Large Deviations and Sum Rules for Orthogonal Polynomials

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Lecture 1: OPRL, OPUC and Sum Rules

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- Lecture 2: Meromorphic Herglotz Functions and Proof of KS Sum Rule
- Lecture 3: The Theory of Large Deviations
- Lecture 4: GNR Proof of Sum Rules

References

[OPUC] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, AMS Colloquium Series **54.1**, American Mathematical Society, Providence, RI, 2005.

[OPUC2] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, AMS Colloquium Series, **54.2**, American Mathematical Society, Providence, RI, 2005.

[SzThm] B. Simon, *Szegő's Theorem and Its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2011.

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- Isospectral manifold for the harmonic oscillator

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For example, the domain of definition of the harmonic oscillator isospectral "manifold" is unknown. It is not even known if it is connected!

OPs

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Indeed, my own interest came from studying discrete Schrödinger operators on $\ell^2(\mathbb{Z})$

$$(Hu)_n = u_{n+1} + u_{n-1} + V u_n$$

and the realization that when restricted to $\mathbb{Z}_+$, one had a special case of OPRL.

OPRL basics

μ will be a probability measure on $\mathbb{R}$. We'll always suppose that μ has bounded support $[a,b]$ which is not a finite set of points. (We then say that μ is non-trivial.) This implies that $1, x, x^2, \ldots$ are independent since $\int |P(x)|^2 \, d\mu = 0 \Rightarrow \mu$ is supported on the zeroes of P.

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Apply Gram Schmidt to $1, x, \dots$ and get monic polynomials

$$P_j(x) = x^j + \alpha_{j,1} x^{j-1} + \dots$$

and orthonormal (ON) polynomials

$$p_j = P_j / \|P_j\|$$

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If $d\mu = d\theta/2\pi$ on $\partial\mathbb{D}$, the span of $\{z^j\}_{j=0}^{\infty}$ is not dense in L^2 (but is only H^2). Perhaps, surprisingly, there are measures $d\mu$ on $\partial\mathbb{D}$ for which they are dense (e.g., μ purely singular).

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More significantly, the argument we'll give for our recursion relation fails if $\mathrm{supp}(d\mu) \not\subset \mathbb{R}$.

OPRL basics

Since P_k is monic and $\{P_j\}_{j=0}^{k+1}$ span polynomials of degree at most $k+1$, we have

$$xP_k = P_{k+1} + \sum_{j=0}^{k} B_{k,j}\, P_j$$

Clearly

$$B_{k,j} = \langle P_j, xP_k \rangle / \|P_j\|^2$$

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If $j < k-1$, this is zero.

If $j = k-1$, $\langle P_{k-1}, xP_k\rangle = \langle xP_{k-1}, P_k\rangle = \|P_k\|^2$.

OPRL basics

Thus ($P_{-1} \equiv 0$); $\{a_j\}_{j=1}^{\infty}$, $\{b_j\}_{j=1}^{\infty}$: *Jacobi recursion*

$$xP_N = P_{N+1} + b_{N+1}P_N + a_N^2 P_{N-1}$$

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This, in turn, implies $p_n = P_n/a_1 \dots a_n$ obeys

$$xp_n = a_{n+1}p_{n+1} + b_{n+1}p_n + a_n p_{n-1}$$

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In the orthonormal basis $\{p_n\}_{n=0}^{\infty}$, multiplication by x has the matrix

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

called a *Jacobi matrix*.

Favard's Theorem

Since

$$b_n = \int x p_{n-1}^2(x)\, d\mu, \quad a_n = \int x p_{n-1}(x) p_n(x)\, d\mu$$

$\mathrm{supp}(\mu) \subset [-R, R] \Rightarrow |b_n| \leq R,\ |a_n| \leq R.$

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Conversely, if $\sup_n \big(|a_n| + |b_n|\big) = \alpha < \infty$, J is a bounded matrix of norm at most 3α. In that case, the spectral theorem implies there is a measure $d\mu$ so that

$$\langle (1, 0, \ldots)^t, J^\ell (1, 0, \ldots)^t \rangle = \int x^\ell d\mu(x)$$

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$$\langle (1, 0, \dots)^t, J^\ell (1, 0, \dots)^t \rangle = \int x^\ell d\mu(x)$$

If one uses Gram-Schmidt to orthonormalize $\{J^\ell (1, 0, \dots)^t\}_{\ell=0}^\infty$, one finds μ has Jacobi matrix exactly given by J.

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Favard's Theorem. *There is a one–one correspondence between bounded Jacobi parameters*

$$\{a_n, b_n\}_{n=1}^{\infty} \in \big[(0,\infty) \times \mathbb{R}\big]^{\infty}$$

and non-trivial probability measures, μ, of bounded support via:

$$\mu \Rightarrow \{a_n, b_n\} \quad \textit{(OP recursion)}$$

$$\{a_n, b_n\} \Rightarrow \mu \quad \textit{(Spectral Theorem)}$$

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There are also results for μ's with unbounded support so long as $\int x^n \, d\mu < \infty$. In this case, $\{a_n, b_n\} \Rightarrow \mu$ may not be unique because J may not be essentially self-adjoint on vectors of finite support.

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We denote the quantity inside the limit as $m_N(z; a_1, \dots, a_{N-1}; b_1, \dots, b_N)$.

Coefficient stripping

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$$\begin{aligned} D_N(z; a_1, \dots, b_N) = (b_1 - z) & D_{N-1}(z; a_2, \dots, b_N) \\ & - a_1^2 D_{N-2}(z; a_3, \dots, b_N) \end{aligned}$$

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Dividing by $D_{N-1}(z; a_2, \dots, b_N)$ and taking $N \to \infty$ after using the formula for m as a ratio of D's, we see that

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The convergence theorems for continued fractions lets one go from Jacobi parameters to measure. One consequence of the single stripping formula is that the poles of m_1 (i.e. the pure points of μ_1) are precisely the zeros of m.

OPUC basics

Let $d\mu$ be a non-trivial probability measure on $\partial\mathbb{D}$. As in the OPRL case, we use Gram-Schmidt to define monic OPs, $\Phi_n(z)$ and ON OP's $\varphi_n(z)$.

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In the OPRL case, $P_{n+1} - xP_n \perp \{1, x, \ldots, x^{n-2}\}$.

OPUC basics

In the OPUC case, $\Phi_{n+1} - z\Phi_n \perp \{z, \dots, z^n\}$, since

$$\langle z\Phi, z^j \rangle = \langle \Phi, z^{j-1} \rangle$$

if $j \geq 1$.

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if $j \geq 1$.

In the OPRL case, we used deg $P = n$ and $P \perp \{1, x, \dots, x^{n-2}\} \Rightarrow P = c_1 P_n + c_2 P_{n-1}$.

In the OPUC case, we want to characterize deg $P = n$, $P \perp \{z, z^2, \dots, z^n\}$.

OPUC basics

Define * on degree n polynomials to themselves by

$$Q^*(z) = z^n \, \overline{Q\left(\frac{1}{\bar{z}}\right)}$$

(bad but standard notation!) or

$$Q(z) = \sum_{j=0}^{n} c_j z^j \Rightarrow Q^*(z) = \sum_{j=0}^{n} \bar{c}_{n-j} \, z^j$$

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Then, * is antiunitary and so for deg $Q = n$

$$Q \perp \{1, \ldots, z^{n-1}\} \Leftrightarrow Q = c \, \Phi_n$$

is equivalent to

$$Q \perp \{z, \ldots, z^n\} \Leftrightarrow Q = c \, \Phi_n^*$$

Szegő recursion and Verblunsky coefficients

Thus, we see, there are parameters $\{\alpha_n\}_{n=0}^{\infty}$ (called Verblunsky coefficients) so that

$$\Phi_{n+1}(z) = z\Phi_n - \overline{\alpha}_n \Phi_n^*(z)$$

This is the Szegő Recursion (History: Szegő and Geronimus in 1939; Verblunsky in 1935–36)

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Applying * for deg $n+1$ polynomials to this yields

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z \Phi_n$$

Thus, we see, there are parameters $\{\alpha_n\}_{n=0}^{\infty}$ (called Verblunsky coefficients) so that

$$\Phi_{n+1}(z) = z\Phi_n - \overline{\alpha}_n \Phi_n^*(z)$$

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The strange looking $-\bar{\alpha}_n$ rather than say $+\alpha_n$ is to have the α_n be the Schur parameter of the Schur function of μ (Geronimus); also the Verblunsky parameterization then agrees with α_n. These are discussed in [OPUC1].

Szegő recursion and Verblunsky coefficients

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This plus $\Phi_{n+1} = z\Phi_n - \bar{\alpha}_n \Phi_n^*(z)$ implies

$$-\overline{\Phi_{n+1}(0)} = \alpha_n$$

i.e., Φ_n determines α_{n-1}.

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For measures with exactly n pure points, there are only n non–trivial OPs, and n α's. $\alpha_{n-1} \in \partial\mathbb{D}$. One has $\|\Phi_k\| = \rho_0 \dots \rho_{k-1}$ where $\rho_j = \sqrt{1 - |\alpha_j|^2}$

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Szegő's Theorem: Toeplitz version

Szegő's Theorem concerns probability measures on $\partial\mathbb{D}$ of the form

$$d\mu(\theta) = w(\theta)\frac{d\theta}{2\pi} + d\mu_s(\theta)$$

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While this is true in general, Szegő only proved it when $d\mu_s = 0$.

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In 1920, Szegő realized that, because a Toeplitz matrix is just the Gram matrix of $\{z^j\}_{j=0}^{n-1}$, it is also the Gram matrix of $\{\Phi_j\}_{j=0}^{n-1}$ which is diagonal so

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But the recursion relation was only published by Szegő in 1939, so he didn't have a form in term's of α_n and ρ_n.

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In particular, $\sum_{j=0}^{\infty} |\alpha_j|^2 < \infty \Rightarrow \Sigma_{ac} = \partial\mathbb{D}$.

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only holds under the a priori condition that μ is supported inside $[-2, 2]$

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only holds under the a priori condition that μ is supported inside $[-2,2]$ and this is not simply expressible in terms of the Jacobi parameters; for example, it doesn't only depend on the parameters near ∞ and can be changed by modifying a single a or b.

Killip–Simon Theorem

In 2001 (published 2003), Killip and I proved the following gem which we regard as an OPRL analog of the Verblunsky–Szegő gem where $\{E_j^\pm\}_{j=1}^{N_\pm}$ are the eigenvalues outside $[-2,2]$ (with + above 2 and - below -2):

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if and only if the essential support of μ is $[-2,2]$ and

$$\int_{-2}^{2} \log(w(x))\sqrt{4-x^2}\,dx > -\infty \qquad \sum_{j,\pm} (|E_j^{\pm}| - 2)^{3/2} < \infty$$

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$$\sum |E_n|^p \le C \int |V(x)|^{p+d/2}\,dx$$

Killip–Simon Sum Rule

The gem comes from a sum rule. Let
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Then the Killip-Simon sum rule says

$$Q(\mu) + \sum_{j,\pm} F(E_j^{\pm}) = \sum_{n=1}^{\infty} \tfrac{1}{4} b_n^2 + \tfrac{1}{2} G(a_n)$$

As with the Szegő–Verblunsky sum rule, an important point is that it always holds although both sides may be $+\infty$.

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The gem comes from the fact that $F \geq 0$, vanishes exactly at $E = \pm 2$ and is $\mathrm{O}((|E| - 2)^{3/2})$ there and that $G \geq 0$, vanishes exactly at $a = 1$ and is $\mathrm{O}((a-1)^2)$ there.

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As in the OPUC case, this sum rule implies the existence of Hilbert–Schmidt perturbations with mixed spectrum.

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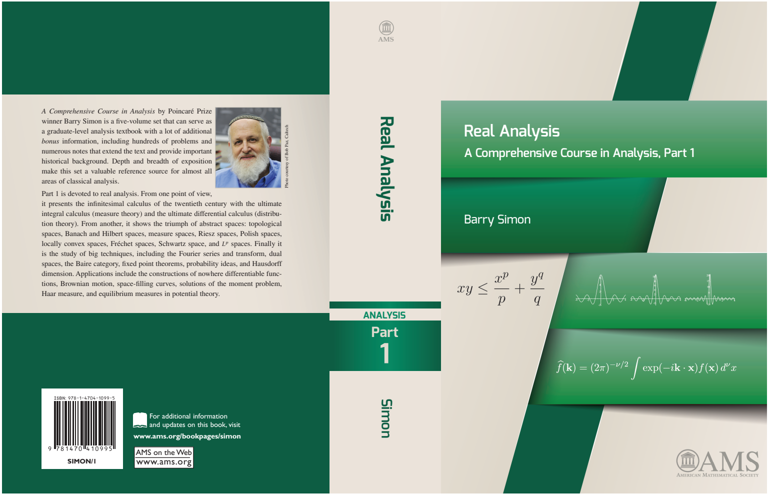

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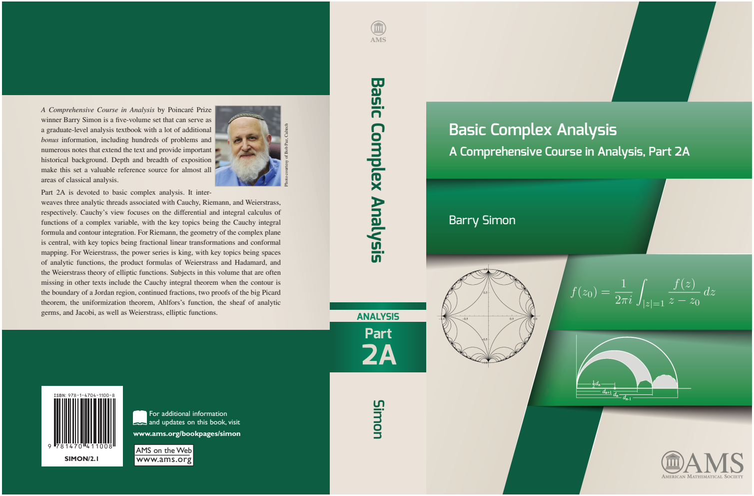

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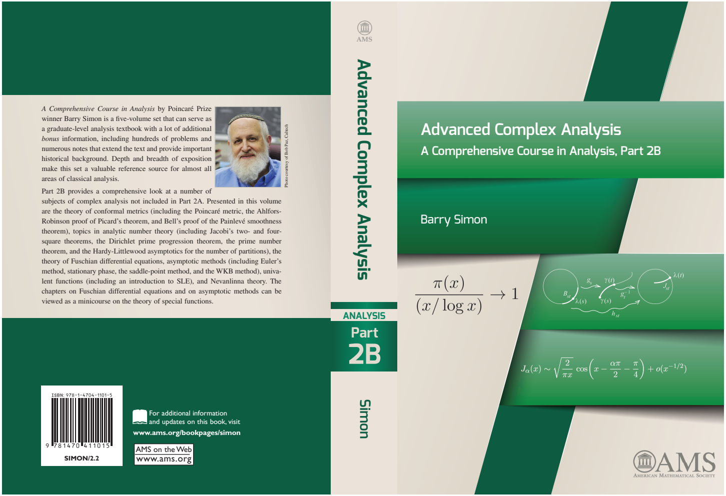

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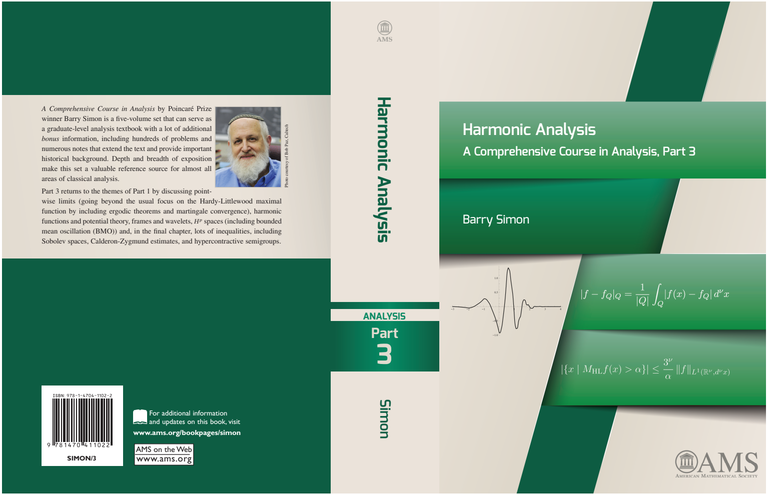

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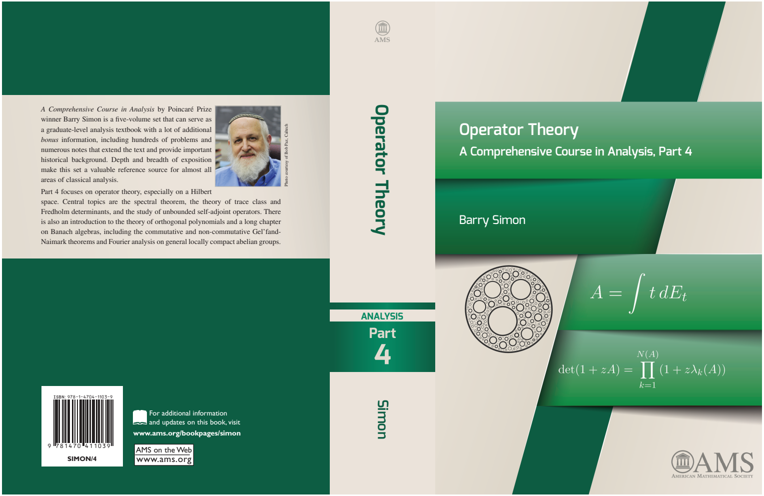

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