Large Deviations and Sum Rules for Orthogonal Polynomials

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Lecture 2: Meromorphic Herglotz Functions and Proof of Killip Simon Sum Rule

- Lecture 1: OPRL, OPUC and Sum Rules
- Lecture 2: Meromorphic Herglotz Functions and Proof of KS Sum Rule
- Lecture 3: The Theory of Large Deviations
- Lecture 4: GNR Proof of Sum Rules

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Killip–Simon functions

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The gem will come from the fact that $F \geq 0$, vanishes exactly at $E = \pm 2$ and is $\mathrm{O}((|E|-2)^{3/2})$ there and that $G \geq 0$, vanishes exactly at $a = 1$ and is $\mathrm{O}((a-1)^2)$ there.

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$$b(z,w) = \begin{cases} z, & \text{if } w = 0 \\ -\frac{|w|(z-w)}{w(1-\bar{w}z)} & \text{if } w \neq 0 \end{cases}$$

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For $0 < p < \infty$, the Hardy class, H^p, is the set of f analytic on $\mathbb{D}$ with $\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty$ and H^∞ is the bounded analytic functions.

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Taking $\log$'s, one gets relations between Taylor coefficients of $\log(f(z)/z)$, certain sums involving $\log$s or powers of zeros and poles and integrals $\cos(n\theta) \log|f(e^{i\theta})|$.

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Notice that the OPUC Szegő integral is precisely $-H(\frac{d\theta}{2\pi} \,|\, \mu)$ and what we called $Q(\mu)$ in the KS sum rule is precisely $H(\nu \,|\, \mu)$ where $d\nu(x) = (2\pi)^{-1}(4-x^2)^{1/2}\chi_{[-2,2]}(x)dx$.

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The point is that $\left[J^{(n)}\right]_n$ is the free Jacobi matrix, which has no eigenvalues outside $[-2, 2]$, no non–trivial Jacobi parameters and the the free m–function. By looking at the n–times iterated sum rule for $J^{(n)}$, we find that

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The point is that $\left[J^{(n)}\right]_n$ is the free Jacobi matrix, which has no eigenvalues outside $[-2, 2]$, no non–trivial Jacobi parameters and the the free m–function. By looking at the n–times iterated sum rule for $J^{(n)}$, we find that

$$\sum_{j=1}^{n} \left[\tfrac{1}{4} b_j^2 + \tfrac{1}{2} G(a_j)\right] = Q(J^{(n)}) + \sum_{j,\pm} F(E_j^{\pm}(J^{(n)}))$$

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4. What does the function
$$F(E)=\tfrac{1}{4}[\beta^2-\beta^{-2}-\log\beta^4];\quad E=\beta+\beta^{-1}$$
mean?

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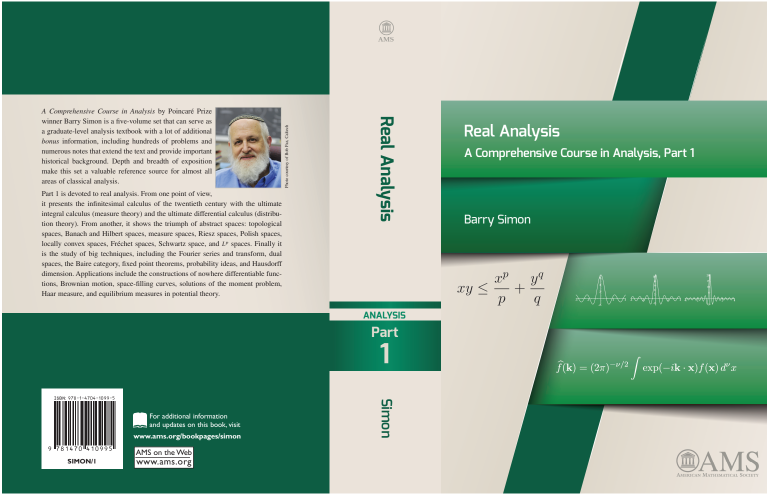

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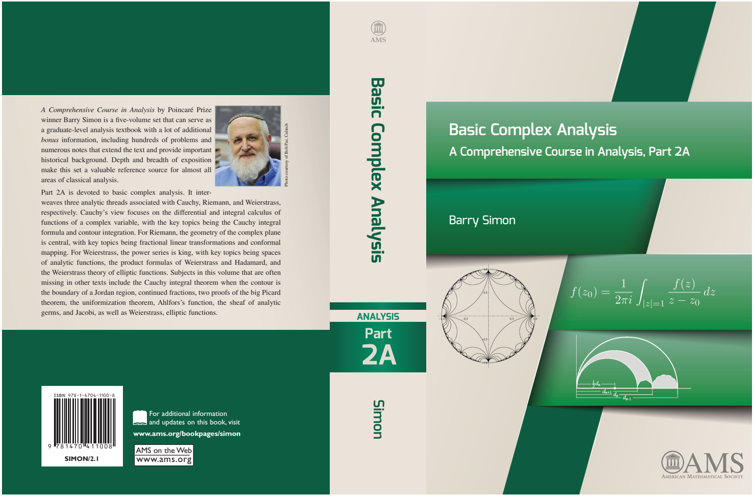

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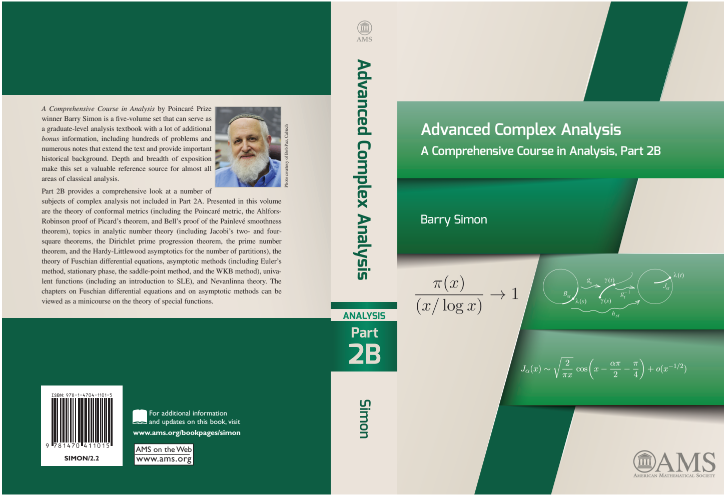

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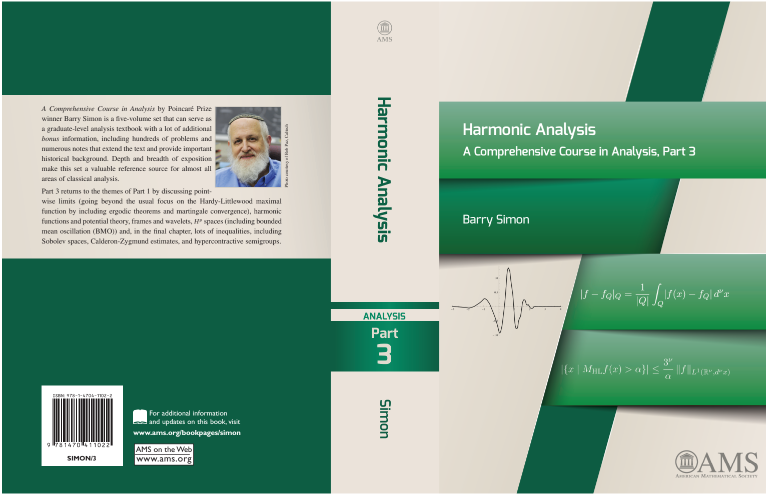

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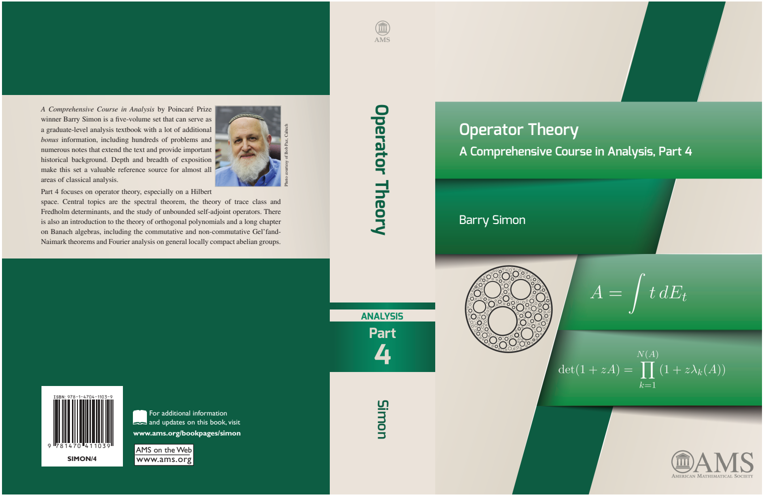

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