



# Large Deviations and Sum Rules for Orthogonal Polynomials

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Lecture 1: OPRL, OPUC and Sum Rules

What is spectral  
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OPRL basics

Favard's Theorem

m-Function  
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OPUC basics

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- Lecture 1: OPRL, OPUC and Sum Rules
- Lecture 2: Meromorphic Herglotz Functions and Proof of KS Sum Rule
- Lecture 3: The Theory of Large Deviations
- Lecture 4: GNR Proof of Sum Rules



# References

[OPUC] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, AMS Colloquium Series **54.1**, American Mathematical Society, Providence, RI, 2005.

[OPUC2] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, AMS Colloquium Series, **54.2**, American Mathematical Society, Providence, RI, 2005.

[SzThm] B. Simon, *Szegő's Theorem and Its Descendants: Spectral Theory for  $L^2$  Perturbations of Orthogonal Polynomials*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2011.

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# What is spectral theory?

Spectral theory is the general theory of the relation of the fundamental parameters of an object and its “spectral” characteristics.

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Spectral characteristics means eigenvalues or scattering data or, more generally, spectral measures.

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- Can you hear the shape of a drum ?

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Examples include

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- Isospectral manifold for the harmonic oscillator

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# What is spectral theory?

The *direct problem* goes from the object to spectra.

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The *direct problem* goes from the object to spectra.

The *inverse problem* goes backwards.

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The *direct problem* goes from the object to spectra.

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The direct problem is typically easy while the inverse problem is typically hard.

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The *direct problem* goes from the object to spectra.

The *inverse problem* goes backwards.

The direct problem is typically easy while the inverse problem is typically hard.

For example, the domain of definition of the harmonic oscillator isospectral “manifold” is unknown. It is not even known if it is connected!

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# OPs

Orthogonal polynomials on the real line (OPRL) and on the unit circle (OPUC) are particularly useful because the inverse problems are easy—indeed the inverse problem is the OP definition as we'll see.

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# OPs

Orthogonal polynomials on the real line (OPRL) and on the unit circle (OPUC) are particularly useful because the inverse problems are easy—indeed the inverse problem is the OP definition as we'll see.

OPs also enter in many application—both specific polynomials and the general theory.

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# OPs

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OPs also enter in many application—both specific polynomials and the general theory.

Indeed, my own interest came from studying discrete Schrödinger operators on  $\ell^2(\mathbb{Z})$

$$(Hu)_n = u_{n+1} + u_{n-1} + Vu_n$$

and the realization that when restricted to  $\mathbb{Z}_+$ , one had a special case of OPRL.

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# OPRL basics

$\mu$  will be a probability measure on  $\mathbb{R}$ . We'll always suppose that  $\mu$  has bounded support  $[a, b]$  which is not a finite set of points. (We then say that  $\mu$  is non-trivial.) This implies that  $1, x, x^2, \dots$  are independent since  $\int |P(x)|^2 d\mu = 0 \Rightarrow \mu$  is supported on the zeroes of  $P$ .

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Apply Gram Schmidt to  $1, x, \dots$  and get monic polynomials

$$P_j(x) = x^j + \alpha_{j,1}x^{j-1} + \dots$$

and orthonormal (ON) polynomials

$$p_j = P_j / \|P_j\|$$

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More generally we can do the same for any probability measure of bounded support on  $\mathbb{C}$ .

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# OPRL basics

More generally we can do the same for any probability measure of bounded support on  $\mathbb{C}$ .

One difference from the case of  $\mathbb{R}$ , the linear combination of  $\{x^j\}_{j=0}^{\infty}$  are dense in  $L^2(\mathbb{R}, d\mu)$  by Weierstrass. This may or may not be true if  $\text{supp}(d\mu) \not\subset \mathbb{R}$ .

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If  $d\mu = d\theta/2\pi$  on  $\partial\mathbb{D}$ , the span of  $\{z^j\}_{j=0}^{\infty}$  is not dense in  $L^2$  (but is only  $H^2$ ). Perhaps, surprisingly, there are measures  $d\mu$  on  $\partial\mathbb{D}$  for which they are dense (e.g.,  $\mu$  purely singular).

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More significantly, the argument we'll give for our recursion relation fails if  $\text{supp}(d\mu) \not\subset \mathbb{R}$ .

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# OPRL basics

Since  $P_k$  is monic and  $\{P_j\}_{j=0}^{k+1}$  span polynomials of degree at most  $k+1$ , we have

$$xP_k = P_{k+1} + \sum_{j=0}^k B_{k,j} P_j$$

Clearly

$$B_{k,j} = \langle P_j, xP_k \rangle / \|P_j\|^2$$

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Now we use

$$\langle P_j, xP_k \rangle = \langle xP_j, P_k \rangle$$

(need  $d\mu$  on  $\mathbb{R}$ !!)

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If  $j < k-1$ , this is zero.

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(need  $d\mu$  on  $\mathbb{R}$ !!)

If  $j < k-1$ , this is zero.

If  $j = k-1$ ,  $\langle P_{k-1}, xP_k \rangle = \langle xP_{k-1}, P_k \rangle = \|P_k\|^2$ .

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# OPRL basics

Thus ( $P_{-1} \equiv 0$ );  $\{a_j\}_{j=1}^{\infty}$ ,  $\{b_j\}_{j=1}^{\infty}$  : *Jacobi recursion*

$$xP_N = P_{N+1} + b_{N+1}P_N + a_N^2P_{N-1}$$

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$$b_N \in \mathbb{R}, \quad a_N = \|P_N\|/\|P_{N-1}\|$$

These are called *Jacobi parameters*.

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$$b_N \in \mathbb{R}, \quad a_N = \|P_N\|/\|P_{N-1}\|$$

These are called *Jacobi parameters*. This implies  $\|P_N\| = a_N a_{N-1} \dots a_1$  (since  $\|P_0\| = 1$ ).

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These are called *Jacobi parameters*. This implies  $\|P_N\| = a_N a_{N-1} \dots a_1$  (since  $\|P_0\| = 1$ ).

This, in turn, implies  $p_n = P_n/a_1 \dots a_n$  obeys

$$xp_n = a_{n+1}p_{n+1} + b_{n+1}p_n + a_np_{n-1}$$

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# OPRL basics

We have thus solved the inverse problem, i.e.,  $\mu$  is the spectral data and  $\{a_n, b_n\}_{n=1}^{\infty}$  are the descriptors of the object.

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In the orthonormal basis  $\{p_n\}_{n=0}^{\infty}$ , multiplication by  $x$  has the matrix

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

called a *Jacobi matrix*.

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# Favard's Theorem

Since

$$b_n = \int x p_{n-1}^2(x) d\mu, \quad a_n = \int x p_{n-1}(x) p_n(x) d\mu$$

$$\text{supp}(\mu) \subset [-R, R] \Rightarrow |b_n| \leq R, |a_n| \leq R.$$

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$$b_n = \int x p_{n-1}^2(x) d\mu, \quad a_n = \int x p_{n-1}(x) p_n(x) d\mu$$

$\text{supp}(\mu) \subset [-R, R] \Rightarrow |b_n| \leq R, |a_n| \leq R.$

Conversely, if  $\sup_n (|a_n| + |b_n|) = \alpha < \infty$ ,  $J$  is a bounded matrix of norm at most  $3\alpha$ . In that case, the spectral theorem implies there is a measure  $d\mu$  so that

$$\langle (1, 0, \dots)^t, J^\ell (1, 0, \dots)^t \rangle = \int x^\ell d\mu(x)$$

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If one uses Gram-Schmidt to orthonormalize  $\{J^\ell (1, 0, \dots)^t\}_{\ell=0}^\infty$ , one finds  $\mu$  has Jacobi matrix exactly given by  $J$ .

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# Favard's Theorem

We have thus proven Favard's Theorem (his paper was in 1935; really due to Stieltjes in 1894 or to Stone in 1932).

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We have thus proven Favard's Theorem (his paper was in 1935; really due to Stieltjes in 1894 or to Stone in 1932).

**Favard's Theorem.** *There is a one–one correspondence between bounded Jacobi parameters*

$$\{a_n, b_n\}_{n=1}^{\infty} \in [(0, \infty) \times \mathbb{R}]^{\infty}$$

*and non-trivial probability measures,  $\mu$ , of bounded support via:*

$$\mu \Rightarrow \{a_n, b_n\} \quad (\text{OP recursion})$$

$$\{a_n, b_n\} \Rightarrow \mu \quad (\text{Spectral Theorem})$$

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$$\mu \Rightarrow \{a_n, b_n\} \quad (\text{OP recursion})$$

$$\{a_n, b_n\} \Rightarrow \mu \quad (\text{Spectral Theorem})$$

There are also results for  $\mu$ 's with unbounded support so long as  $\int x^n d\mu < \infty$ . In this case,  $\{a_n, b_n\} \Rightarrow \mu$  may not be unique because  $J$  may not be essentially self-adjoint on vectors of finite support.

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# m-function

By the above construction, one has that

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# m-function

By the above construction, one has that

$$m(z) \equiv \int \frac{d\mu(x)}{x-z} = \langle \delta_1, (J-z)^{-1} \delta_1 \rangle$$

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# m-function

By the above construction, one has that

$$m(z) \equiv \int \frac{d\mu(x)}{x-z} = \langle \delta_1, (J-z)^{-1} \delta_1 \rangle$$

Let  $J_N$  be the  $N \times N$  matrix obtained by keeping the top  $N$  rows and leftmost  $N$  columns of  $J$ . Then it is easy to prove that

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$$m(z) = \lim_{N \rightarrow \infty} \langle \delta_1, (J_N - z)^{-1} \delta_1 \rangle$$

We denote the quantity inside the limit as  $m_N(z; a_1, \dots, a_{N-1}; b_1, \dots, b_N)$ .

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# Coefficient stripping

Let  $D_N(z; a_1, \dots, a_{N-1}; b_1, \dots, b_N) = \det(J_N - z)$  so that Cramér's rule says that

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$$m_N(z; a_1, \dots, b_N) = \frac{D_{N-1}(z; a_2, \dots, a_{N-1}; b_2, \dots, b_N)}{D_N(z; a_1, \dots, a_{N-1}; b_1, \dots, b_N)}$$

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where we look at the determinant of a once stripped matrix obtained by removing the first row and column.

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where we look at the determinant of a once stripped matrix obtained by removing the first row and column. By expanding  $\det(J_N - z)$  in minors in the first column we get

$$\begin{aligned} D_N(z; a_1, \dots, b_N) &= (b_1 - z)D_{N-1}(z; a_2, \dots, b_N) \\ &\quad - a_1^2 D_{N-2}(z; a_3, \dots, b_N) \end{aligned}$$

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where we look at the determinant of a once stripped matrix obtained by removing the first row and column. By expanding  $\det(J_N - z)$  in minors in the first column we get

$$D_N(z; a_1, \dots, b_N) = (b_1 - z)D_{N-1}(z; a_2, \dots, b_N) - a_1^2 D_{N-2}(z; a_3, \dots, b_N)$$

Dividing by  $D_{N-1}(z; a_2, \dots, b_N)$  and taking  $N \rightarrow \infty$  after using the formula for  $m$  as a ratio of  $D$ 's, we see that

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$$m(z)^{-1} = b_1 - z - a_1^2 m_1(z)$$

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$$m(z)^{-1} = b_1 - z - a_1^2 m_1(z)$$

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$$m(z) = \frac{1}{b_1 - z + \frac{a_1^2}{b_2 - z + \frac{a_2^2}{b_3 - z + \dots}}}$$

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The convergence theorems for continued fractions lets one go from Jacobi parameters to measure.

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The convergence theorems for continued fractions lets one go from Jacobi parameters to measure. One consequence of the single stripping formula is that the poles of  $m_1$  (i.e. the pure points of  $\mu_1$ ) are precisely the zeros of  $m$ .

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# OPUC basics

Let  $d\mu$  be a non-trivial probability measure on  $\partial\mathbb{D}$ . As in the OPRL case, we use Gram-Schmidt to define monic OPs,  $\Phi_n(z)$  and ON OP's  $\varphi_n(z)$ .

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In the OPRL case, if  $z$  is multiplication by the underlying variable,  $z^* = z$ . This is replaced by  $z^*z = 1$ .

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In the OPRL case, if  $z$  is multiplication by the underlying variable,  $z^* = z$ . This is replaced by  $z^*z = 1$ .

In the OPRL case,  $P_{n+1} - xP_n \perp \{1, x, \dots, x^{n-2}\}$ .

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# OPUC basics

In the OPUC case,  $\Phi_{n+1} - z\Phi_n \perp \{z, \dots, z^n\}$ , since

$$\langle z\Phi, z^j \rangle = \langle \Phi, z^{j-1} \rangle$$

if  $j \geq 1$ .

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if  $j \geq 1$ .

In the OPRL case, we used  $\deg P = n$  and  $P \perp \{1, x, \dots, x^{n-2}\} \Rightarrow P = c_1 P_n + c_2 P_{n-1}$ .

In the OPUC case, we want to characterize  $\deg P = n$ ,  $P \perp \{z, z^2, \dots, z^n\}$ .

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# OPUC basics

Define  $*$  on degree  $n$  polynomials to themselves by

$$Q^*(z) = z^n \overline{Q\left(\frac{1}{\bar{z}}\right)}$$

(bad but standard notation!) or

$$Q(z) = \sum_{j=0}^n c_j z^j \Rightarrow Q^*(z) = \sum_{j=0}^n \bar{c}_{n-j} z^j$$

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$$Q(z) = \sum_{j=0}^n c_j z^j \Rightarrow Q^*(z) = \sum_{j=0}^n \bar{c}_{n-j} z^j$$

Then,  $*$  is antiunitary and so for  $\deg Q = n$

$$Q \perp \{1, \dots, z^{n-1}\} \Leftrightarrow Q = c \Phi_n$$

is equivalent to

$$Q \perp \{z, \dots, z^n\} \Leftrightarrow Q = c \Phi_n^*$$

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# Szegő recursion and Verblunsky coefficients

Thus, we see, there are parameters  $\{\alpha_n\}_{n=0}^{\infty}$  (called Verblunsky coefficients) so that

$$\Phi_{n+1}(z) = z\Phi_n - \bar{\alpha}_n\Phi_n^*(z)$$

This is the Szegő Recursion (History: Szegő and Geronimus in 1939; Verblunsky in 1935–36)

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Applying  $*$  for deg  $n + 1$  polynomials to this yields

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z \Phi_n$$

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Applying  $*$  for deg  $n + 1$  polynomials to this yields

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z \Phi_n$$

The strange looking  $-\bar{\alpha}_n$  rather than say  $+\alpha_n$  is to have the  $\alpha_n$  be the Schur parameter of the Schur function of  $\mu$  (Geronimus); also the Verblunsky parameterization then agrees with  $\alpha_n$ . These are discussed in [OPUC1].

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# Szegő recursion and Verblunsky coefficients

$\Phi_n$  monic  $\Rightarrow$  constant term in  $\Phi_n^*$  is 1  $\Rightarrow \Phi_n^*(0) = 1$ .

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# Szegő recursion and Verblunsky coefficients

$\Phi_n$  monic  $\Rightarrow$  constant term in  $\Phi_n^*$  is 1  $\Rightarrow \Phi_n^*(0) = 1$ .

This plus  $\Phi_{n+1} = z\Phi_n - \bar{\alpha}_n\Phi_n^*(z)$  implies

$$-\overline{\Phi_{n+1}(0)} = \alpha_n$$

i.e.,  $\Phi_n$  determines  $\alpha_{n-1}$ .

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# Szegő recursion and Verblunsky coefficients

For OPRL, we saw  $\|P_{n+1}\|/\|P_n\| = a_{n+1}$ . We are looking for the analog for OPUC.

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$$\text{Szegő Recursion} \Rightarrow \Phi_{n+1} + \bar{\alpha}_n \Phi_n^* = z \Phi_n$$

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$$\text{Szegő Recursion} \Rightarrow \Phi_{n+1} + \bar{\alpha}_n \Phi_n^* = z \Phi_n$$

$$\Phi_{n+1} \perp \Phi_n^* \Rightarrow \|\Phi_{n+1}\|^2 + |\alpha_n|^2 \|\Phi_n^*\|^2 = \|z \Phi_n\|^2$$

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$$\Phi_{n+1} \perp \Phi_n^* \Rightarrow \|\Phi_{n+1}\|^2 + |\alpha_n|^2 \|\Phi_n^*\|^2 = \|z \Phi_n\|^2$$

Multiplication by  $z$  unitary plus  $*$  antiunitary  $\Rightarrow$

$$\|\Phi_{n+1}\|^2 = \rho_n^2 \|\Phi_n\|^2; \quad \rho_n^2 = 1 - |\alpha_n|^2$$

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# Szegő recursion and Verblunsky coefficients

For OPRL, we saw  $\|P_{n+1}\|/\|P_n\| = a_{n+1}$ . We are looking for the analog for OPUC.

Szegő Recursion  $\Rightarrow \Phi_{n+1} + \bar{\alpha}_n \Phi_n^* = z\Phi_n$

$$\Phi_{n+1} \perp \Phi_n^* \Rightarrow \|\Phi_{n+1}\|^2 + |\alpha_n|^2 \|\Phi_n^*\|^2 = \|z\Phi_n\|^2$$

Multiplication by  $z$  unitary plus  $*$  antiunitary  $\Rightarrow$

$$\|\Phi_{n+1}\|^2 = \rho_n^2 \|\Phi_n\|^2; \quad \rho_n^2 = 1 - |\alpha_n|^2$$

which implies  $|\alpha_n| < 1$  (i.e.,  $\alpha_n \in \mathbb{D}$ ) and

$$\|\Phi_n\| = \rho_{n-1} \cdots \rho_0$$

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# Szegő recursion and Verblunsky coefficients

$$\begin{pmatrix} \varphi_{n+1} \\ \varphi_{n+1}^* \end{pmatrix} = A_n(z) \begin{pmatrix} \varphi_n \\ \varphi_n^* \end{pmatrix} x; \quad A_n = \rho_n^{-1} \begin{pmatrix} z & -\bar{\alpha}_n \\ -\alpha_n z & 1 \end{pmatrix}$$

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# Verblunsky's Theorem

There is a one-one correspondence, called the *Verblunsky map*, from measures of infinite support and sequences in  $\mathbb{D}$ .

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$$\alpha_j = \beta_j; \quad j = 0, \dots, n-1 \text{ and } \alpha_j = 0; \quad j \geq n.$$

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# Verblunsky's Theorem

For measures with exactly  $n$  pure points, there are only  $n$  non-trivial OPs, and  $n$   $\alpha$ 's.  $\alpha_{n-1} \in \partial\mathbb{D}$ . One has  $\|\Phi_k\| = \rho_0 \dots \rho_{k-1}$  where  $\rho_j = \sqrt{1 - |\alpha_j|^2}$

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# Szegő's Theorem: Toeplitz version

Szegő's Theorem concerns probability measures on  $\partial\mathbb{D}$  of the form

$$d\mu(\theta) = w(\theta) \frac{d\theta}{2\pi} + d\mu_s(\theta)$$

where  $d\mu_s$  is singular w.r.t.  $d\theta$ .

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where  $d\mu_s$  is singular w.r.t.  $d\theta$ . The Toeplitz determinant  $D_n(d\mu)$  is the  $n \times n$  determinant with

$$c_{k\ell} \equiv \int e^{i(k-\ell)\theta} d\mu(\theta) = \langle e^{-ik\cdot}, e^{-i\ell\cdot} \rangle_{L^2(d\mu)}$$

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In 1915, Szegő proved that

$$\lim_{n \rightarrow \infty} D_n(d\mu)^{1/n} = \exp \left[ \int \log(w(\theta)) \frac{d\theta}{2\pi} \right]$$

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While this is true in general, Szegő only proved it when  $d\mu_s = 0$ .

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# Szegő's Theorem: OPUC version

In 1920, Szegő realized that, because a Toeplitz matrix is just the Gram matrix of  $\{z^j\}_{j=0}^{n-1}$ , it is also the Gram matrix of  $\{\Phi_j\}_{j=0}^{n-1}$  which is diagonal so

$$D_n = \prod_{j=0}^{n-1} \|\Phi_j\|^2$$

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so using that  $\|\Phi_j\|$  is monotone decreasing (by a variational argument), one has an equivalent form of his theorem, namely

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But the recursion relation was only published by Szegő in 1939, so he didn't have a form in terms of  $\alpha_n$  and  $\rho_n$ .

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# Szegő's Theorem: Szegő-Verblunsky sum rule

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$$\sum_{j=0}^{\infty} \log(1 - |\alpha_j|^2) = \int \log(w(\theta)) \frac{d\theta}{2\pi}$$

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It is critical that this always holds although both sides may be  $-\infty$ .

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It is critical that this always holds although both sides may be  $-\infty$ . This implies what I've called a "spectral theory gem"

$$\sum_{j=0}^{\infty} |\alpha_j|^2 < \infty \iff \int \log(w(\theta)) \frac{d\theta}{2\pi} > -\infty$$

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In particular,  $\sum_{j=0}^{\infty} |\alpha_j|^2 < \infty \Rightarrow \Sigma_{ac} = \partial\mathbb{D}$ .

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# Szegő's Theorem: Szegő-Verblunsky sum rule

What makes the gems so interesting is that they allow arbitrary singular parts of the measures so long as the Szegő condition holds, i.e.  $\int \log(w(\theta)) \frac{d\theta}{2\pi} > -\infty$ .

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In the late 1990's unaware of the OPUC literature, my research group was studying  $1D$  Schrodinger operators,  $-\frac{d^2}{dx^2} + V(x)$  and the difference between  $L^1$  and  $L^2$  conditions.

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# Szegő's Theorem: Szegő-Verblunsky sum rule

What makes the gems so interesting is that they allow arbitrary singular parts of the measures so long as the Szegő condition holds, i.e.  $\int \log(w(\theta)) \frac{d\theta}{2\pi} > -\infty$ . If  $\sum_{j=0}^{\infty} |\alpha_j| < \infty$ , one can show that there is a scattering theory and strong asymptotic completeness holds in that there is only a.c. spectrum. The VS sum rules implies in going from  $\ell^1$  to  $\ell^2$  Verblunsky coefficients, one can have arbitrary mixed spectral types.

In the late 1990's unaware of the OPUC literature, my research group was studying  $1D$  Schrodinger operators,  $-\frac{d^2}{dx^2} + V(x)$  and the difference between  $L^1$  and  $L^2$  conditions. Deift-Killip had proven there was a.c. spectrum for  $L^2$  and showing there were examples with mixed spectrum was one of the problems in my list at the 2000 ICMP.

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# Szegő Condition

Here is one version of Szegő's Theorem for OPRL.

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# Szegő Condition

Here is one version of Szegő's Theorem for OPRL. The map  $z \mapsto z + z^{-1}$  maps  $\partial\mathbb{D}$  to  $[-2, 2]$  (via  $e^{i\theta} \mapsto 2 \cos \theta$ )

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$$\liminf_{n \rightarrow \infty} \prod_{j=1}^n a_j = \sqrt{2} \exp \left( \int_{-2}^2 \log |\pi s(x) w(x)| s(x) \frac{dx}{4\pi} \right)$$

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The condition for the finiteness of the integral is called the *Szegő condition*:

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$$\int_{-2}^2 \log |w(x)| (4 - x^2)^{-1/2} dx > -\infty$$

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This doesn't yield a gem because

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only holds under the a priori condition that  $\mu$  is supported inside  $[-2, 2]$  and this is not simply expressible in terms of the Jacobi parameters; for example, it doesn't only depend on the parameters near  $\infty$  and can be changed by modifying a single  $a$  or  $b$ .

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# Killip–Simon Theorem

In 2001 (published 2003), Killip and I proved the following gem which we regard as an OPRL analog of the Verblunsky–Szegő gem where  $\{E_j^\pm\}_{j=1}^{N_\pm}$  are the eigenvalues outside  $[-2, 2]$  (with  $+$  above 2 and  $-$  below  $-2$ ):

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**Killip–Simon Theorem** If  $d\mu = w(x)dx + d\mu_s$  is a measure of compact support on  $\mathbb{R}$  and  $\{a_n, b_n\}_{n=1}^\infty$  its Jacobi parameters, then

$$\sum_{j=1}^{\infty} |a_j - 1|^2 + b_j^2 < \infty$$

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**Killip–Simon Theorem** If  $d\mu = w(x)dx + d\mu_s$  is a measure of compact support on  $\mathbb{R}$  and  $\{a_n, b_n\}_{n=1}^\infty$  its Jacobi parameters, then

$$\sum_{j=1}^{\infty} |a_j - 1|^2 + b_j^2 < \infty$$

if and only if the essential support of  $\mu$  is  $[-2, 2]$  and

$$\int_{-2}^2 \log(w(x)) \sqrt{4 - x^2} dx > -\infty \quad \sum_{j,\pm} (|E_j^\pm| - 2)^{3/2} < \infty$$

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# Killip–Simon Theorem

This result on Jacobi Hilbert-Schmidt perturbations of the free Jacobi matrix should be compared with a celebrated theorem of von-Neumann that any bounded self-adjoint operator has a Hilbert-Schmidt perturbation with only dense point spectrum!

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We called  $\int_{-2}^2 \log(w(x))\sqrt{4-x^2} dx > -\infty$  *the quasi-Szegő condition* since the square root appeared to the  $+1/2$  power rather than the  $-1/2$  in the Szegő condition.

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$$\sum |E_n|^p \leq C \int |V(x)|^{p+d/2} dx$$

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# Killip–Simon Sum Rule

The gem comes from a sum rule. Let

$$Q(\mu) = \frac{1}{2\pi} \int_0^{2\pi} \log \left( \frac{\sin(\theta)}{\operatorname{Im} m(2 \cos(\theta))} \right) \sin^2(\theta) d\theta,$$

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$$F(E) \equiv \frac{1}{4} [\beta^2 - \beta^{-2} - \log(\beta^4)] \quad E = \beta + \beta^{-1} \quad |\beta| > 1$$

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Then the Killip–Simon sum rule says

$$Q(\mu) + \sum_{j,\pm} F(E_j^\pm) = \sum_{n=1}^{\infty} \frac{1}{4} b_n^2 + \frac{1}{2} G(a_n)$$

As with the Szegő–Verblunsky sum rule, an important point is that it always holds although both sides may be  $+\infty$ .

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# Killip-Simon Sum Rule

The gem comes from the fact that  $F \geq 0$ , vanishes exactly at  $E = \pm 2$  and is  $O((|E| - 2)^{3/2})$  there and that  $G \geq 0$ , vanishes exactly at  $a = 1$  and is  $O((a - 1)^2)$  there.

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The positivity of the terms is essential to be sure that there aren't cancelations.

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As in the OPUC case, this sum rule implies the existence of Hilbert-Schmidt perturbations with mixed spectrum.

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Barry Simon

$xy \leq \frac{x^p}{p} + \frac{y^q}{q}$

$\hat{f}(\mathbf{k}) = (2\pi)^{-v/2} \int \exp(-i\mathbf{k} \cdot \mathbf{x}) f(\mathbf{x}) d^v x$

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Simon

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**Basic Complex Analysis**  
A Comprehensive Course in Analysis, Part 2A

Barry Simon

**ANALYSIS**  
Part 2A

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$$f(z_0) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z - z_0} dz$$

*A Comprehensive Course in Analysis* by Poincaré Prize winner Barry Simon is a five-volume set that can serve as a graduate-level analysis textbook with a lot of additional bonus information, including hundreds of problems and numerous notes that extend the text and provide important historical background. Depth and breadth of exposition make this set a valuable reference source for almost all areas of classical analysis.

Part 2A is devoted to basic complex analysis. It interweaves three analytic threads associated with Cauchy, Riemann, and Weierstrass, respectively. Cauchy's view focuses on the differential and integral calculus of functions of a complex variable, with the key topics being the Cauchy integral formula and contour integration. For Riemann, the geometry of the complex plane is central, with key topics being fractional linear transformations and conformal mapping. For Weierstrass, the power series is king, with key topics being spaces of analytic functions, the product formulas of Weierstrass and Hadamard, and the Weierstrass theory of elliptic functions. Subjects in this volume that are often missing in other texts include the Cauchy integral theorem when the contour is the boundary of a Jordan region, continued fractions, two proofs of the big Picard theorem, the uniformization theorem, Ahlfors's function, the sheaf of analytic germs, and Jacobi, as well as Weierstrass, elliptic functions.

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# And Now a Word from Our Sponsor

What is spectral theory?

OPs

OPRL basics

Favard's Theorem

m-Function Expansion

OPUC basics

Szegő recursion and Verblunsky coefficients

Szegő-Verblunsky Sum Rule

Szegő for OPRL

Killip-Simon Sum Rule

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**Advanced Complex Analysis**

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## Advanced Complex Analysis

A Comprehensive Course in Analysis, Part 2B

Barry Simon

$$\frac{\pi(x)}{(x/\log x)} \rightarrow 1$$

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + o(x^{-1/2})$$

---

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Part 2B provides a comprehensive look at a number of subjects of complex analysis not included in Part 2A. Presented in this volume are the theory of conformal metrics (including the Poincaré metric, the Ahlfors-Robinson proof of Picard's theorem, and Bell's proof of the Painlevé smoothness theorem), topics in analytic number theory (including Jacob's two- and four-square theorems, the Dirichlet prime progression theorem, the prime number theorem, and the Hardy-Littlewood asymptotics for the number of partitions), the theory of Fuchsian differential equations, asymptotic methods (including Euler's method, stationary phase, the saddle-point method, and the WKB method), univalent functions (including an introduction to SLE), and Nevanlinna theory. The chapters on Fuchsian differential equations and on asymptotic methods can be viewed as a minicourse on the theory of special functions.

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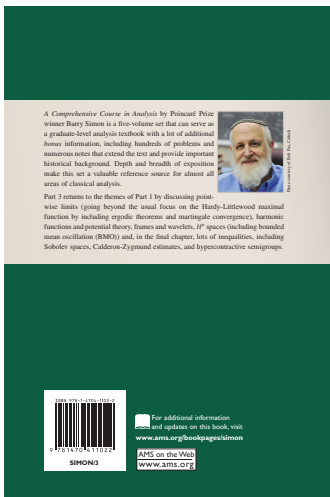
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Harmonic Analysis

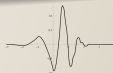
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Harmonic Analysis  
A Comprehensive Course in Analysis, Part 3

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$$\|f - f_Q\|_Q = \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx$$

$$|\{x \mid M_{H^1} f(x) > \alpha\}| \leq \frac{3^n}{\alpha} \|f\|_{L^1(\mathbb{R}^n)}$$



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**Operator Theory**

**Operator Theory**  
A Comprehensive Course in Analysis, Part 4

Barry Simon

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**Part 4**

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*A Comprehensive Course in Analysis* by Poincaré Prize winner Barry Simon is a five-volume set that can serve as a graduate-level analysis textbook with a lot of additional bonus information, including hundreds of problems and numerous notes that extend the text and provide important historical background. Depth and breadth of exposition make this set a valuable reference source for almost all areas of classical analysis.

Part 4 focuses on operator theory, especially on a Hilbert space. Central topics are the spectral theorem, the theory of trace class and Fredholm determinants, and the study of unbounded self-adjoint operators. There is also an introduction to the theory of orthogonal polynomials and a long chapter on Banach algebras, including the commutative and non-commutative Gelfand-Naimark theorems and Fourier analysis on general locally compact abelian groups.

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$$A = \int t dE_t$$

$$\det(1 + zA) = \prod_{k=1}^{N(A)} (1 + z\lambda_k(A))$$

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