



# Large Deviations and Sum Rules for Orthogonal Polynomials

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Lecture 2: Meromorphic Herglotz Functions and Proof of Killip  
Simon Sum Rule

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reprise

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- Lecture 1: OPRL, OPUC and Sum Rules
- Lecture 2: Meromorphic Herglotz Functions and Proof of KS Sum Rule
- Lecture 3: The Theory of Large Deviations
- Lecture 4: GNR Proof of Sum Rules



# References

[OPUC2] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, AMS Colloquium Series, **54.2**, American Mathematical Society, Providence, RI, 2005.

[SzThm] B. Simon, *Szegő's Theorem and Its Descendants: Spectral Theory for  $L^2$  Perturbations of Orthogonal Polynomials*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2011.

[CA] B. Simon, *A Comprehensive Course in Analysis, Part 2A, Basic Complex Analysis*, American Mathematical Society, Providence, R.I., 2015.

[HA] B. Simon, *A Comprehensive Course in Analysis, Part 3, Harmonic Analysis*, American Mathematical Society, Providence, R.I., 2015.

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# Killip–Simon functions

Given a measure  $d\mu = w(x) dx + d\mu_s$  with pure points  $\{E_j^\pm\}_{j=1}^{N^\pm}$  outside  $[-2, 2]$  (with  $+$  above 2 and  $-$  below -2) one defines (with  $m(z) = \int (x - z)^{-1} d\mu(x)$  and  $m(x) \equiv \lim_{y \downarrow 0} m(x + iy)$ ).

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$$Q(\mu) = \frac{1}{2\pi} \int_0^{2\pi} \log \left( \frac{\sin(\theta)}{\operatorname{Im} m(2 \cos(\theta))} \right) \sin^2(\theta) d\theta$$

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$$G(a) = a^2 - 1 - \log(a^2) \text{ and}$$

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$$G(a) = a^2 - 1 - \log(a^2) \text{ and}$$

$$F(E) \equiv \frac{1}{4} [\beta^2 - \beta^{-2} - \log(\beta^4)] \quad E = \beta + \beta^{-1} \quad |\beta| > 1$$

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The gem will come from the fact that  $F \geq 0$ , vanishes exactly at  $E = \pm 2$  and is  $O((|E| - 2)^{3/2})$  there and that  $G \geq 0$ , vanishes exactly at  $a = 1$  and is  $O((a - 1)^2)$  there.

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# Killip–Simon sum rule

The Killip–Simon rule says that

$$Q(\mu) + \sum_{j,\pm} F(E_j^\pm) = \sum_{n=1}^{\infty} \frac{1}{4} b_n^2 + \frac{1}{2} G(a_n)$$

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An important point is that it always holds although both sides may be  $+\infty$ .

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An important point is that it always holds although both sides may be  $+\infty$ . Our main goal in Lecture 2 will be to sketch a variant of the original proof of this sum rule. I know of many proofs of Szegő's Theorem but until recently all proofs of the Killip–Simon sum rule were variants of this proof. We'll need a Poisson-Jensen formula for certain meromorphic functions, so I start by recalling the classical PJ formula in the subtle form found by Smirnov and Beurling.

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# Nevanlinna Class and Blaschke Products

I'll remind you of the classical results without proofs which you can find in my *Basic Complex Analysis*, Sections 9.8 and 9.9 and *Harmonic Analysis*, Sections 5.3, 5.6 and 5.7.

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Let  $f$  be an analytic function on the unit disk,  $\mathbb{D}$ . We say that  $f \in N$ , the Nevanlinna class, if and only if,

$$\sup_{0 < r < 1} \int_0^{2\pi} \log_+ |f(re^{i\theta})| d\theta < \infty.$$

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$$f \in N \Rightarrow \sum (1 - |z_j|) < \infty$$

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and this implies that the Blaschke product  $B(z) = \prod_{j=1}^N b(z, z_j)$  converges absolutely on the unit disk to an analytic function vanishing precisely at the  $z_j$ .

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and this implies that the Blaschke product  $B(z) = \prod_{j=1}^N b(z, z_j)$  converges absolutely on the unit disk to an analytic function vanishing precisely at the  $z_j$ . Here:

$$b(z, w) = \begin{cases} z, & \text{if } w = 0 \\ -\frac{|w|(z-w)}{w(1-\bar{w}z)} & \text{if } w \neq 0 \end{cases}$$

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# Hardy Class and Smirnov–Beurling Representation

For  $0 < p < \infty$ , the Hardy class,  $H^p$ , is the set of  $f$  analytic on  $\mathbb{D}$  with  $\sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty$  and  $H^\infty$  is the bounded analytic functions.

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For  $p \geq 1$ ,  $h^p$  is defined like  $H^p$  but its elements,  $u$ , are real-valued harmonic, rather than analytic functions.

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For  $p \geq 1$ ,  $h^p$  is defined like  $H^p$  but its elements,  $u$ , are real-valued harmonic, rather than analytic functions. If  $u \in h^1$ , then the measure  $u(re^{i\theta})d\theta/2\pi$  has a weak-\* limit  $d\mu$

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# Hardy Class and Smirnov–Beurling Representation

Moreover, the a.e. pointwise limit  $u^*$  exists and  $u^*(e^{i\theta})d\theta/2\pi$  is the a.c. part of  $d\mu$

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# Meromorphic Herglotz Functions

By a *meromorphic Herglotz* function, we mean a function meromorphic on  $\mathbb{D}$ , real on  $(-1, 1)$  with  $\operatorname{Im} z > 0 \Rightarrow \operatorname{Im} f(z) > 0$ .

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One can prove that in  $\mathbb{D} \cap \mathbb{C}_+$ , one has that  $|\arg z B(z)| \leq 2\pi$  so that  $\arg(f(z)/zB(z))$  is bounded on  $\mathbb{D}$ .

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Taking log's, one gets relations between Taylor coefficients of  $\log(f(z)/z)$ , certain sums involving logs or powers of zeros and poles and integrals  $\cos(n\theta) \log |f(e^{i\theta})|$ .

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# Case Step-by-Step Sum Rules

Recall that  $m(z) = \int d\mu(x)/(x - z)$ . It defines a Herglotz function on  $\mathbb{C}_+$ , real on  $\mathbb{R}$ .

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The above procedure thus yields a relation between polynomials of Jacobi parameters, the difference of functions of the eigenvalues of  $J$  and  $J_1$  and integral of  $\log |M(e^{i\theta})|$ .

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# $P_2$ Sum Rule

What results is a step-by-step sum rule which if iterated with boundary terms dropped yields the formal sum rules stated by Case

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# Step-by-Step $P_2$ Sum Rule

The explicit result is:

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# Step-by-Step $P_2$ Sum Rule

The explicit result is:

$$\frac{1}{4}b_1^2 + \frac{1}{2}G(a_1) = Q(J|J_1) + \sum_{j,\pm} \left[ F(E_j^\pm(J)) - F(E_j^\pm(J_1)) \right]$$

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By interlacing, the sum of  $F$  terms is always convergent. And one can prove that the integral defining  $Q$  is always convergent.

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$$\sum_{j=1}^n \frac{1}{4}b_j^2 + \frac{1}{2}G(a_j) = Q(J|J_n) + \sum_{j,\pm} \left[ F(E_j^\pm(J)) - F(E_j^\pm(J_n)) \right]$$

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# Upper Bound on Jacobi Sum

We'll prove the sum rule by proving two inequalities. First that

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If either  $Q(J)$  or  $\sum_{j, \pm} F(E_j^{\pm}(J))$  is infinite, then there is nothing to prove.

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# Lower Semicontinuity of the KL Divergence

Given a pair of probability measures,  $\mu$  and  $\nu$  on the same space, one defines their *Kullback–Leibler (KL) divergence* by

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$$H(\nu | \mu) = \begin{cases} \int \log \left( \frac{d\nu}{d\mu} \right) d\nu, & \text{if } \nu \text{ is } \mu\text{-a.c.} \\ \infty, & \text{otherwise.} \end{cases}$$

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One has  $H(\nu | \mu) \geq 0$  with equality only if  $\mu = \nu$ .

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Notice that the OPUC Szegő integral is precisely  $-H\left(\frac{d\theta}{2\pi} | \mu\right)$

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Notice that the OPUC Szegő integral is precisely  $-H\left(\frac{d\theta}{2\pi} | \mu\right)$  and what we called  $Q(\mu)$  in the KS sum rule is precisely  $H(\nu | \mu)$  where

$$d\nu(x) = (2\pi)^{-1}(4 - x^2)^{1/2}\chi_{[-2,2]}(x)dx.$$

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# Lower Semicontinuity of the KL Divergence

An important property of the KL divergence is lower semicontinuity.

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An important property of the KL divergence is lower semicontinuity. One proves the following variational principle

$$H(\nu | \mu) = \sup_f \left( - \int f d\mu(x) + \int [1 + \log(f(x))] d\nu(x) \right)$$

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$$H(\nu | \mu) = \sup_f \left( - \int f d\mu(x) + \int [1 + \log(f(x))] d\nu(x) \right)$$

where the sup is taken over all strictly positive continuous functions.

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$$H(\nu | \mu) = \sup_f \left( - \int f d\mu(x) + \int [1 + \log(f(x))] d\nu(x) \right)$$

where the sup is taken over all strictly positive continuous functions. If  $d\nu = g d\mu$  with  $g$  continuous and strictly positive, then the quantity in the sup when  $f = g$  is  $H$  and Jensen's inequality implies the sup is always great than  $H$ .

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**$H(\nu | \mu)$  is jointly convex and jointly lower semicontinuous in  $\mu$  and  $\nu$**

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# Lower Bound on Jacobi Sum

We defined  $J_n$  as what one gets by stripping off the first  $n$  Jacobi pairs, i.e. subtracting from the left. Complementary is  $J^{(n)}$  which builds up by adding on the right, i.e. it has Jacobi parameters:

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$$a_k^{(n)}, b_k^{(n)} = \begin{cases} a_k, & b_k, & \text{if } k = 1, \dots, n \\ 0, & 1, & \text{if } k \geq n \end{cases}$$

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The point is that  $[J^{(n)}]_n$  is the free Jacobi matrix, which has no eigenvalues outside  $[-2, 2]$ , no non-trivial Jacobi parameters and the the free  $m$ -function. By looking at the  $n$ -times iterated sum rule for  $J^{(n)}$ , we find that

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$$\sum_{j=1}^n \left[ \frac{1}{4} b_j^2 + \frac{1}{2} G(a_j) \right] = Q(J^{(n)}) + \sum_{j, \pm} F(E_j^{\pm}(J^{(n)}))$$

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# Lower Bound on Jacobi Sum

We get a lower bound on these equal term by replacing the full eigenvalue sum which might have more and more terms as  $n$  increases by the sum for  $j = 1, \dots, K$  for  $K$  fixed.

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This complements the upper bound and proves the full KS sum rule.

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This complements the upper bound and proves the full KS sum rule. Note the importance of the positivity of  $F$  and  $G$  in proving the lower bound.

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# Mysteries

While the gem one gets from the  $P_2$  sum rule is simple and elegant, the proof has lots of mysteries:

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# Mysteries

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- 1 Why are there any positive combinations?

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While the gem one gets from the  $P_2$  sum rule is simple and elegant, the proof has lots of mysteries:

- 1 Why are there any positive combinations?
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- 4 What does the function

$$F(E) = \frac{1}{4}[\beta^2 - \beta^{-2} - \log \beta^4]; \quad E = \beta + \beta^{-1}$$

mean?

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Real Analysis

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$xy \leq \frac{x^p}{p} + \frac{y^q}{q}$

$\hat{f}(\mathbf{k}) = (2\pi)^{-n/2} \int \exp(-i\mathbf{k} \cdot \mathbf{x}) f(\mathbf{x}) d^n x$

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 **Basic Complex Analysis**

**Barry Simon**

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*A Comprehensive Course in Analysis* by Poincaré Prize winner Barry Simon is a five-volume set that can serve as a graduate-level analysis textbook with a lot of additional bonus information, including hundreds of problems and numerous notes that extend the text and provide important historical background. Depth and breadth of exposition make this set a valuable reference source for almost all areas of classical analysis.

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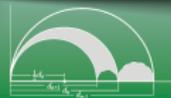
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$$f(z_0) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z - z_0} dz$$



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**Advanced Complex Analysis**

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## Advanced Complex Analysis

A Comprehensive Course in Analysis, Part 2B

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$$\frac{\pi(x)}{(x/\log x)} \rightarrow 1$$

$$J_u(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + o(x^{-1/2})$$

A *Comprehensive Course in Analysis* by Poincaré Prize winner Barry Simon is a five-volume set that can serve as a graduate-level analysis textbook with a lot of additional bonus information, including hundreds of problems and numerous notes that extend the text and provide important historical background. Depth and breadth of exposition make this set a valuable reference source for almost all areas of classical analysis.

Part 2B provides a comprehensive look at a number of subjects of complex analysis not included in Part 2A. Presented in this volume are the theory of conformal metrics (including the Poincaré metric, the Ahlfors–Robinson proof of Picard’s theorem, and Bell’s proof of the Painlevé smoothness theorem), topics in analytic number theory (including Jacob’s two- and four-square theorems, the Dirichlet prime progression theorem, the prime number theorem, and the Hardy–Littlewood asymptotics for the number of partitions), the theory of Fuchsian differential equations, asymptotic methods (including Euler’s method, stationary phase, the saddle-point method, and the WKB method), univalent functions (including an introduction to SLE), and Nevanlinna theory. The chapters on Fuchsian differential equations and on asymptotic methods can be viewed as a minicourse on the theory of special functions.

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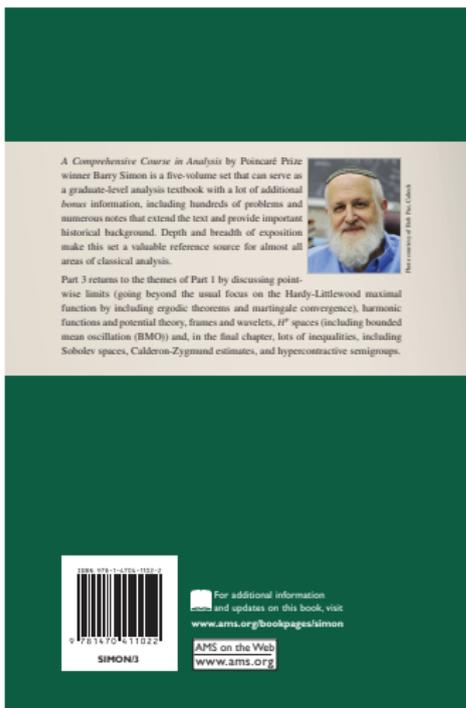
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Harmonic Analysis

ANALYSIS

Part 3

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Harmonic Analysis  
A Comprehensive Course in Analysis, Part 3

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$$\|f - f_Q\|_Q = \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx$$

$$\{|x| \mid M_{H^1} f(x) > \alpha\} \leq \frac{3^n}{\alpha} \|f\|_{L^1(\mathbb{R}^n, dx)}$$



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A Comprehensive Course in Analysis, Part 4

Barry Simon

$A = \int t dE_t$

$$\det(1 + zA) = \prod_{k=1}^{N(A)} (1 + z\lambda_k(A))$$

ANALYSIS  
Part 4

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*A Comprehensive Course in Analysis by Poincaré Prize winner Barry Simon is a five-volume set that can serve as a graduate-level analysis textbook with a lot of additional bonus information, including hundreds of problems and numerous notes that extend the text and provide important historical background. Depth and breadth of exposition make this set a valuable reference source for almost all areas of classical analysis.*

Part 4 focuses on operator theory, especially on a Hilbert space. Central topics are the spectral theorem, the theory of trace class and Fredholm determinants, and the study of unbounded self-adjoint operators. There is also an introduction to the theory of orthogonal polynomials and a long chapter on Banach algebras, including the commutative and non-commutative Gelfand–Naimark theorems and Fourier analysis on general locally compact abelian groups.

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