Large Deviations and Sum Rules for Orthogonal Polynomials

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Lecture 3: The Theory of Large Deviations

Rate Functions
Cramér's Theorem
Contraction Principle
Projective Limits
Killip–Nenciu Theorem

- Lecture 1: OPRL, OPUC and Sum Rules
- Lecture 2: Meromorphic Herglotz Functions and Proof of KS Sum Rule
- Lecture 3: The Theory of Large Deviations
- Lecture 4: GNR Proof of Sum Rules

References

[DS] J. Deuschel and D. Stroock, *Large Deviations*, Academic Press, Boston, 1989.

[DZ] A. Dembo and O. Zeitouni *Large Deviations and Applications*, Springer, Berlin, 1998.

[KN] R. Killip and I. Nenciu, *Matrix models for circular ensembles*, Int. Math. Res. Not., **50** (2004), 2665–2701.

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2. The proofs are straight–forward.

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We start with the proof of the upper bound on probabilities.

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where the independence of the X_i's was used in the last equality. Choosing $\lambda = \lambda_x$ completes the proof of the large deviations upper bound.

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$$\begin{aligned}\mathbb{E}_\nu(X_1) &= \int y e^{\lambda_x y-\Lambda(\lambda_x)}\,d\mathbb{P}_1(y)\\ &= e^{-\Lambda(\lambda_x)}\frac{d}{d\lambda}\,e^{\Lambda(\lambda)}\Big|_{\lambda=\lambda_x}\\ &= \Lambda'(\lambda_x) = x\end{aligned}$$

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Clearly, $\Lambda(\lambda) = \infty$ for $\lambda \neq 0$ and $\Lambda(0) = 0$. For any x, the $\sup$ of $\lambda x - \Lambda(\lambda)$ occurs at $\lambda = 0$ so the Legendre transform is $\equiv 0$ showing the above calculation is consistent with Cramér's theorem.

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Example 3 (Exponential distribution) Relevant to our considerations later is the average of exponential random variables. So let $\{X_j\}_{j=1}^{\infty}$ be independent, identically distributed random variables (iidrv) with density $\chi_{[0,\infty)}(x)e^{-x}dx$. The cumulant generating function is

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For exponential iidrv, $n^{-1}\sum_{j=1}^n X_n$ has distribution $G_{n-1,n}$, so this example allows one to also read off a LDP for suitable gamma distributions.

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Remark This goes beyond the direct use of Cramér in two ways. First, we note that if real valued Z_N have a LDP with speed N and rate I, then αZ_N has a LDP with speed αN and rate $\alpha I(\cdot/\alpha)$ by a trivial calculation. Secondly, if $\alpha_N = \ell_N/N$ and $\alpha_N \to 1$, then $\alpha_N^{-1} Y_N$ has a LDP with speed N and rate I if Y_N does and the rate function is continuous.

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Then the family of measures on Y defined by $\mathbb{P}_N^{(f)}(A) = \mathbb{P}_N(f^{-1}[A])$ obeys a LDP with speed a_N and good rate function $I^{(f)}$.

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so the LDP bounds for $\mathbb{P}_N$ carry over to such bounds for $\mathbb{P}_N^{(f)}$.

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Then I is a good rate function and $\{\mathbb{P}_N\}_{N=1}^{\infty}$ obeys a LDP with speed a_N and rate function I.

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which shows that $I_j(\pi_j(x))$ is monotone in j so the $\sup$ in the formula for I is a limit.

Some Examples

Example $[\mathbb{R}^\infty]$ Take $X_j = \mathbb{R}^j$,
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Example [$\mathcal{M}_{+,1}(\partial\mathbb{D})$] Let $\mathbb{P}$ be a measure on $\mathcal{M}_{+,1}(\partial\mathbb{D})$, the probability measures on the unit circle. Given $\mu \in \mathcal{M}_{+,1}(\partial\mathbb{D})$ and $j = 1, 2, \dots$, let $\pi_j(\mu)$ be the point in $\mathbb{R}^{2^j}$ with coordinates $\mu(I_k^{(j)}), k = 1, \dots, 2^j$ where $I_k^{(j)} = \{e^{2\pi i\theta} \mid \frac{k-1}{2^j} \leq \theta < \frac{k}{2^j}\}$.

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$$\pi_{j+1,j}(y)_\ell = y_{2\ell-1} + y_\ell \qquad \ell = 1, \dots, 2^j$$

Thus, to get a LDP for $\mathcal{M}_{+,1}(\partial\mathbb{D})$, we need only prove 2^j–dimensional LDPs.

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Killip–Nenciu Theorem The measure induced by Haar measure on $\mathbb{U}(n)$ on the Verblunsky coefficients $\alpha_0, \dots, \alpha_{n-2} \in \mathbb{D}, \alpha_{n-1} \in \partial\mathbb{D}$ is given given by

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$$d\widetilde{\mathbb{P}}_n(\alpha_0, \ldots, \alpha_{n-1}) = \prod_{j=0}^{n-1} d\kappa_{n-2-j}(\alpha_j)$$

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GGT is a name I introduced in OPUC1 after Geronimus (who had it first in his work on OPUC), Gragg (who rediscovered it in work in numerical linear algebra) and Teplyaev (who used it in his study of Anderson localization for OPUC).

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Let $\sigma : G/H \to G$ be a choice of representative from each coset, i.e. $\pi(\sigma(x)) = x$. Then $\Sigma : G/H \times H \to G$, defined by $\Sigma(x, h) = \sigma(x)h$, is a bijection.

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so $U\Sigma(x, W) = \Sigma(Ux, W_{U,x}W)$ which, given the invariance of $\mu_{G/H}$ under the action of G and of μ_H under left multiplication by elements of H, implies the image of the product measure is invariant under multiplication by U (by integrating first over W and then x).

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Returning to $\mathbb{U}(n)$, fix a unit vector $e_1 \in \mathbb{C}_1^n$ (it may be helpful to think of e_1 as the first vector, $\delta_1 = (1, 0, \ldots, 0)$, of the canonical basis of $\mathbb{C}^n$).

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$$\mathbf{1} - 2\langle e_1 - f, \cdot\rangle(e_1 - f)/\|e_1 - f\|^2$$

To find its matrix form in the e_1, e_2 basis, we note that its first column must be $\begin{pmatrix} \bar{\beta} \\ \kappa \end{pmatrix}$ since it takes e_1 to f. Its second column is then determined by orthonormality and the desire to have determinant -1 (i.e. a reflection). Thus

$$\Theta(\beta) = \begin{pmatrix} \bar{\beta} & \kappa \\ \kappa & -\beta \end{pmatrix}$$

We define the *Householder reflection*, $\sigma(f)$, on $\mathbb{C}^n$ to be $\Theta(\beta) \oplus \mathbf{1}_{n-2}$ under $\mathbb{C}^n = \mathcal{H}_f \oplus \mathcal{H}_f^\perp$ (where $\mathbf{1}_k$ is the size k identity matrix).

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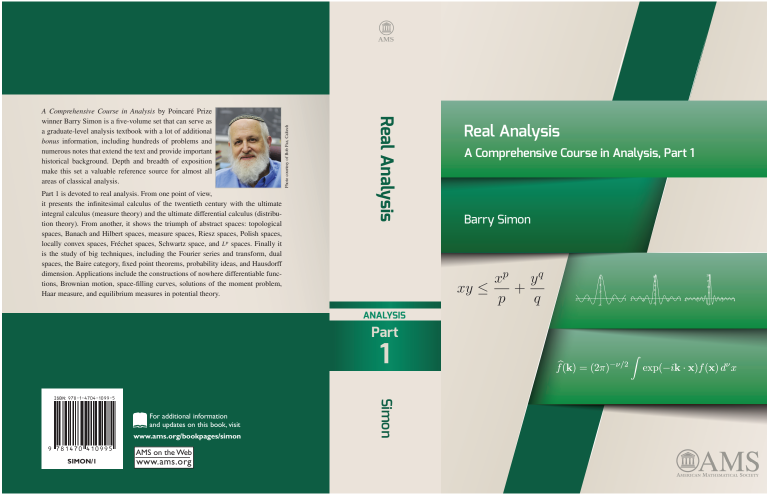

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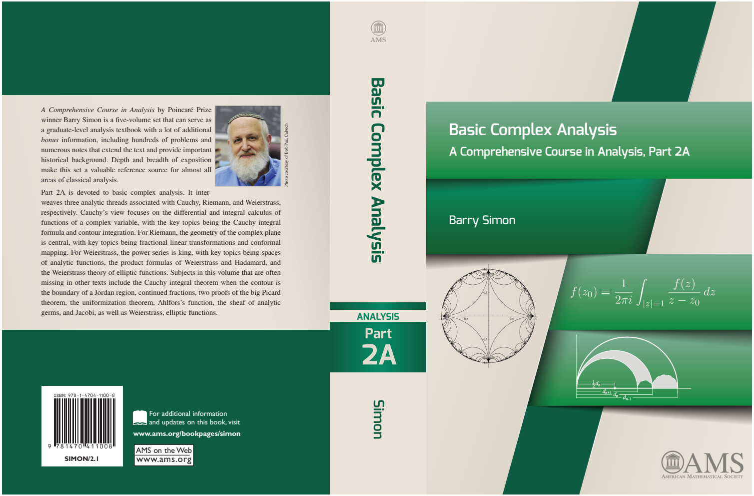

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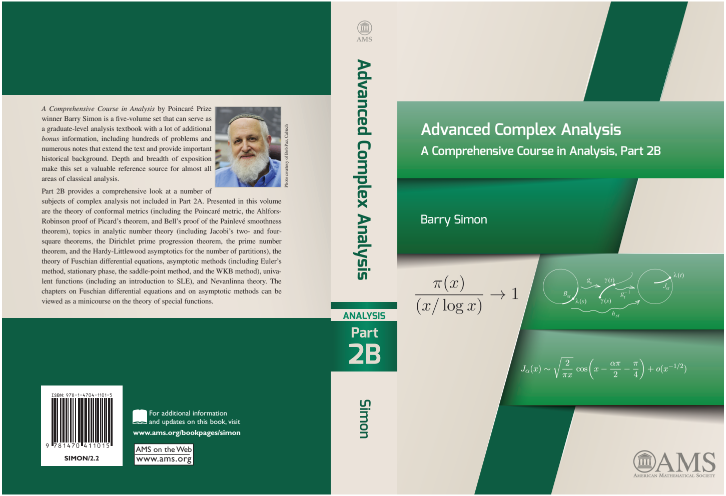

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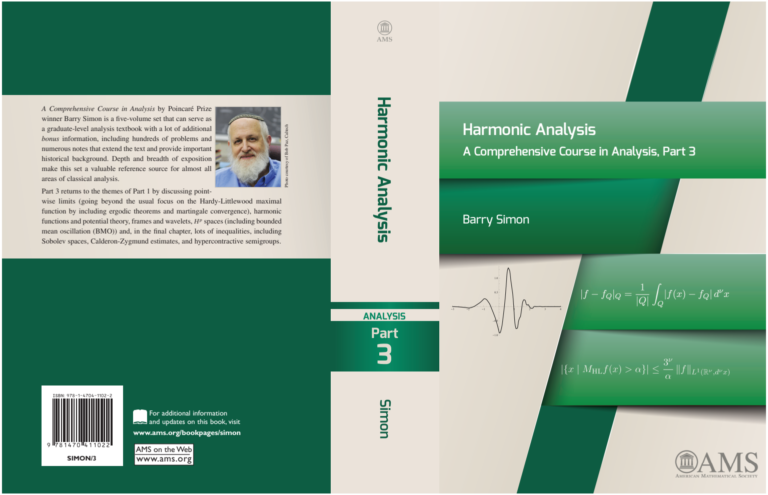

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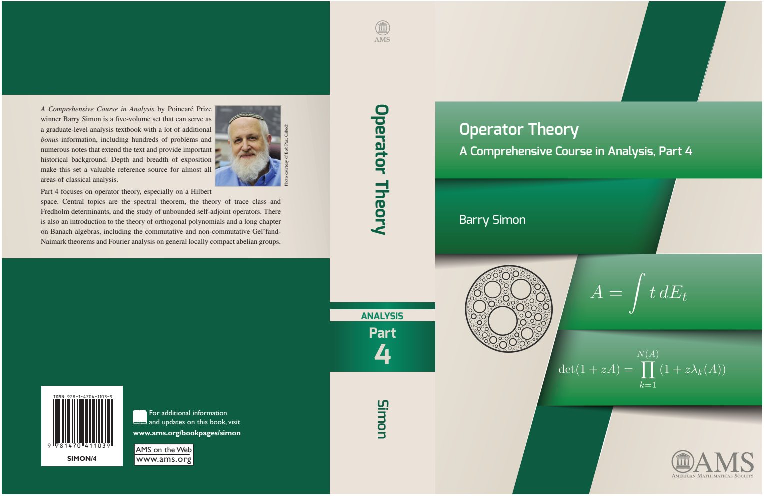

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