arthur Wightman, called the "no goo hypothesis".

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In 1977-78, two breakthroughs changed the focus of my research. The physicist, Phil Anderson had found in 1958, unexpected localization of electrons in one–dimensional random systems.

Schrödinger Operators

I remember discussing this with David Ruelle in the early 1970's who thought that perhaps, there was singular continuous spectrum in such systems but, in fact, Goldsheid, Molchanov and Pastur, in 1977 were able to prove a particular model of $-\frac{d^2}{dx^2} + V$ with random V had dense point spectrum. This has since become such an industry that, for example, last year, the Newton Institute had a 6–month program on random (and almost periodic) Schrödinger operators.

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The other development was the discovery of Pearson in 1978 that sparse, slowly decaying, one dimensional systems had singular continuous spectrum – the dividing line was L^2 decay vs. slower.

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Slowly Decaying Potentials

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Slowly Decaying Potentials

In the 1990's motivated by work of Gordon and del Rio, I was able to prove that if V decays slower than $n^{-\alpha}; \alpha < 1/2$, then generically the spectrum is actually singular continuous! It has been known for many years (for the analog $-\frac{d^2}{dx^2} + V$ since the 1940s) that if $V \in \ell^1$, then the spectrum on $(-2, 2)$ is purely absolutely continuous with only discrete spectrum outside $[-2, 2]$. In terms of power decay, this means that $|V(n)| < Cn^{-\alpha}; \alpha > 1$. That left open what happens in general if $1 \geq \alpha \geq 1/2$ and in about 1995, I realized this was a natural and very interesting question.

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I've been blessed with lots of fine graduate students during my career and, fortunately, two of the best I've had showed up in Pasadena about that time – Sasha Kiselev, who'd been an undergrad in St. Petersburg, and two years later, Rowan Killip from New Zealand.

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I gave Kiselev the question of seeing if there was always a.c. spectrum when $|V(n)| < Cn^{-\alpha}; \alpha \in [1, 1/2)$ and he originally settled the problem positively for $\alpha \in [1, 2/3)$ and eventually, the whole interval.

Slowly Decaying Potentials

While I tended to state hypotheses in the form $|V(n)| < Cn^{-\alpha}$, it is often useful to instead state a condition in terms of ℓ^p. By Pearson's results and also my power decay, one knows as soon as $p > 2$, one can lose all a.c. spectrum, so there was a natural conjecture floating around Caltech that one could exactly capture the borderline with a result that there is always a.c. spectrum on $[-2, 2]$ if $V \in \ell^2$.

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Percy Deift (who'd been my student at Princeton around 1975) visited Caltech. As soon as Rowan Killip mentioned to him the L^2 conjecture for the $-\frac{d^2}{dx^2} + V$ analog, Percy, who'd become an expert on completely integrable systems, commented that there was a KdV sum rule with a term $\int |V(x)|^2 \, dx$ term and that might be relevant.

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Mixed Spectral Types

Because of the Kiselev and Deift-Killip results, $|V(n)| < Cn^{-\alpha}$, looked similar for $\alpha > 1$ and for $1/2 < \alpha \leq 1$ in that both have a.c. spectrum on all of $[-2, 2]$.

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Krein Systems

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I didn't realize until several years later, but the analog of my 2000 open problem for OPUC had been solved by Verblunsky in 1936!

Killip–Simon

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I invited Killip to visit Caltech for the summer of 2001 and he came prepared with a number of relevant papers. First, Case not only had the sum rule that Deift–Killip used but an infinite family of them. He only gave formal proofs and didn't write them explicitly for larger values of n (we eventually did using Chebyshev polynomials).

Killip–Simon

Killip also had a review article of Paul Nevai (whose name I knew, but not his work). He focused on a mapping of OPUC to OPRL and the induced form of Szegő's Theorem. The resulting OPRL systems were very special – originally the spectrum was in $[-2, 2]$ with no eigenvalues outside, although due to work of Nevai, it was known how to allow a finite number of eigenvalues outside $[-2, 2]$.

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Killip–Simon

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Killip–Simon

Killip and I realized that to get an equivalence, we wanted a sum rule – an equality with coefficient information on one side and spectral information on the other. We'd prove one side was finite if and only if the other was. But to get information out only from finiteness, one needed the terms in the sum rule to all be positive.

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Killip–Simon

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Killip–Simon Theorem

We realized that if we could prove the sum rule, we'd get the following lovely Theorem:

Killip–Simon Theorem Let $\{a_n, b_n\}$ be a set of Jacobi parameters for a half line Jacobi matrix, J. Let J_0 be the "free" Jacobi matrix ($a_n \equiv 1; b_n \equiv 0$). Then the following are equivalent ($(1) \iff (2\mathrm{a}) - (2\mathrm{c})$)

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(2a) (Blumenthal-Weyl) $\mathrm{ess\ spec}(J) = [-2, 2]$

(2b) (Quasi–Szegő) $\int_{-2}^{2} (4 - x^2)^{1/2} (\log(\rho(x)))dx > -\infty$

(2c) (Lieb–Thirring) $\sum (|E_n| - 2)^{3/2} < \infty$ where $\{E_n\}$ are the eigenvalues of J outside $[-2, 2]$

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This theorem proves the existence of lots of Jacobi matrices with ℓ^2 decay and mixed spectrum. For the theorem has no hypothesis on $d\mu_s$ other than that it is supported on $[-2, 2]$ and of course that measure has total mass 1. If ρ is such that $\int \rho(x)\, dx \equiv 1 - \beta < 1$, any $d\mu_s$ of total mass β can occur without destroying ℓ^2 decay.

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We had no trouble proving the needed sum rule for J's where $J - J_0$ had only finitely many non-zero matrix elements but there were technical issues of taking limits. We proved two inequalities – one was straight–forward. I remember how proud we were when we realized that the quasi-Szegő term was a relative entropy, given by a variational principle and so weakly lower semi–continuous which allowed us to complete the proof of the sum rule.

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It would be an overstatement to say that I fell in love with OPUC but I was charmed and jumped in to understand the subject, so much so that it was a major focus for the next few years.

A Tale of Two Tribes

I made three interesting discoveries about the theory of OPUC early on. The first is that there was an active research community studying spectral problems, not only of OPUC, but also of OPRL. The latter was essentially the Jacobi matrices that had been a major focus of my research group and of many others from the mathematical physics community.

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No Book Literature

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A Playground

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It became clear that I could write dozens of little papers on the differing results but I don't need to pad my bibliography so I decided that one long paper which I naively expected to be about 80 pages was the sensible way to go. That's long for journals I deal with but I was pleased to learn from Paul Nevai that Journal of Approximation Theory would be glad to consider it.

A Tale of Two Volumes

In planning my paper, I realized that while the results would be of interest to spectral theorists, like me a few months before, they'd never seen OPUC and there was no book to refer them to – no place to give them the simple prof of the recursion relations.

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Interesting results kept be added to my mental list and I realized that I probably needed to be thinking about a book.

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I close with the story of two developments that had considerable impact: Verblunsky's work and the CMV matrix.

Szegő's Theorem as a Sum Rule

One result I understood from several points of view in 2001-2002 was Szegő's Theorem. I understood that if one proved it for arbitrary probability measures (in his papers before 1950, Szegő only considered measures with $d\mu_s = 0$), then one had a result exactly analogous to the mixed spectral result of Killip – Simon.

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The literature on this question was very confused. Many claimed that it was first proven in the 1958 book of Grenander and Szegő. Others mentioned some early 1940's work of Kolmogorov and Krein on prediction theory. That was wrong. They only had that the two sides were infinite at once – which was enough for a mixed spectral conclusion. In the mid-1940's, the result was in papers of Geronimus and much of the Russian OP literature gave him credit.

Szegő's Theorem as a Sum Rule

I realized that the true analog of our sum rule was $\sum_{n=0}^{\infty} \log(1 - |\alpha_n|^2) = \int_0^{2\pi} log(w(\theta) \frac{d\theta}{2\pi}$ where α_n were the recursion coefficients and $w(\theta)$ the a.c. weight of the underlying measure.

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Leonid Golinskii responded that he'd never seen the papers because the journal they were in (Proc. LMS) wasn't available to him in his library at his institute in Kharkov but he'd heard that the result might have been in two papers by someone named Verblunsky and he gave me the references from 1935-36.

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Buried in the papers, were also numerous other important theorems rediscovered and credited to others. Among them that there is a strict bijection between between all probability measures on $\partial\mathbb{D}$ with infinite support and all choices of coefficients $\{\alpha_n\}_{n=0}^{\infty} \in \mathbb{D}^{\infty}$.

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Verblunsky died at age 90 in 1996 having spent most of his career at Queen's University Belfast. He left a bequest to the LMS where you can find the Samuel Verblunsky Reading Room. But he never received any recognition for his two great papers nor did he do other work of great note.

Recognizing Verblunsky

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Jacobi Matrix Analogs

Central to my new results (and also the Jacobi matrix theory I was often mimicking) was the important role of the operator of multiplication by z on $L^2(X, d\mu)$ where $X = \mathbb{R}$ for OPUC and $X = \partial\mathbb{D}$ for OPUC. In the OPRL case, this operator is self–adjoint and in OPUC case, it is unitary.

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For OPUC if $\{\varphi_n\}_{n=0}^\infty$ are the orthonormal polynomials, they may or may not be an orthonormal basis for all of $L^2(\partial\mathbb{D}, d\mu)$, so the matrix $\mathcal{G}_{k\ell} = \langle \varphi_k, z\, \varphi_\ell \rangle$ may or may not be unitary.

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CMV basis

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This was clearly an ideal tool for studying perturbations and, within a week, working together with Golinskii by email, we had a number of results of which the most interesting was that if μ and $\tilde{\mu}$ are two measures with Verblunsky coefficients $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\tilde{\alpha}_n\}_{n=0}^{\infty}$ and if $|\alpha_n - \tilde{\alpha}_n| < \infty$, then μ and $\tilde{\mu}$ have the same a.c. supports.

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CMV matrix

Leonid got permission from the authors and shortly thereafter I got the beautiful preprint from three Spanish mathematicians: Cantero, Moral and Velázquez. They had a lovely factorization of $\mathcal{C}$ into a product of two tridiagonal matrices. I decided to name the matrix $\mathcal{C}$ the *CMV matrix* and the name has stuck.

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Writing these books was a lot of fun!

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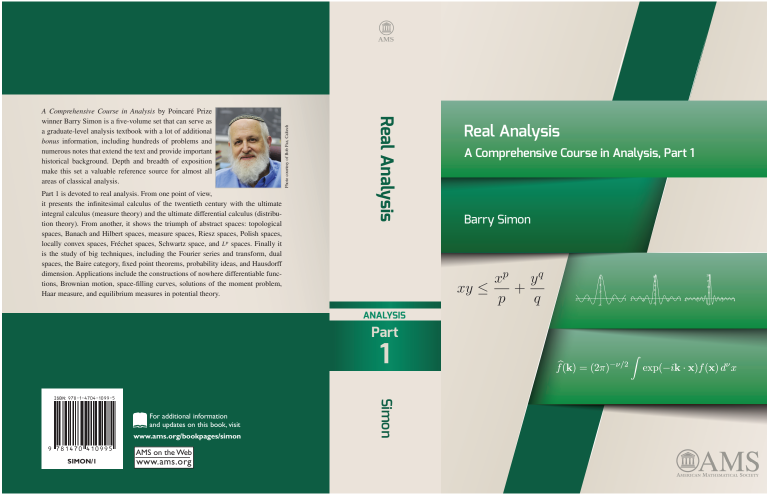

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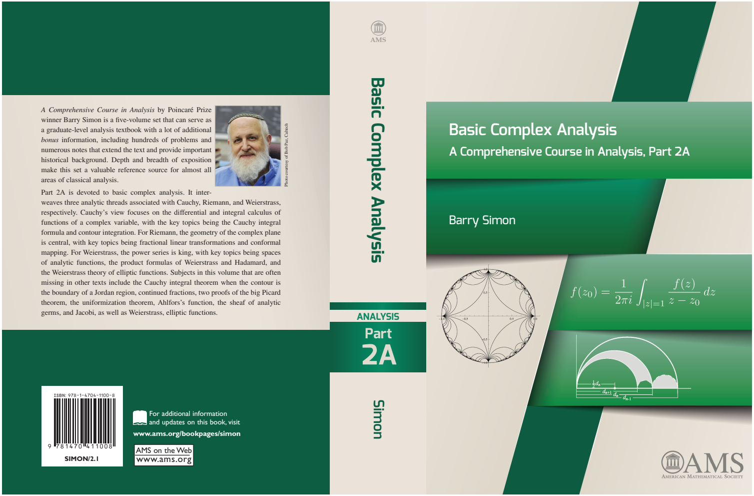

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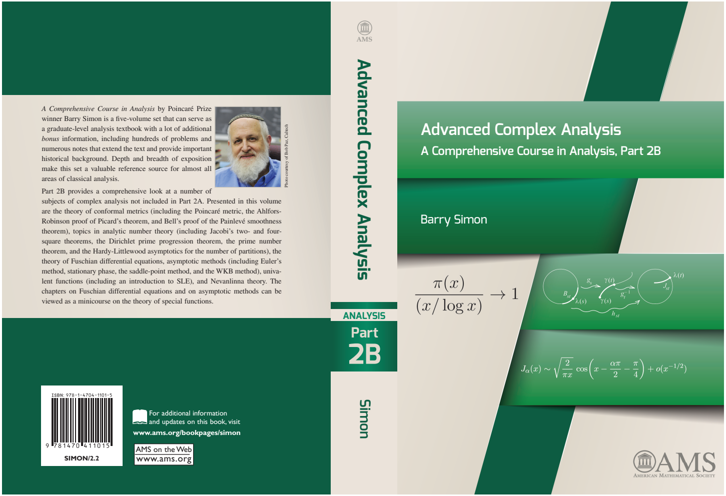

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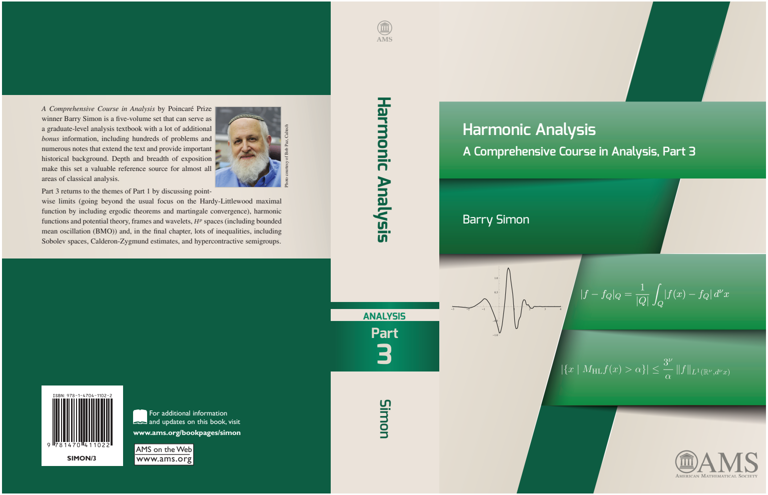

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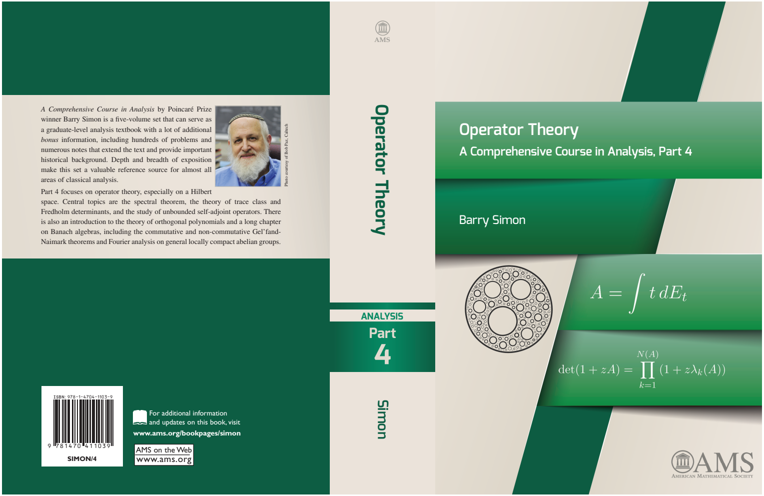

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